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EXISTENCE AND UNIQUENESS OF AN INVERSE PROBLEM FOR A WAVE EQUATION WITH DYNAMIC BOUNDARY CONDITION

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ABSTRACT. In this paper, an initial boundary value problem for a wave equation with dynamic boundary condition is considered. Giving an additional condition, a time-dependent coefficient is determined and existence and uniqueness theorem for small times is proved.

Keywords: Wave equation, Inverse problem, Fourier method.

AMS Subject Classification: 35R30, 35L05, 34B09

1. INTRODUCTION

The pioneering model of the hyperbolic equations and one of the most important equation of mathematical physics is the wave equation. Wave equations occur in many fields such as electromagnetic theory, acoustics, hydrodynamics, elasticity and quantum theory, see [4] and [5].

Consider the following initial-boundary value problem for one dimensional wave equation

$$u_{tt} = u_{xx} + a(t)u(x,t) + f(x,t), \quad (x,t) \in \overline{D}_T, \tag{1}$$

$$u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x),\ 0 \le x \le 1, \tag{2}$$

$$u(0,t) = 0,\ 0 \le t \le T, \tag{3}$$

$$mu_{tt}(1,t) + du_x(1,t) + ku(1,t) = 0,\ 0 \le t \le T, \tag{4}$$

where $D_T = \{(x,t):\ 0 < x < 1,\ 0 < t \le T\}$ for some fixed $T > 0$, f, φ, ψ are given functions and m, d, k are given numbers which are not simultaneously zero.

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This model can be used in vibration of an uniform elastic bar subjected to distributed force $f(x,t)$ per unit length, the functions $\varphi(x)$ and $\psi(x)$ are the axial displacement and axial velocity of the bar, respectively. The boundary condition (3) means that the left end of the bar is fixed while the boundary condition (4), known as dynamic boundary condition, describes the right end of the bar is connected to a mass m and string where $ku(1,t)$ is spring force and $mu_{tt}(1,t)$ is the inertia force, [4]. Such type of boundary conditions also arise in a model of flexible membrane which boundary affected by vibration only in a region, [7].

When the function $a(t)$ is given, the problem of finding the displacement $u(x,t)$ from the equation (1), initial condition (2) and the boundary conditions (3) and (4) is called the direct (forward) problem. The well-posedness of the direct problem has been established in [19] and with another boundary conditions (i.e. integral, non-local etc.) has been studied in [2], [18].

When the function $a(t)$ for $t \in [0,T]$ is unknown, the inverse problem is formulated as finding the pair of functions $\{a(t), u(x,t)\}$ which satisfy the equation (1), initial conditions (2), boundary conditions (3) and (4), and the additional condition

$$u(x_0,t) = h(t),\ x_0 \in (0,1),\ 0 \le t \le T. \tag{5}$$

The inverse problems of determining the time dependent coefficient $a(t)$ is scarce. The inverse problems for the wave equation with different boundary conditions and space dependent coefficients are considered in [10], [16], [20], [24] and more recently in [9], [17]. The inverse problem for the wave equation with time dependent coefficient with integral condition is investigated in [15] and with non-classical boundary condition is studied in [1]. The time-dependent source function of a time-fractional wave equation with integral condition in a bounded domain is determined in [21]. Since the inverse problems for linear wave equations with dynamic boundary conditions are scarce, it is important to note that the paper [22] considers the inverse source problem for a time-fractional wave equation of the order $1 < \beta < 2$ with dynamic boundary condition. Notice that for $\beta = 2$, the time-fractional equation becomes classical wave equation. Although the authors use the variational formulation to determine both the solution of the equation and the source term and prove the existence and uniqueness of the solution in the suitable functional spaces in [22], we present the fixed-point system via Fourier series, which brings along computations that are technically simple, to obtain the solution of the inverse coefficient problem.

In present paper, we consider an initial boundary value problem for a wave equation with dynamic boundary condition. Giving an additional condition,we determine the time-dependent coefficient and prove the existence and uniqueness theorem for small T.

The article is organized as following: In Section 2, we present auxiliary spectral problem of this problem and its properties. In Section 3, the series expansion method in terms of eigenfunctions converts the inverse problem to a fixed point problem in a suitable Banach space. Under some consistency, regularity conditions on initial and boundary data the existence and uniqueness of the inverse problem is shown by the way that the fixed point problem has unique solution for small T.

2. Auxiliary Spectral Problem

Since the function a is space independent, m, d, k are constants and the boundary conditions (3) and (4) are linear and homogeneous, the method of separation of variables is suitable for investigating this problem.

The auxiliary spectral problem of the problem is

$$\begin{cases} X''(x)+\lambda X(x)=0,\ 0\le x\le 1, \\ X(0)=0, \\ (m\lambda-k)X(1)=dX'(1). \end{cases} \tag{6}$$

The problems on vibration of a homogeneous loaded string, torsional vibrations of a rod with a pulley at the one end, heat propagation in a rod with lumped heat capacity at one end lead to this spectral problem.

Since the boundary condition includes the spectral parameter, this problem differs from the classical Sturm-Liouville problems. It makes impossible to apply the classical results on eigenfunction expansion. Thus we need the explicit availability of basis for the expansion in terms of eigenfunctions of the auxiliary spectral problem (6). The spectral analysis of such type of problems was started by [23], and after that [3], [6], [11], [12].

Consider the spectral problem (6) with $md>0$. This problem has the eigenvalues $\lambda_n=\mu_n^2$, $n=0,1,2,...$ are real and simple, and form an unbounded increasing sequence. The eigenfunctions $X_n(x)$ corresponding to λ_n has n simple zeros in the interval $(0,1)$. The eigenvalues and eigenfunctions have the following asymptotic behaviour [8]:

$$\sqrt{\lambda_n}=\mu_n=n\pi+O(\frac{1}{n}),\quad X_n(x)=\sin(n\pi x)+O(\frac{1}{n}),$$

for sufficiently large n.

It was shown in [13] that the system $\{X_n(x)\}$, $(n=0,1,...;n\ne n_0)$ forms a Riesz basis in $L_2[0,1]$ where n_0 be arbitrary non-negative integer. The system $\{Y_n(x)\}$, $(n=0,1,...;n\ne n_0)$ which has the form

$$Y_n(x)=\frac{1}{\|X_n\|^2_{L_2[0,1]}+\frac{m}{d}X_n^2(1)}\left(X_n(x)-\frac{X_n(1)}{X_{n_0}(1)}X_{n_0}(x)\right)$$

is biorthogonal to the system $\{X_n(x)\}$, $(n=0,1,...;n\ne n_0)$.

The following Bessel-type inequalities are true for the system $\{X_n(x)\}$, $(n=0,1,...;n\ne n_0)$, see [8].

Lemma 2.1. *(Bessel-type inequalities) Let $g(x)\in L_2[0,1]$, then the estimates*

$$\sum_{\substack{n=0\\ n\ne n_0}}^{\infty}|(g,X_n)|^2\le C_1\|g\|^2_{L_2[0,1]},\quad \sum_{\substack{n=0\\ n\ne n_0}}^{\infty}|(g,Y_n)|^2\le C_2\|g\|^2_{L_2[0,1]}$$

hold for some positive constant C_i, $i=1,2$, where $(g,X_n)=\int_0^1 g(x)X_n(x)dx$ and $(g,Y_n)=\int_0^1 g(x)Y_n(x)dx$ are the usual inner products in $L_2[0,1]$.

Let us denote

$$S_{n_0}:=\Big\{g(x)\in C^4[0,1],\ g(0)=g''(0)=0,$$

$$g(1)=g'(1)=g''(1)=g'''(1)=0,\ \int_0^1 g(x)X_{n_0}(x)dx=0\Big\}.$$

Lemma 2.2. *If $g(x) \in S_{n_0}$, then we have*

$$\mu_n^4(g, X_n) = (g^{(4)}, X_n), \ \ \mu_n^4(g, Y_n) = (g^{(4)}, Y_n), \tag{7}$$

$$\sum_{\substack{n=0\\ n\neq n_0}}^{\infty} \left|\mu_n^2\left(g, X_n\right)\right| \leq C_3 \left\|g\right\|_{C^4[0,1]}, \ \sum_{\substack{n=0\\ n\neq n_0}}^{\infty} \left|\mu_n^2\left(g, Y_n\right)\right| \leq C_4 \left\|g\right\|_{C^4[0,1]}, \tag{8}$$

$$\sum_{\substack{n=0\\ n\neq n_0}}^{\infty} \left|\mu_n\left(g, X_n\right)\right| \leq C_5 \left\|g\right\|_{C^4[0,1]}, \ \sum_{\substack{n=0\\ n\neq n_0}}^{\infty} \left|\mu_n\left(g, Y_n\right)\right| \leq C_6 \left\|g\right\|_{C^4[0,1]}, \tag{9}$$

$$\sum_{\substack{n=0\\ n\neq n_0}}^{\infty} \left|\left(g, X_n\right)\right| \leq C_7 \left\|g\right\|_{C^4[0,1]}, \ \sum_{\substack{n=0\\ n\neq n_0}}^{\infty} \left|\left(g, Y_n\right)\right| \leq C_8 \left\|g\right\|_{C^4[0,1]}, \tag{10}$$

where C_i, $i = \overline{3,8}$ are some positive constant.

Proof. Since $\lambda_n X_n(x) = -X_n''(x)$ and $X_n(0) = 0$, the equalities (7) can be obtained by applying four times integration by parts in (6) considering that $g(x) \in S_{n_0}$. The estimate $\sum_{\substack{n=0\\ n\neq n_0}}^{\infty} \left|\mu_n^2\left(g, X_n\right)\right| \leq C_3 \left\|g\right\|_{C^4[0,1]}$ is obtained from the Lemma 2.1, equation (7) by using Schwartz inequality. Then the convergence of the series $\sum_{\substack{n=0\\ n\neq n_0}}^{\infty} \left|\mu_n^2\left(g, X_n\right)\right|$ is equivalent to the convergence of

$$\sum_{\substack{n=0\\ n\neq n_0}}^{\infty} \left|\mu_n^2\left(g, Y_n\right)\right| = \sum_{\substack{n=0\\ n\neq n_0}}^{\infty} \frac{\left|\mu_n^2\left(g, X_n\right)\right|}{\left\|X_n\right\|_{L_2[0,1]}^2 + \frac{m}{d} X_n^2(1)}.$$

Finally, the estimates (9) and (10) are hold because for sufficiently large p the series $\sum_{n=p}^{\infty} \left|\mu_n^2\left(g, X_n\right)\right|$ is the majorant for the series $\sum_{n=p}^{\infty} \left|\mu_n\left(g, X_n\right)\right|$ and $\sum_{n=p}^{\infty} \left|\left(g, X_n\right)\right|$. □

Let us introduce the functional space

$$B_{\beta,T}^{\alpha} = \left\{ u(x,t) = \sum_{\substack{n=0\\ n\neq n_0}}^{\infty} u_n(t) X_n(x) : \ u_n(t) \in C[0,T], \right.$$
$$\left. J_T(u) = \left[\sum_{\substack{n=0\\ n\neq n_0}}^{\infty} \left(\mu_n^{\alpha} \left\|u_n(t)\right\|_{C[0,T]} \right)^{\beta} \right]^{1/\beta} < +\infty \right\}$$

with the norm $\left\|u(x,t)\right\|_{B_{\beta,T}^{\alpha}} \equiv J_T(u)$ which relates the Fourier coefficients of the function $u(x,t)$ by the eigenfunctions $X_n(x)$, $n = 1, 2, ...$ where $\alpha \geq 0$ and $\beta \geq 1$. It is shown in [14] that $B_{\beta,T}^{\alpha}$ is Banach space. Obviously $E_T^{\alpha} = B_{\beta,T}^{\alpha} \times C[0,T]$ with the norm $\|z\|_{E_T^{\alpha}} = \|u(x,t)\|_{B_{\beta,T}^{\alpha}} + \|a(t)\|_{C[0,T]}$ is also Banach space, where $z = \{a(t), u(x,t)\}$.

In this paper, we will use the functional space $B_{1,T}^{1}$ for convenience.

3. Solution of the Inverse Problem

In this section, we will examine the existence and uniqueness of the solution of the inverse initial-boundary value problem for the equation(1) with time-dependent coefficient.

Definition 3.1. *The pair $\{a(t), u(x,t)\}$ from the class $C[0,T] \times C^2(\overline{D}_T)$ for which the conditions (1)-(5) are satisfied is called the classical solution of the inverse problem (1)-(5).*

For a given $a(t)$, $t \in [0,T]$, to construct the formal solution of the direct problem (1)-(4) we will use the generalised Fourier method. Based on this method, let us seek the solution

$$u(x,t) = \sum_{\substack{n=0 \\ n \neq n_0}}^{\infty} u_n(t) X_n(x) \tag{11}$$

where $u_n(t) = \int_0^1 u(x,t) Y_n(x) dx$.

The functions $u_n(t)$,$(n = 0, 1, ...; n \neq n_0)$ satisfy the Cauchy problem

$$\begin{cases} u_n''(t) + \mu_n^2 u_n(t) = F_n(t; a, u), \\ u_n(0) = \varphi_n, \quad u_n'(0) = \psi_n, \end{cases} ,(n = 0, 1, ...; n \neq n_0)$$

where $F_n(t; a, u) = a(t)u_n(t) + f_n(t)$, $f_n(t) = \int_0^1 f(x,t) Y_n(x) dx$, $\varphi_n = \int_0^1 \varphi(x) Y_n(x) dx$, $\psi_n = \int_0^1 \psi(x) Y_n(x) dx$.

Solving these Cauchy problems, we obtain

$$\begin{aligned} u_n(t) &= \varphi_n \cos(\mu_n t) + \frac{1}{\mu_n} \psi_n \sin(\mu_n t) \\ &+ \frac{1}{\mu_n} \int_0^t F_n(\tau; a, u) \sin(\mu_n (t - \tau)) \, d\tau \end{aligned} \tag{12}$$

Substituting (12) into (11), we have the formal solution

$$\begin{aligned} u(x,t) &= \sum_{\substack{n=0 \\ n \neq n_0}}^{\infty} \left[\varphi_n \cos(\mu_n t) + \frac{1}{\mu_n} \psi_n \sin(\mu_n t) \right. \\ &\left. + \frac{1}{\mu_n} \int_0^t F_n(\tau; a, u) \sin(\mu_n (t - \tau)) \, d\tau \right] X_n(x). \end{aligned} \tag{13}$$

To obtain the coefficient $a(t)$, consider the additional condition (5) in the equation (1), i.e.

$$a(t) = \frac{1}{h(t)} \left[h''(t) - f(x_0, t) - u_{xx}(x_0, t) \right].$$

Considering (12) into the equation (11) with the second partial derivative by x, we can easily get $u_{xx}(x_0,t)$. Thus we have

$$a(t) = \frac{1}{h(t)}\left[h''(t) - f(x_0,t) + \sum_{\substack{n=0\\ n\neq n_0}}^{\infty} \mu_n^2 \left\{\varphi_n \cos(\mu_n t) + \frac{1}{\mu_n}\psi_n \sin(\mu_n t) + \frac{1}{\mu_n}\int_0^t F_n(\tau;a,u)\sin(\mu_n(t-\tau))\,d\tau\right\} X_n(x_0)\right]. \tag{14}$$

Thus, the solution of problem (1)-(5) is reduced to the solution of system (13)-(14) with respect to the unknown functions $\{a(t), u(x,t)\}$.

From the definition of the classical solution of problem (1)-(5), the following lemma is proved.

Lemma 3.1. *If $\{a(t), u(x,t)\}$ is any solution of problem (1)-(5), then the functions*

$$u_n(t) = \int_0^1 u(x,t)Y_n(x)dx, n = 1, 2, ...$$

satisfy the equation (12) in $[0,T]$.

From Lemma 3.1, it follows that to prove the uniqueness of the solution of the problem (1)-(5) is equivalent to prove the uniqueness of the solution of system (13)-(14).

Let us denote $z = [a(t), u(x,t)]^T$ and consider the operator equation

$$z = \Phi(z). \tag{15}$$

The operator Φ is determined in the set of functions z and has the form $[\phi_1, \phi_2]^T$, where

$$\phi_1(z) = \frac{1}{h(t)}\left[h''(t) - f(x_0,t) + \sum_{\substack{n=0\\ n\neq n_0}}^{\infty} \mu_n^2 \left\{\varphi_n \cos(\mu_n t) + \frac{1}{\mu_n}\psi_n \sin(\mu_n t) + \frac{1}{\mu_n}\int_0^t F_n(\tau;a,u)\sin(\mu_n(t-\tau))\,d\tau\right\} X_n(x_0)\right], \tag{16}$$

$$\phi_2(z) = \sum_{\substack{n=0\\ n\neq n_0}}^{\infty} \left[\varphi_n \cos(\mu_n t) + \frac{1}{\mu_n}\psi_n \sin(\mu_n t) + \frac{1}{\mu_n}\int_0^t F_n(\tau;a,u)\sin(\mu_n(t-\tau))\,d\tau\right] X_n(x). \tag{17}$$

Let us show that Φ maps E_T^1 onto itself continuously. In other words, we need to show $\phi_1(z) \in C[0,T]$ and $\phi_2(z) \in B_{1,T}^1$ for arbitrary $z = [a(t), u(x,t)]^T$ with $a(t) \in C[0,T]$, $u(x,t) \in B_{1,T}^1$.

We will use the following assumptions on the data of problem (1)-(5):

(A_1)**:** $\varphi(x), \psi(x) \in S_{n_0}$,

(A_2)**:** $h(t) \in C^2[0,T]$, $h(0) = \varphi(x_0),\ h'(0) = \psi(x_0),\ h(t) \neq 0$,

(A_3)**:** $f(x,t) \in C(\overline{D}_T);\ \ f(x,t) \in S_{n_0}, \forall t \in [0,T]$.

First, let us show that $\phi_1(z) \in C[0,T]$. Under the assumptions (A$_1$)-(A$_3$), we obtain from (16)

$$\max_{0 \le t \le T} |\phi_1(z)| \le R_1(T) + R_2(T)\, \|a(t)\|_{C[0,T]}\, \|u(x,t)\|_{B^1_{1,T}} \tag{18}$$

where $R_1(T) = \frac{1}{\|h(t)\|_{C[0,T]}} (\|h''(t)\|_{C[0,T]} + \|f(x_0,t)\|_{C[0,T]} + C_4\, \|\varphi(x)\|_{C^4[0,1]} + C_6(\|\psi(x)\|_{C^4[0,1]} + T\, \|f(x,\cdot)\|_{C^4[0,1]}))$, $R_2(T) = \frac{T}{\|h(t)\|_{C[0,T]}}$. Since the right hand side is bounded, $\phi_1(z) \in C[0,T]$.

Now, let us show that $\phi_2(z) \in B^1_{1,T}$, i.e. we need to show

$$J_T(\phi_2) = \sum_{\substack{n=0 \\ n \neq n_0}}^{\infty} \mu_n\, \|\phi_{2n}(t)\|_{C[0,T]} < +\infty,$$

where

$$\phi_{2n}(t) = \varphi_n \cos(\mu_n t) + \frac{1}{\mu_n} \psi_n \sin(\mu_n t) + \frac{1}{\mu_n} \int_0^t F_n(\tau; a, u) \sin(\mu_n (t - \tau))\, d\tau.$$

After some manipulations under the assumptions (A$_1$)-(A$_3$), we get

$$\sum_{\substack{n=0 \\ n \neq n_0}}^{\infty} \mu_n\, \|\phi_{2n}(t)\|_{C[0,T]} \le \widetilde{R}_1(T) + \widetilde{R}_2(T)\, \|a(t)\|_{C[0,T]}\, \|u(x,t)\|_{B^1_{1,T}} \tag{19}$$

where $\widetilde{R}_1(T) = \left[C_6\, \|\varphi(x)\|_{C^4[0,1]} + C_8(\|\psi(x)\|_{C^4[0,1]} + T\, \|f(x,\cdot)\|_{C^4[0,1]}) \right]$, $\widetilde{R}_2(T) = T$.

Thus $J_T(\phi_2) < +\infty$ and ϕ_2 is belongs to the space $B^1_{1,T}$.

Now, let z_1 and z_2 be any two elements of E^1_T. We know that $\|\Phi(z_1) - \Phi(z_2)\|_{E^1_T} = \|\phi_1(z_1) - \phi_1(z_2)\|_{C[0,T]} + \|\phi_2(z_1) - \phi_2(z_2)\|_{B^1_{1,T}}$. Here $z_i = \left[a^i(t), u^i(x,t)\right]^T$, $i = 1, 2$.

Under the assumptions (A$_1$)-(A$_3$) and considering (18)-(19), we obtain

$$\|\Phi(z_1) - \Phi(z_2)\|_{E^1_T} \le A(T) C(a^1, u^2)\, \|z_1 - z_2\|_{E^1_T}$$

where $A(T) = T\left(1 + \frac{1}{\|h(t)\|_{C[0,T]}}\right)$ and $C(a^1, u^2)$ is the constant includes the norms of $\left\|a^1(t)\right\|_{C[0,T]}$ and $\left\|u^2(x,t)\right\|_{B^1_{1,T}}$.

For sufficiently small T, $0 < A(T) < 1$. This implies that the operator Φ is contraction mapping which maps E^1_T onto itself continuously. Then according to Banach fixed point theorem there exists a unique solution of (15).

Thus, we proved the following theorem:

Theorem 3.1 (Existence and uniqueness)**.** *Let the assumptions (A$_1$)-(A$_3$) be satisfied. Then, the inverse problem (1)-(5) has a unique solution for small T.*

4. Conclusion

The inverse problems for linear wave equations with dynamic boundary conditions connected with recovery of the coefficient are scarce. The paper consider the of inverse problem of recovering a time-dependent coefficient in an initial-boundary value problem for a wave equation. The series expansion method in terms of eigenfunction of a Sturm-Liouville problem converts the considered inverse problem to a fixed point problem in a suitable Banach space. Under some consistency and regularity conditions on initial and boundary data, the existence and uniqueness of inverse problem is shown by using the Banach fixed point theorem.

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