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WEIGHTED STATISTICAL CONVERGENCE OF ORDER α IN PARANORMED SPACES

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Abstract. In this study, we introduce and examine the concept of weighted statistical convergence of order α in paranormed spaces. Also some relations between weighted statistical convergence of order α and $\left[\left(\overline{N},p_n\right),g\right]_r^\alpha$ −summability are given.

Keywords: Weighted statistical convergence, Paranormed space, Cesàro summability, Density.

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1. Introduction

The idea of statistical convergence was given by Zygmund [26] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Steinhaus [25] and Fast [13] and then reintroduced independently by Schoenberg [23]. Over the years and under different names, statistical convergence has been discussed in the Theory of Fourier Analysis, Ergodic Theory, Number Theory, Measure Theory, Trigonometric Series, Turnpike Theory and Banach Spaces. Later on it was further investigated from the sequence spaces point of view and linked with summability theory by Cinar et al. [5], Colak [6], Connor [7], Et et al. ([10],[11],[12],[22]), Fridy [14], Işık et al. ([16],[17]), Mursaleen [19], Salat [21], Srivastava and Et [24] and many others. For some more fundamental and current topics, please refer to [4, 8].

Let $\mathbb{N}$ be the set of all natural numbers and $K \subseteq \mathbb{N}$ and $K(n) = \{k \leq n : k \in K\}$. The natural density of K is defined by $\delta(K) = \lim_n \frac{1}{n}|K(n)|$ if limit exists. The vertical bars indicate the number of the elements in enclosed set.

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The number sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$ the set $K(\varepsilon) = \{k \leq n : |x_k - L| \geq \varepsilon\}$ has natural density zero.

Weighted statistical convergence was first defined by Karakaya and Chishti [18] and the concept was modified by Mursaleen et al. [20]. Recently Ghosal [15] was revised the definition of weighted statistical convergence as follows.

Let (p_n) be a sequence of real numbers such that $\liminf p_n > 0$ and $P_n = p_1 + p_2 + p_3 + ... + p_n$ for all $n \in \mathbb{N}$. A sequence $x = (x_n)$ is said to be weighted statistically convergent of order α (where $0 < \alpha \leq 1$) to L if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{P_n^\alpha} |\{k \leq P_n \ : p_k |x_k - L| \geq \varepsilon\}| = 0.$$

In this case we write $S_{\overline{N}}^\alpha - \lim x = L$. By $S_{\overline{N}}^\alpha$, we denote the set of all weighted statistically convergent sequences of order α.

Alotaibi and Alroqi [1] was defined $g-$convergence and $g-$statistical convergence in paranormed spaces and later on it was further investigated by Alghamdi and Mursaleen [2], Arani et al. [3] and Ercan [9].

2. Main results

In this section we give the main results of this article.

Definition 1 Let (p_n) be a sequence of real numbers such that $\liminf p_n > 0$ and $P_n = p_1 + p_2 + p_3 + ... + p_n$ for all $n \in \mathbb{N}$. A sequence $x = (x_n)$ is said to be weighted statistically convergent of order α $(0 < \alpha \leq 1)$ (or $S_{\overline{N}}^\alpha(g)-$statistically convergent) to L in (X, g), if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{P_n^\alpha} |\{k \leq P_n : p_k g(x_k - L) \geq \varepsilon\}| = 0,$$

where $P_n^\alpha = (P_n)^\alpha$. In this case we write $S_{\overline{N}}^\alpha(g) - \lim x = L$ or $x_k \to L\left(S_{\overline{N}}^\alpha(g)\right)$. We denote the set of all weighted statistically convergent sequences of order α by $S_{\overline{N}}^\alpha(g)$. If we take $\alpha = 1$, we write $S_{\overline{N}}(g)$ instead of $S_{\overline{N}}^\alpha(g)$. Here and in what follows, (X, g) will denote a paranormed space with paranorm g.

Definition 2 Let (p_k) be a sequence of nonnegative real numbers such that $p_1 > 0$ and $P_n = \sum_{k=1}^{n} p_k \to \infty$ as $n \to \infty$, $r > 0$ be a real number. A sequence $x = (x_n)$ is said to be weighted $\left(\overline{N}, p_n\right)-$summable of order α $(0 < \alpha \leq 1)$ (or $\left[\left(\overline{N}, p_n\right), g\right]_r^\alpha-$summable) to L in (X, g), if

$$\lim_{n\to\infty} \frac{1}{P_n^\alpha} \sum_{k=0}^{n} p_k g(x_k - L)^r = 0$$

and we write $x_k \to L\left(\left[\left(\overline{N}, p_n\right), g\right]_r^\alpha\right)$. We denote the set of all weighted $\left(\overline{N}, p_n\right)-$summable sequences of order α by $\left[\left(\overline{N}, p_n\right), g\right]_r^\alpha$. If we take $\alpha = 1$, we write $\left[\left(\overline{N}, p_n\right), g\right]_r$ instead of $\left[\left(\overline{N}, p_n\right), g\right]_r^\alpha$ and $r = 1$, we write $\left[\left(\overline{N}, p_n\right), g\right]^\alpha$ instead of $\left[\left(\overline{N}, p_n\right), g\right]_r^\alpha$. In the special cases $r = 1$ and $\alpha = 1$ we write $\left[\left(\overline{N}, p_n\right), g\right]$ instead of $\left[\left(\overline{N}, p_n\right), g\right]_r^\alpha$.

The proof of each of the following results is straightforward, so we choose to state these results without proof.

Theorem 3 If a sequence $x=(x_k)$ is weighted statistical convergence of order α in (X,g), then $S^{\alpha}_{\overline{N}}(g)$ −limit is unique.

Theorem 4 Let $S^{\alpha}_{\overline{N}}(g)-\lim x=L_1$ and $S^{\alpha}_{\overline{N}}(g)-\lim y=L_2$. Then

i) $S^{\alpha}_{\overline{N}}(g)-\lim(x\pm y)=L_1\pm L_2$

ii) $S^{\alpha}_{\overline{N}}(g)-\lim cx=cL, c\in\mathbb{R}$.

Theorem 5 Let x be a $\left[\left(\overline{N},p_n\right),g\right]^{\alpha}_{r}$ −summable sequence to L. If the following assertions hold, then x is $S^{\alpha}_{\overline{N}}(g)$ −statistically convergent to L.

i) $0<r<1$ and $0\leq g(x_k-L)<1$,

ii) $1\leq r<\infty$ and $1\leq g(x_k-L)<\infty$.

Proof. Since $x=(x_k)$ is $\left[\left(\overline{N},p_n\right),g\right]^{\alpha}_{r}$ −summable to L we have

$$\frac{1}{P^{\alpha}_n}\sum_{k=1}^{n}p_k g(x_k-L)^r=0.$$

From (i) and (ii) we can write

$$p_k g(x_k-L)^r\geq p_k g(x_k-L).$$

For any sequence (x_k) in (X,g) and $\varepsilon>0$ we have

$$\begin{aligned}\sum_{k=1}^{n}p_k g(x_k-L)^r &\geq \sum_{k=1}^{n}p_k g(x_k-L)\\ &\geq |\{k\leq P_n : p_k g(x_k-L)\geq\varepsilon\}|\,\varepsilon\end{aligned}$$

and so that

$$\frac{1}{P^{\alpha}_n}|\{k\leq P_n : p_k g(x_k-L)\geq\varepsilon\}|\,\varepsilon\leq\frac{1}{P^{\alpha}_n}\sum_{k=1}^{n}p_k g(x_k-L)^r\to 0.$$

This means that $x=(x_k)$ is $S^{\alpha}_{\overline{N}}(g)$ −statistically convergent to L.

Theorem 6 Let x be a $S_{\overline{N}}(g)$ −statistically convergent sequence and $p_k g(x_k-L)\leq M$. If the following assertions hold, then x is $\left[\left(\overline{N},p_n\right),g\right]^{\alpha}_{r}$ −summable sequence to L.

i) $0<r<1$ and $1\leq M<\infty$,

ii) $1\leq r<\infty$ and $0\leq M<1$.

Proof. Suppose that $x=(x_k)$ is a $S_{\overline{N}}(g)$ −statistically convergent sequence to L. Then for every $\varepsilon>0$ we have $\delta_{\bar{N}}(K(\varepsilon))=0$, where $K(\varepsilon)=\{k\in\mathbb{N}: p_k g(x_k-L)\geq\varepsilon\}$. Write $K_{P_n}(\varepsilon)=\{k\leq P_n : p_k g(x_k-L)\geq\varepsilon\}$. Since $p_k g(x_k-L)\leq M$ $(k=1,2,...)$ we have

$$\begin{aligned}\frac{1}{P_n}\sum_{k=1}^{n}p_k g(x_k-L)^r &= \frac{1}{P_n}\sum_{\substack{k=1\\ k\notin K_{P_n}(\varepsilon)}}^{n}p_k g(x_k-L)^r+\frac{1}{P_n}\sum_{\substack{k=1\\ k\in K_{P_n}(\varepsilon)}}^{n}p_k g(x_k-L)^r\\ &\leq \varepsilon+M\frac{K_{P_n}(\varepsilon)}{P_n}\to 0.\end{aligned}$$

Hence $x_k\to L\left(\left[\left(\overline{N},p_n\right),g\right]_r\right)$.

Theorem 7 Let $\lim\limits_{n\to\infty}\dfrac{p_{n+1}}{P^{\alpha}_n}=0$ and $S^{\alpha}_{\overline{N}}(g)-\lim x=L$, then $S^{\alpha}(g)-\lim x=L$.

Proof. Let $S_{\overline{N}}^{\alpha}(g)-\lim x=L$, $\liminf p_n>c>0$ and n be a sufficiently large number, then there exists a positive integer m such that $P_m<n\leq P_{m+1}$. Then for $\varepsilon>0$,

$$\begin{aligned} & \frac{1}{n^{\alpha}}\left|\left\{k\leq n: g\left(x_k-L\right)\geq\varepsilon\right\}\right| \\ \leq\ & \frac{1}{P_m^{\alpha}}\left|\left\{k\leq P_{m+1}: p_k g\left(x_k-L\right)\geq c\varepsilon\right\}\right| \\ =\ & \frac{1}{P_m^{\alpha}}\left|\left\{k\leq P_m: p_k g\left(x_k-L\right)\geq c\varepsilon\right\}\right|+\frac{p_{m+1}}{P_m^{\alpha}} \end{aligned}$$

Consequently $S^{\alpha}(g)-\lim x=L$.

The following example shows that in general the converse of Theorem 7 is not true.

Example 8 Let $g(x)=|x|$ and define a sequence $x=(x_n)$ by

$$x_n=\begin{cases} 1, & n=k^2 \\ \frac{1}{\sqrt{n}}, & \text{otherwise} \end{cases}, k\in\mathbb{N}.$$

It is clear that x is statistically convergent sequence of order α to 0, but not weighted statistically convergent sequence of order α to 0 (If we take $p_n=n$ for all $n\in\mathbb{N}$ and $\frac{1}{2}<\alpha\leq 1$).

Theorem 9 Let α and β are fixed real numbers such that $0<\alpha\leq\beta\leq 1$. Then the inclusion $S_{\overline{N}}^{\alpha}(g)\subseteq S_{\overline{N}}^{\beta}(g)$ is strict for some α and β such that $\alpha<\beta$.

Proof. The inclusion part of the proof follows from the following inequality:

$$\frac{1}{P_n^{\beta}}\left|\left\{k\leq P_n: p_k g\left(x_k-L\right)\geq\varepsilon\right\}\right|\leq\frac{1}{P_n^{\alpha}}\left|\left\{k\leq P_n: p_k g\left(x_k-L\right)\geq\varepsilon\right\}\right|.$$

To prove that the inclusions is strict, consider a paranormed space X with paranorm $g(x)=|x|$, $p_n=n$ for all $n\in\mathbb{N}$ and also choose a sequence $x=(x_n)$ defined by

$$x_n=\begin{cases} 1 & n=k^2 \\ \frac{1}{\sqrt{n}} & n\neq k^2 \end{cases}, k\in\mathbb{N}.$$

Then we have

$$g(x_n)=\begin{cases} 1 & n=k^2 \\ \frac{1}{\sqrt{n}} & n\neq k^2 \end{cases}, k\in\mathbb{N}.$$

Hence $x\in S_{\overline{N}}^{\beta}(g)$ for $\frac{1}{2}<\beta\leq 1$, but $x\notin S_{\overline{N}}^{\alpha}(g)$ for $0<\alpha\leq\frac{1}{2}$.

Corollary 10 If we take $\beta=1$ then $S_{\overline{N}}^{\alpha}(g)\subseteq S_{\overline{N}}(g)$ strictly holds.

Theorem 11 Let α and β are fixed real numbers such that $0<\alpha\leq\beta\leq 1$. Then the inclusion $\left[\left(\overline{N},p_n\right),g\right]_r^{\alpha}\subseteq\left[\left(\overline{N},p_n\right),g\right]_r^{\beta}$ is strict for some α and β such that $\alpha<\beta$.

Proof. The inclusion part of the proof follows from the following inequality:

$$\frac{1}{P_n^{\beta}}\sum_{k=0}^{n}p_k g\left(x_k-L\right)^r\leq\frac{1}{P_n^{\alpha}}\sum_{k=0}^{n}p_k g\left(x_k-L\right)^r$$

To show that the inclusion is strict, choose $g(x) = |x|$, $p_n = 1$ for all $n \in \mathbb{N}$ and define a sequence $x = (x_k)$ such that

$$x_k = \begin{cases} 1, & \text{if } k \text{ is square} \\ 0, & \text{otherwise} \end{cases}.$$

Then $x \in \left[\left(\overline{N}, p_n\right), g\right]_r^\beta$ for $\frac{1}{2} < \beta \leq 1$ but $x \notin \left[\left(\overline{N}, p_n\right), g\right]_r^\alpha$ for $0 < \alpha \leq \frac{1}{2}$.

3. Conclusion

The concept of weighted statistical convergence was introduced and studied by Karakaya and Chishti [18] in 2009 and then this concept was improved by Mursaleen et al. [20] in 2012. Later, Ghosal [15] redefined the concept of weighted statistical convergence in 2016. Usin generalized difference operator Δ^m, where $m \in \mathbb{N}$, the set of positve integers, researchers who are working in this area can study the concepts of Δ^m-weighted statistical convergenc and Δ^m-weighted $\left(\overline{N}, p_n\right)-$summability of order$\alpha$, where $0 < \alpha \leq 1$.

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