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HYPERSURFACES OF A FINSLER SPACE WITH DEFORMED BERWALD-INFINITE SERIES METRIC

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ABSTRACT. The rationale of this paper is to study the Finslerian hypersurfaces of a Finsler space with the special deformed Berwald-Infinite series metric. Further examined under which condition, the Finslerian hypersurfaces of this special metric becomes a hyperplane of first, second and third kinds.

Keywords: Finslerian hypersurface, Infinite series metric and Berwald metric.

AMS Subject Classification: 53B40, 53C60.

1. Introduction

In 1929, Berwald [1] introduced a very famous Finsler metric which was defined on unit ball $B^n(1)$ with all the straight line segments and its geodesics has constant flag curvature $K = 0$ in the form of

$$L = \frac{\{\sqrt{1-|x|^2|y|^2+\langle x,y\rangle^2}+\langle x,y\rangle\}^2}{\{1-|x|^2\}^2\sqrt{1-|x|^2|y|^2+\langle x,y\rangle^2}} \tag{1}$$

From the contemporary point of view, the above Berwald's metric belongs to a special kind of Finsler metric which is called Berwald type metric and defined as $\frac{(\alpha+\beta)^2}{\alpha}$[10] and authors of the papers introduced very interesting geometrical result in the field of Finsler geometry.

In 2004, Lee and Park [6] introduced a r-th series (α,β)-metric

$$L(\alpha,\beta) = \beta\sum_{k=0}^{r}(\frac{\alpha}{\beta})^k \tag{2}$$

where they assume $\alpha < \beta$. If $r = 1$ then $L = \alpha + \beta$ is a Randers metric. If $r = 2$ then $L = \alpha+\beta+\frac{\alpha^2}{\beta}$ is an amalgamation of Randers metric and Kropina metric. If $r = \infty$ then above metric is expressed as

$$L(\alpha,\beta) = \frac{\beta^2}{\beta-\alpha} \tag{3}$$

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and the metric (3) named as infinite series (α, β)-metric. This metric is very imperative due to the difference of Randers and Matsumoto metric.

In 1985 Matsumoto introduced the concept of Finslerian hypersurface and further he defined three types of hypersurfaces that were called a hyperplane of the first, second and third kinds. Further many authors studied these hyperplanes in different changes of the Finsler metric [2, 3, 4, 5, 8, 9] and obtained various interesting geometrical results.

In the present paper, the deformed Berwald-Infinite series metric is introduced and basic geometrical properties for the Finsler space with this metric has been obtained. Further it has been examined that under which condition the hypersurfaces of this special metric becomes a hyperplane of first, second and third kinds.

2. Preliminaries

Berwald-Infinite series metric is a combination of Berwald and Infinite series metric which is defined as

Definition 2.1. *Let F^n be an n-dimensional Finsler space consisting of an n-dimensional differentiable manifold M^n equipped with a fundamental function L defined as*

$$L(\alpha, \beta) = \frac{(\alpha+\beta)^2}{\alpha} + \frac{\beta^2}{(\beta-\alpha)} \tag{4}$$

then the metric L is known as Berwald-Infinite series metric and the Finsler space $F^n = \{M^n, L(\alpha, \beta)\}$ equiped with this metric is known as Berwald-infinite series Finsler space.

Differentiating equation (4) partially with respect to α and β are given by

$$\begin{cases} L_\alpha = \frac{(\alpha^2-\beta^2)(\beta-\alpha)^2+\alpha^2\beta^2}{\alpha^2(\beta-\alpha)^2}, \\ L_\beta = \frac{2(\alpha+\beta)(\beta-\alpha)^2+\alpha\beta^2-2\alpha^2\beta}{\alpha(\beta-\alpha)^2}, \\ L_{\alpha\alpha} = \frac{2\{\alpha^3+(\beta-\alpha)^3\}\beta^2}{\alpha^3(\beta-\alpha)^3}, \\ L_{\beta\beta} = \frac{2\{\alpha^3+(\beta-\alpha)^3\}}{\alpha(\beta-\alpha)^3}, \\ L_{\alpha\beta} = \frac{-2\{\alpha^3+(\beta-\alpha)^3\}\beta}{\alpha^2(\beta-\alpha)^3} \end{cases}$$

where $L_\alpha = \frac{\partial L}{\partial \alpha}, \quad L_\beta = \frac{\partial L}{\partial \beta}, \quad L_{\alpha\alpha} = \frac{\partial L\alpha}{\partial \alpha}, \quad L_{\beta\beta} = \frac{\partial L_\beta}{\partial \beta}, \quad L_{\alpha\beta} = \frac{\partial L_\alpha}{\partial \beta}$.

In Finsler space $F^n = \{M^n, L(\alpha, \beta)\}$ the normalized element of support $l_i = \dot{\partial}_i L$ and angular metric tensor h_{ij} are given by:

$$l_i = \alpha^{-1} L_\alpha Y_i + L_\beta b_i \tag{5}$$

$$h_{ij} = p a_{ij} + q_0 b_i b_j + q_{-1}(b_i Y_j + b_j Y_i) + q_{-2} Y_i Y_j \tag{6}$$

where $Y_i = a_{ij} y^j$, and the quantities p, q_0, q_{-1}, q_{-2} are given by

$$\begin{cases} p = \frac{(\beta^3-\alpha^3+2\alpha\beta^2-\alpha^2\beta)(\alpha^4-\beta^4+2\alpha\beta^3-2\alpha^3\beta+\alpha^2\beta^2)}{\alpha^4(\beta-\alpha)^3}, \\ q_0 = \frac{2(\beta^3-\alpha^3+2\alpha\beta^2-\alpha^2\beta)(3\alpha^2+\beta^2-3\alpha\beta)\beta}{\alpha^2(\beta-\alpha)^4}, \\ q_{-1} = \frac{2(\beta^3-\alpha^3+2\alpha\beta^2-\alpha^2\beta)(-3\alpha^2-\beta^2+3\alpha\beta)\beta^2}{\alpha^4(\beta-\alpha)^4}, \\ q_{-2} = \frac{(\beta^3-\alpha^3+2\alpha\beta^2-\alpha^2\beta)(\alpha^5-3\alpha^4\beta+3\alpha^3\beta^2+7\alpha^2\beta^3-9\alpha\beta^4+3\beta^5)}{\alpha^6(\beta-\alpha)^4} \end{cases} \tag{7}$$

Note: $0, -1, -2$ in the subscript represents homoginity of the respective terms.

Fundamental metric tensor $g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2$ and its reciprocal tensor g^{ij} for $L = L(\alpha, \beta)$ are given by

$$g_{ij} = pa_{ij} + p_0 b_i b_j + p_{-1}(b_i Y_j + b_j Y_i) + p_{-2} Y_i Y_j \tag{8}$$

where

$$\begin{cases} p_0 = q_0 + L_\beta^2, \\ p_{-1} = q_{-1} + L^{-1} p L_\beta, \\ p_{-2} = q_{-2} + p^2 L^{-2} \end{cases} \tag{9}$$

The reciprocal tensor g^{ij} of g_{ij} is given by

$$g^{ij} = p^{-1} a^{ij} - s_0 b^i b^j - s_{-1}(b^i y^j + b^j y^i) - s_{-2} y^i y^j \tag{10}$$

where $b^i = a^{ij} b_j$ *and* $b^2 = a_{ij} b^i b^j$.

$$\begin{cases} s_0 = \frac{1}{\tau p}\{pp_0 + (p_0 p_{-2} - p_{-1}^2)\alpha^2\}, \\ s_{-1} = \frac{1}{\tau p}\{pp_{-1} + (p_0 p_{-2} - p_{-1}^2)\beta\}, \\ s_{-2} = \frac{1}{\tau p}\{pp_{-2} + (p_0 p_{-2} - p_{-1}^2) b^2\}, \\ \tau = p(p + p_0 b^2 + p_{-1}\beta) + (p_0 p_{-2} - p_{-1}^2)(\alpha^2 b^2 - \beta^2) \end{cases} \tag{11}$$

The hv-torsion tensor $C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$ is given by [9]

$$2pC_{ijk} = p_{-1}(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + \gamma_1 m_i m_j m_k \tag{12}$$

where,

$$\gamma_1 = p\frac{\partial p_0}{\partial \beta} - 3p_{-1} q_0, \qquad m_i = b_i - \alpha^{-2}\beta Y_i \tag{13}$$

where m_i is a non-vanishing covariant vector orthogonal to the element of support y^i. Thus, it is given by

Proposition 2.1. *Let F^n be an n-dimensional Finsler space equipped with a Berwald-Infinite series metric L then its normalised supporting element l_i, angular metric h_{ij}, fundamentiacl metric tensor g_{ij} and its reciprocal metric $g^{ij}$is given by (5), (6), (8) and (10) respectively.*

Proposition 2.2. *Let F^n be an n-dimensional Finsler space equipped with a Berwald-Infinite series metric L then its hv-torsion tensor is given by (12).*

Let $\{^{\,i}_{jk}\}$ be the component of christoffel symbols of the associated Riemannian space R^n and $\bigtriangledown_k$ be the covariant derivative with respect to x^k relative to this christoffel symbol. Now it defines,

$$2E_{ij} = b_{ij} + b_{ji}, \qquad 2F_{ij} = b_{ij} - b_{ji} \tag{14}$$

where $b_{ij} = \bigtriangledown_j b_i$.

Let $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, \Gamma^i_{jk})$ be the cartan connection of F^n. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \{^{\,i}_{jk}\}$ of the special Finsler space F^n is given by

$$\begin{cases} D^i_{jk} = B^i E_{jk} + F^i_k B_j + F^i_j B_k + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} - C^i_{jm} A^m_k \\ \qquad -C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} + \lambda^s (C^i_{jm} C^m_{sk} + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}) \end{cases} \tag{15}$$

where

$$\begin{cases} B_k = p_0 b_k + p_{-1} Y_k, \\ B^i = g^{ij} B_j, \\ F_i^k = g^{kj} F_{ji}, \\ B_{ij} = \frac{\{p_{-1}(a_{ij} - \alpha^{-2} Y_i Y_j) + \frac{\partial p_0}{\partial \beta} m_i m_j\}}{2}, \\ B_i^k = g^{kj} B_{ji}, \\ A_k^m = B_k^m E_{00} + B^m E_{k0} + B_k F_0^m + B_0 F_k^m, \\ \lambda^m = B^m E_{00} + 2B_0 F_0^m, \\ B_0 = B_i y^i \end{cases} \tag{16}$$

where $'0'$ denote contraction with y^i except for the quantities p_0, q_0 and s_o.

Proposition 2.3. *Let F^n be an n-dimensional Finsler space equipped with a Berwald-Infinite series metric L then the difference tensor of its Cartan connection is given by (15).*

3. Induced Cartan Connection

Let F^{n-1} be a hypersurface of F^n given by the equation $x^i = x^i(u^\alpha)$ where $\{\alpha = 1, 2, 3...(n-1)\}$. The element of support y^i of F^n is to be taken tangential to F^{n-1}, that is [7],

$$y^i = B_\alpha^i(u) v^\alpha \tag{17}$$

The metric tensor $g_{\alpha\beta}$ and hv-tensor $C_{\alpha\beta\gamma}$ of F^{n-1} are given by

$$g_{\alpha\beta} = g_{ij} B_\alpha^i B_\beta^j, \qquad C_{\alpha\beta\gamma} = C_{ijk} B_\alpha^i B_\beta^j B_\gamma^k$$

and at each point (u^α) of F^{n-1}, a unit normal vector $N^i(u,v)$ is defined by

$$g_{ij}\{x(u,v), y(u,v)\} B_\alpha^i N^j = 0, \qquad g_{ij}\{x(u,v), y(u,v)\} N^i N^j = 1$$

Angular metric tensor $h_{\alpha\beta}$ of the hypersurfaces are given by

$$h_{\alpha\beta} = h_{ij} B_\alpha^i B_\beta^j, \qquad h_{ij} B_\alpha^i N^j = 0, \qquad h_{ij} N^i N^j = 1 \tag{18}$$

(B_i^α, N_i) inverse of (B_α^i, N^i) is given by

$$B_i^\alpha = g^{\alpha\beta} g_{ij} B_\beta^j, \qquad B_\alpha^i B_i^\beta = \delta_\alpha^\beta, \qquad B_i^\alpha N^i = 0, \qquad B_\alpha^i N_i = 0$$

$$N_i = g_{ij} N^j, \qquad B_i^k = g^{kj} B_{ji}, \qquad B_\alpha^i B_j^\alpha + N^i N_j = \delta_j^i$$

The induced connection $IC\Gamma = (\Gamma_{\beta\gamma}^{*\alpha}, G_\beta^\alpha, C_{\beta\gamma}^\alpha)$ of F^{n-1} from the Cartan's connection $C\Gamma = (\Gamma_{jk}^{*i}, \Gamma_{0k}^{*i}, C_{jk}^{*i})$ is given by [7].

$$\Gamma_{\beta\gamma}^{*\alpha} = B_i^\alpha (B_{\beta\gamma}^i + \Gamma_{jk}^{*i} B_\beta^j B_\gamma^k) + M_\beta^\alpha H_\gamma$$

$$G_\beta^\alpha = B_i^\alpha (B_{0\beta}^i + \Gamma_{0j}^{*i} B_\beta^j), \qquad C_{\beta\gamma}^\alpha = B_i^\alpha C_{jk}^i B_\beta^j B_\gamma^k$$

where

$$M_{\beta\gamma} = N_i C_{jk}^i B_\beta^j B_\gamma^k, \qquad M_\beta^\alpha = g^{\alpha\gamma} M_{\beta\gamma}, \qquad H_\beta = N_i (B_{0\beta}^i + \Gamma_{oj}^{*i} B_\beta^j)$$

and

$$B_{\beta\gamma}^i = \frac{\partial B_\beta^i}{\partial u^\gamma}, \qquad B_{0\beta}^i = B_{\alpha\beta}^i v^\alpha$$

The quantities $M_{\beta\gamma}$ and H_β are called the second fundamental v-tensor and normal curvature vector respectively [7]. The second fundamental h-tensor $H_{\beta\gamma}$ is defined as [7]

$$H_{\beta\gamma} = N_i (B_{\beta\gamma}^i + \Gamma_{jk}^{*i} B_\beta^j B_\gamma^k) + M_\beta H_\gamma \tag{19}$$

where

$$M_\beta = N_i C_{jk}^i B_\beta^j N^k \tag{20}$$

The relative h and v-covariant derivatives of projection factor B^i_α with respect to $IC\Gamma$ are given by

$$B^i_{\alpha|\beta} = H_{\alpha\beta}N^i, \qquad B^i_\alpha|_\beta = M_{\alpha\beta}N^i$$

It is obvious form the equation (3.3), that $H_{\beta\gamma}$ is generally not symmetric and

$$H_{\beta\gamma} - H_{\gamma\beta} = M_\beta H_\gamma - M_\gamma H_\beta \tag{21}$$

The above equation yield

$$H_{0\gamma} = H_\gamma, \quad H_{\gamma 0} = H_\gamma + M_\gamma H_0 \tag{22}$$

Subseuent lemmas are used, which are due to Matsumoto [7] in the trailing section

Lemma 3.1. *The normal curvature $H_0 = H_\beta v^\beta$ vanishes if and only if the normal curvature vector H_β vanishes.*

Lemma 3.2. *A hypersurface $F^{(n-1)}$ is a hyperplane of the first kind with respect to connection $C\Gamma$ if and only if $H_\alpha = 0$.*

Lemma 3.3. *A hypersurface $F^{(n-1)}$ is a hyperplane of the second kind with respect to connection $C\Gamma$ if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$.*

Lemma 3.4. *A hypersurface $F^{(n-1)}$ is a hyperplane of the third kind with respect to connection $C\Gamma$ if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = M_{\alpha\beta} = 0$.*

4. Hypersurface $F^{(n-1)}(c)$ of a deformed Berwald-Infinite series Finsler space

Let's consider a Finsler space with the deformed Berwald-Infinite series metric $L(\alpha,\beta) = \frac{(\alpha+\beta)^2}{\alpha} + \frac{\beta^2}{(\beta-\alpha)}$, where, $\alpha^2 = a_{ij}(x)y^iy^j$ is a Riemannian metric, $\beta = b_i(x)y^i$ is one-form metric and vector field $b_i(x) = \frac{\partial b}{\partial x^i}$ is a gradient of some scalar function $b(x)$. This necessitates the consideration of hypersurface $F^{(n-1)}(c)$ given by equation $b(x) = c$, a constant [9].

From the parametric equation $x^i = x^i(u^\alpha)$ of $F^{n-1}(c)$, it gives

$$\frac{\partial b(x)}{\partial u^\alpha} = 0$$

$$\frac{\partial b(x)}{\partial x^i}\frac{\partial x^i}{\partial u^\alpha} = 0$$

$$b_iB^i_\alpha = 0$$

Above information show that $b_i(x)$ are covarient component of a normal vector field of hypersurface $F^{n-1}(c)$. Further, it gives

$$b_iB^i_\alpha = 0 \quad and \quad b_iy^i = 0 \quad i.e \quad \beta = 0 \tag{23}$$

and induced metric $L(u,v)$ of $F^{n-1}(c)$ is given by

$$L(u,v) = a_{\alpha\beta}v^\alpha v^\beta, a_{\alpha\beta} = a_{ij}B^i_\alpha B^j_\beta \tag{24}$$

which is a Riemannian metric.

Substituting $\beta = 0$ in the equations (5), (6) and (8) then it gives

$$\begin{cases} p = 1, & q_0 = 0, & q_{-1} = 0, & q_{-2} = -\alpha^{-2} \\ p_0 = 4, & p_{-1} = 2\alpha^{-1}, & p_{-2} = 0, & \tau = 1, \\ s_0 = 0, & s_{-1} = \frac{2}{\alpha}, & s_{-2} = \frac{-4b^2}{\alpha^2} \end{cases} \tag{25}$$

From equation (8), it gives

$$g^{ij} = a^{ij} - \frac{2}{\alpha}(b^i y^j + b^j y^i) + \frac{4b^2}{\alpha^2} y^i y^j \tag{26}$$

Thus along $F^{n-1}(c)$, equations (26) and (23) leads to

$$g^{ij} b_i b_j = b^2$$

So it is written as

$$b_i(x(u)) = bN_i, \qquad b^2 = a^{ij} b_i b_j \tag{27}$$

Where b is the length of the vector b^i

Again equations (26) and (27) give

$$b^i = a^{ij} b_j = N^i + \frac{2b^2 y^i}{\alpha} \tag{28}$$

Thus, it is shown by theorem

Theorem 4.1. *The induced Riemannian metric in a Finslerian hypersurface $F^{(n-1)}(c)$ of the Finsler space F^n equipped with a deformed Berwald-Infinite series metric is given by (24) and its scalar function $b(x)$ is given by (27) and (28).*

Now the angular metric tensor h_{ij} and metric tensor g_{ij} of F^n are given by

$$h_{ij} = a_{ij} - \frac{1}{\alpha^2} Y_i Y_j \quad \text{and} \quad g_{ij} = a_{ij} + 4b_i b_j + \frac{2}{\alpha}(b_i Y_j + b_j Y_i) \tag{29}$$

Equations (23), (29) and (18) follow that if $h^{(a)}_{\alpha\beta}$ denote the angular metric tensor of the Riemannian $a_{ij}(x)$ then it is written along $F^{(n-1)}(c)$, $h_{\alpha\beta} = h^{(a)}_{\alpha\beta}$.

Thus along $F^{(n-1)}(c)$, $\quad \frac{\partial p_0}{\partial \beta} = \frac{-6}{\alpha}$.

From equation (11), it gives

$$r_1 = \frac{-6}{\alpha}, \qquad m_i = b_i$$

Then hv-torsion tensor becomes

$$C_{ijk} = \frac{1}{\alpha}(h_{ij} b_k + h_{jk} b_i + h_{ki} b_j) - \frac{3}{\alpha} b_i b_j b_k \tag{30}$$

in the deformed Berwald-Infinite series Finsler hypersurface $F^{(n-1)}(c)$. Due to fact, from equations (18), (19), (21), (23) and (30) are given by

$$M_{\alpha\beta} = \frac{b}{\alpha} h_{\alpha\beta} \quad \text{and} \quad M_\alpha = 0 \tag{31}$$

Therefore equation (22) follows that $H_{\alpha\beta}$ is symmetric. Thus, it is written by

Theorem 4.2. *The second fundamental v-tensor in a Finslerian hypersurface $F^{(n-1)}(c)$ of the Finsler space F^n, equipped with a deformed Berwald-Infinite series metric is given by (31) and the second fundamental h-tensor $H_{\alpha\beta}$ is symmetric.*

Now from equation (23) it becomes $b_i B^i_\alpha = 0$, it gives

$$b_{i|\beta} B^i_\alpha + b_i B^i_{\alpha|\beta} = 0$$

Therefore, from equation (21) and using $b_{i|\beta} = b_{i|j} B^j_\beta + b_i|_j N^j H_\beta$,it gives

$$b_{i|j} B^i_\alpha B^j_\beta + b_{i|j} B^i_\alpha N^j H_\beta + b_i H_{\alpha\beta} N^i = 0 \tag{32}$$

Since $b_i|_j = -b_h C^h_{ij}$, gives

$$b_{i|j}B^i_\alpha N^j = 0$$

Therefore from equation (32), it gives

$$bH_{\alpha\beta} + b_{i|j}B^i_\alpha B^j_\beta = 0 \tag{33}$$

because $b_{i|j}$ is symmetric.
Now contracting equation (33) with v^β and using equation (17),which gives

$$bH_\alpha + b_{i|j}B^i_\alpha y^j = 0 \tag{34}$$

Again contracting by v^α equation (34) and using (17) it gives

$$bH_0 + b_{i|j}y^i y^j = 0 \tag{35}$$

From lemma (3.1) and (3.2), it is clear that the deformed Berwald-Infinite series hypersuface $F^{(n-1)}(c)$ is a hyperplane of first kind if and only if $H_0 = 0$. Thus from (35) it is obvious that $F^{(n-1)}(c)$ is a hyperplane of first kind if and only if $b_{i|j}y^iy^j = 0$. This $b_{i|j}$ being the covariant derivative with respect to $C\Gamma$ of F^n is defined on y^i, but $b_{ij} = \bigtriangledown_j b_i$ is the covariant derivative with respect to Riemannian connection $\{^{\,i}_{jk}\}$ constructed from $a_{ij}(x)$. Hence b_{ij} does not depend on y^i. It is significant to consider the difference of $b_{i|j} - b_{ij}$, where $b_{ij} = \bigtriangledown_j b_i$ is presented subsequently. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \{^{\,i}_{jk}\}$ is given by equation (13). Since b_i is a gradient vector, then from equation (12) it gives

$$E_{ij} = b_{ij} \qquad F_{ij} = 0 \qquad and \qquad F^i_j = 0$$

Thus equation (13) reduces to

$$\begin{cases} D^i_{jk} = B^i b_{jk} + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m}g^{im}B_{jk} - C^i_{jm}A^m_k - C^i_{km}A^m_j \\ \qquad +C_{jkm}A^m_s g^{is} + \lambda^s(C^i_{jm}C^m_{sk} + C^i_{km}C^m_{sj} - C^m_{jk}C^i_{ms}) \end{cases} \tag{36}$$

where

$$\begin{cases} B_i = 4b_i + 2\alpha^{-1}Y_i, \\ B^i = \frac{2}{\alpha}y^i, \\ B_iB^i = 4, \\ \lambda^m = B^m b_{00}, \\ B_{ij} = \frac{1}{\alpha}(a_{ij} - \frac{Y_iY_j}{\alpha^2}) - \frac{3}{\alpha}b_ib_j, \\ B^i_j = \frac{1}{\alpha}(\delta^i_j - \alpha^{-2}Y^iY_j) - \frac{3}{\alpha}b^ib_j - \frac{6(1-b^2)}{\alpha^2}b_jy^i, \\ A^m_k = B^m_k b_{00} + B^m b_{k0} \end{cases} \tag{37}$$

In view of equations (25) and (26), the relation in (14) becomes to, by virture of euation (37). It is shown that $B^i_0 = 0$, $B_{i0} = 0$ which leads $A^m_0 = B^m b_{00}$.

Now contracting equation (36) by y^k, it gives

$$D^i_{j0} = B^i b_{j0} + B^i_j b_{00} - B^m C^i_{jm} b_{00}$$

Again contracting the above equation with respect to y^j, it gives

$$D^i_{00} = B^i b_{00} = \frac{2}{\alpha}y^i b_{00}$$

Paying attention to equation (23), along $F^{(n-1)}(c)$, which gives

$$b_iD^i_{j0} = \frac{1}{\alpha}(1-3b^2)b_j - \frac{2}{\alpha}b_iy^mC^i_{jm}b_{00} \tag{38}$$

Now contracting equation (38) by y^j, it gives

$$b_iD^i_{00} = 0 \tag{39}$$

From equations (19), (27), (28), (31) and $M_\alpha = 0$, it gives

$$b_i b^m C^i_{jm} B^j_\alpha = b^2 M_\alpha = 0$$

Thus the relation $b_{i|j} = b_{ij} - b_r D^r_{ij}$ the equation (38) and (39) gives

$$b_{i|j} y^i y^j = b_{00} - b_r D^r_{00} = b_{00}$$

Consequently equations (34) and (35) may be written as

$$\begin{cases} \alpha H_\alpha + b_{i0} B^i_\alpha = 0, \\ b H_0 + b_{00} = 0 \end{cases} \tag{40}$$

Thus the condition $H_0 = 0$ is equivalent to $b_{00} = 0$. Using the fact $\beta = b_i y^i = 0$ the condition $b_{00} = 0$ can be written as $b_{ij} y^i y^j = b_i y^i b_j y^j$ for some $c_j(x)$. Thus it can be written,

$$2b_{ij} = b_i c_j + b_j c_i \tag{41}$$

Now from equations (23) and (41), gives

$$b_{00} = 0, \quad b_{ij} B^i_\alpha B^j_\beta = 0, \quad b_{ij} B^i_\alpha y^j = 0$$

Hence equation (40) gives $H_\alpha = 0$, again from equations (41) and (37), it can be written $b_{i0} b^i = \frac{c_0 b^2}{2}$, $\lambda^m = 0$, $A^i_j B^j_\beta = 0$ and $B_{ij} B^i_\alpha B^j_\beta = \frac{1}{\alpha} h_{\alpha\beta}$.

Now using equations (19), (26), (27), (28), (31) and (36) then write the equation

$$b_r D^r_{ij} B^i_\alpha B^j_\beta = -\frac{c_0 b^3}{2\alpha} h_{\alpha\beta} \tag{42}$$

Thus, the equation (33) reduces to

$$H_{\alpha\beta} + \frac{c_0 b^2}{2\alpha} h_{\alpha\beta} = 0 \tag{43}$$

Hence the hypersurface $F^{(n-1)}(c)$ is umbilic. Thus

Lemma 4.1. *The second fundamental tensor in a Finslerian hypersurface $F^{(n-1)}(c)$ of the Finsler space F^n, equipped with a deformed Berwald-Infinite series metric is directly proportional to its angular metric tensor.*

Theorem 4.3. *The necessary and sufficient condition for a Finslerian hypersurface $F^{(n-1)}(c)$ of the Finsler space F^n, equipped with a deformed Berwald-Infinite series metric to a hyperplane of first kind is (41).*

Now from lemma (3.3), $F^{(n-1)}(c)$ is a hyperplane of second kind if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$. Thus equation (42) gives us

$$c_0 = c_i(x) y^i = 0$$

Therefore there exist a function $\psi(x)$ such that

$$c_i(x) = \psi(x) b_i(x)$$

Therefore equation (41) gives

$$2b_{ij} = b_i(x)\psi(x)b_j(x) + b_j(x)\psi(x)b_i(x)$$

This can also be written as

$$b_{ij} = \psi(x) b_i b_j$$

Thus

Theorem 4.4. *The necessary and sufficient condition in a Finslerian hypersurface $F^{(n-1)}(c)$ of the Finsler space F^n, equipped with a deformed Berwald-Infinite series metric to be a hyperplane of second kind is (43).*

Again lemma (4.4), together with (31) and $M_\alpha = 0$, show that $F^{(n-1)}(c)$ does not become a hyperplane of third kind. Thus

Theorem 4.5. *The Finslerian hypersurface $F^{(n-1)}(c)$ of the Finsler space F^n, equipped with a deformed Berwald-Infinite series metric is not a hyperplane of the third kind.*

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