ON TOTAL VERTEX IRREGULARITY STRENGTH OF SOME CLASSES OF TADPOLE CHAIN GRAPHS

I. ROSYIDA\textsuperscript{1} AND D. INDRIATI\textsuperscript{2}, §

Abstract. A total $k$-labeling $f$ that assigns $V \cup E$ into $\{1, 2, \ldots, k\}$ on graph $G$ is named vertex irregular if $wt_f(u) \neq wt_f(v)$ for dissimilar vertices $u, v$ in $G$ with the weights $wt_f(u) = f(u) + \sum_{ux \in E(G)} f(ux)$. We call the minimum number $k$ utilized in total labeling $f$ as a total vertex irregularity strength of $G$, symbolized by $\text{tvs}(G)$. In this research, we focus on tadpole chain graphs that are chain graphs which contain tadpole graphs in their blocks. We investigate $\text{tvs}$ of some classes of tadpole chain graphs, i.e., $T_r(4, n)$ and $T_r(5, n)$ with length $r$. Some formulas are derived as follows: $\text{tvs}(T_r(4, n)) = \left\lceil \frac{(n+1)r+3}{3} \right\rceil$ and $\text{tvs}(T_r(5, n)) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil$.

Keywords: Vertex irregular total $k$-labeling, tvs, tadpole graph, chain graph.

AMS Subject Classification: 83-02, 99A00

1. Introduction

We assume that $G(V, E)$ is a finite, undirected, and simple graph. A mapping $f$ that assigns $V(G) \cup E(G)$ into a set of integers is named a total labeling. Further, the integers used in $f$ are called as labels [14]. In addition, the total $k$-labeling $f$ is called a total vertex irregular if the vertex weights $wt_f(u) \neq wt_f(v)$ for distinct vertices $u \neq v$ in $G$ with $wt_f(u) = f(u) + \sum_{ux \in E(G)} f(ux)$. Baca et al. [5] initiated the notion of total vertex irregularity strength of graph $G$, denoted by $\text{tvs}(G)$, that is defined as the minimum number $k$ in such a way that $G$ has a vertex irregular total $k$-labeling.

Báča et al. [5] proposed the lower bound in the following: $\left\lceil \frac{(p+\delta)}{\Delta+1} \right\rceil \leq \text{tvs}(G) \leq p + \Delta - 2\delta + 1$ for any graph $G(V, E)$ where $p = |V(G)|$, $\delta = \min\{d(v)|\forall v \in V(G)\}$, and $\Delta = \max\{d(v)|\forall v \in V(G)\}$, respectively. Whereas, Anholcer, et al. provided another bounds in [3]. Another way to get the lower bound of tvs for any connected graph $G$ was given by Nurdin et al. [11]. Let $G$ be a connected graph where the number of vertices of

\textsuperscript{1} Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Negeri Semarang, Indonesia.

e-mail: iisisnaini@gmail.com; ORCID: https://orcid.org/0000-0002-1282-5988.

\textsuperscript{2} Faculty of Mathematics and Natural Sciences, Universitas Sebelas Maret-Surakarta - Indonesia.

e-mail: diari.indri@yahoo.co.id; ORCID: https://orcid.org/0000-0002-2889-0557.

§ Manuscript received: October, 25, 2019; accepted: April 20, 2020.

TWMS Journal of Applied and Engineering Mathematics, Vol.11, Special Issue © Işık University, Department of Mathematics, 2021; all rights reserved.

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degrees $i$ is $n_i$, for $i = \delta, \delta + 1, \delta + 2, \ldots, \Delta$.

$$tvs(G) \geq \max \left\{ \left\lceil \frac{\delta(G) + n_{\delta(G)}}{\delta(G) + 1} \right\rceil, \left\lceil \frac{\delta(G) + n_{\delta(G)} + n_{\delta(G) + 1}}{\delta(G) + 2} \right\rceil, \ldots, \left\lceil \frac{\delta(G) + \sum_{i=\delta(G)}^{\Delta} n_i}{\Delta(G) + 1} \right\rceil \right\}. \quad (1)$$

Nowadays, Many scholars have found total vertex irregularity strength of some graph classes such as in [5],[7], [10], [1], [2], [9], etc. Meanwhile, the results related to cactus chain graphs has been invented in [4] and [12]. Recently, Rosyida et al. published a result of tvs of $T_r(4,1)$ tadpole chain graph [13]. To continue the result in [13], we investigate tvs of some tadpole chain graphs in this paper, i.e. $tvs(T_r(4, n))$ and $tvs(T_r(5, n))$.

2. Main Results

In this part, we present formulas of tvs of $T_r(4, n)$ and $T_r(5, n)$. Let us consider the chain graph in Definition 2.1.

**Definition 2.1.** A graph which consists of a cycle graph $C_m$ and a path graph $P_n$ connected with a bridge is called as a tadpole graph, denoted by $T(m,n)$. Given a connected graph $G$. A bipartite graph $(B,C)$, where $B$ is a set of blocks in graph $G$ and $C$ consists of cut vertices on each block in $B$, is named a block cut vertex of $G$. The edges of $G$ join cut vertices with those blocks to which they belong. Further, $G$ is called as a chain graph of length $r$ if it contains $r$-blocks such that each pair of two blocks $B_i$ and $B_{i+1}$ has one common cut vertex for which the block cut vertex is a path ([8], [6]). Furthermore, tadpole chain graphs $T_r(4,n)$ and $T_r(5,n)$ are chain graphs which their blocks are $T(4,n)$ and $T(5,n)$, respectively.

2.1. Total vertex irregularity strength of $T_r(5,n)$. The chain graph $T_r(5,n)$ consists of:

- $\{y_1^n, y_2^n, \ldots, y_r^n\}$, i.e. a set of vertices with degrees 1
- $\{y_1^{n-1}, y_2^{n-1}, \ldots, y_r^{n-1}, y_1^{n-2}, y_2^{n-2}, \ldots, y_r^{n-2}, y_1^2, y_2^2, \ldots, y_r^2, y_1^1, y_2^1, \ldots, y_r^1\} \cup \{u_1, u_2, \ldots, u_r, u_{2r+1}, u_{2r+2}\}$, i.e. a set of vertices of degrees 2;
- $\{x_1, x_2, \ldots, x_r\}$ that is a set of vertices with degrees 3; and
- $\{v_1, v_2, \ldots, v_{r-1}\}$ that is a set of vertices with degrees 4.

**Theorem 2.1.** If $T_r(5,1)$ is a tadpole chain graph of length $r(r \geq 2)$, then

$$tvs(T_r(5,1)) = r + 1.$$  

**Proof.** Let $y_1, y_2, \ldots, y_r$ be vertices of $T_r(5,1)$ with degrees 1.

<table>
<thead>
<tr>
<th>Labels of vertices</th>
<th>Labels of edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(u_1) = 1,$</td>
<td>$f(u_{2i-1}u_{2i}) = i, 1 \leq i \leq r,$</td>
</tr>
<tr>
<td>$f(u_{2i-1}) = i - 1, 2 \leq i \leq r,$</td>
<td>$f(u_1u_{2r+1}) = r,$</td>
</tr>
<tr>
<td>$f(u_{2i}) = r, 1 \leq i \leq r,$</td>
<td>$f(u_{2i}u_{2r+2}) = r + 1,$</td>
</tr>
<tr>
<td>$f(u_{2r+1}) = f(u_{2r+2}) = r + 1,$</td>
<td>$f(u_{2i}v_1) = r + 1, 1 \leq i \leq r - 1,$</td>
</tr>
<tr>
<td>$f(v_i) = r - 1, 1 \leq i \leq r - 1,$</td>
<td>$f(u_{2i+1}v_1) = r + 1, 1 \leq i \leq r - 1,$</td>
</tr>
<tr>
<td>$f(x_i) = r + 1, 1 \leq i \leq r,$</td>
<td>$f(v_ix_i) = r + 1, 1 \leq i \leq r - 1,$</td>
</tr>
<tr>
<td>$f(y_i) = 1, 1 \leq i \leq r.$</td>
<td>$f(v_ix_{i+1}) = r + 1, 1 \leq i \leq r - 1,$</td>
</tr>
<tr>
<td></td>
<td>$f(u_{r+1}x_i) = r + 1, f(u_{r+2}x_i) = r + 1,$</td>
</tr>
<tr>
<td></td>
<td>$f(x_1y_i) = i, 1 \leq i \leq r.$</td>
</tr>
</tbody>
</table>
According to Inequality (1), we obtain

\[ tvs(T_r(5,1)) \geq \max \left\{ \left\lfloor \frac{r + 1}{2} \right\rfloor, \left\lfloor \frac{3r + 3}{3} \right\rfloor, \left\lfloor \frac{4r + 3}{4} \right\rfloor, \left\lfloor \frac{5r + 2}{5} \right\rfloor \right\} = \left\lfloor \frac{3r + 3}{3} \right\rfloor = r + 1. \quad (2) \]

To proof that \( tvs(T_r(5,1)) \leq r+1 \), we provide a total \( k \)-labeling \( f : V \cup E \rightarrow \{1, 2, \ldots, k\} \) with \( k = r + 1 \) and define labels of vertices and edges as in Table 1.

Under labeling \( f \), we obtain that each vertex has the weight below:

\[
\begin{align*}
wt_f(y_i) & = i + 1, \quad 1 \leq i \leq r, \\
wt_f(u_1) & = r + 2, \\
wt_f(u_i) & = i + r + 1, \quad 1 \leq i \leq 2r, \\
wt_f(u_{2r+1}) & = 3r + 2, \\
wt_f(u_{r+2}) & = 3(r + 1), \\
wt_f(v_1) & = i + 4r + 3, \quad 1 \leq i \leq r - 1, \\
wt_f(x_i) & = i + 3(r + 1), \quad 1 \leq i \leq r.
\end{align*}
\]

The minimum label of vertices and edges is 1 and the maximum label is \( r + 1 \). Also, it is shown that the vertex weights are all diverse. Hence, we get upper bound \( tvs(T_r(5,1)) \leq r + 1 \). Combining with Lower bound (2), it is proved that \( tvs(T_r(5,1)) = r + 1 \). \( \Box \)

**Theorem 2.2.** If \( T_r(5,n) \) are tadpole chain graphs with length \( r(r \geq 2) \), \( n = 4 \mod 3 \), and \( n \geq 4 \), then \( tvs(T_r(5,n)) = \left\lceil \frac{(n + 2)r + 3}{3} \right\rceil \).

**Proof.** Based on (1), we acquire the lower bound

\[
tvs(T_r(5,n)) \geq \max \left\{ \left\lfloor \frac{r + 1}{2} \right\rfloor, \left\lfloor \frac{(n + 2)r + 3}{3} \right\rfloor, \left\lfloor \frac{(n + 3)r + 3}{4} \right\rfloor, \left\lfloor \frac{(n + 4)r + 2}{5} \right\rfloor \right\} = \left\lfloor \frac{(n + 2)r + 3}{3} \right\rfloor.
\]  

We prove the upper bound through 3 cases.

**Case 1.** For \( n = 4 \).

<table>
<thead>
<tr>
<th>Labels of vertices</th>
<th>Labels of edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(u_{2i-1}) = i + r - 2, 1 \leq i \leq r ),</td>
<td>( f(u_{2i-1}u_{2i}) = i + r + 1, 1 \leq i \leq r ),</td>
</tr>
<tr>
<td>( f(u_{2i}) = (2r + 1) - 2 = 2r - 1, 1 \leq i \leq r ),</td>
<td>( f(u_1u_{2r+1}) = 2r + 1 ),</td>
</tr>
<tr>
<td>( f(u_{2r+1}) = 2r + 1 - 1 = 2r ),</td>
<td>( f(u_{r+2}) = 2r + 1 ),</td>
</tr>
<tr>
<td>( f(u_{2r+2}) = 2r + 1 ),</td>
<td></td>
</tr>
<tr>
<td>( f(v_1) = 1, 1 \leq i \leq r - 1 ),</td>
<td>( f(u_{2i}v_i) = i + r + 1, 1 \leq i \leq r - 1 ),</td>
</tr>
<tr>
<td>( f(x_i) = 2r + 1, 1 \leq i \leq r ),</td>
<td>( f(u_{2i+1}v_i) = 2r + 1, 1 \leq i \leq r - 1 ),</td>
</tr>
<tr>
<td>( f(y_1^1) = 2r - 1, 1 \leq i \leq r - 1 ),</td>
<td>( f(v_1x_i) = 2r - 1, 1 \leq i \leq r - 1 ),</td>
</tr>
<tr>
<td>( f(y_2^1) = 2r + 1 ),</td>
<td>( f(v_1x_{i+1}) = 2r + 1, 1 \leq i \leq r - 1 ),</td>
</tr>
<tr>
<td>( f(y_1^2) = r + 1, 1 \leq i \leq r ),</td>
<td>( f(u_{2r+1}x_1) = 2r + 1, f(u_{2r+2}x_r) = 2r + 1 ),</td>
</tr>
<tr>
<td>( f(y_1^3) = r, 1 \leq i \leq r ),</td>
<td>( f(x_1y_1^1) = i + 2, 1 \leq i \leq r - 1 ),</td>
</tr>
<tr>
<td>( f(y_2^3) = i, 1 \leq i \leq r ).</td>
<td>( f(x_1y_1^3) = r ),</td>
</tr>
</tbody>
</table>

**Table 2.** Labels of vertices and edges of \( T_r(5,4) \)
We assume that $f : V \cup E \rightarrow \{1, 2, \ldots, k\}$ is a total $k$-labeling with $k = \left\lceil \frac{6r + 3}{3} \right\rceil = 2r + 1$. Vertex and edge labels are defined in Table 2.

Under labeling $f$, we derive the weights of vertices below:

<table>
<thead>
<tr>
<th>Labels of vertices</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$f(u_{2i-1}) = i + 2r - 2, 1 \leq i \leq r,$</td>
<td>$f(u_{2i-1}u_{2i}) = i + 2r + 1, 1 \leq i \leq r,$</td>
</tr>
<tr>
<td>$f(u_{2i}) = (3r + 1) - 2 = 3r - 1, 1 \leq i \leq r,$</td>
<td>$f(u_{1}u_{r+1}) = 3r + 1,$</td>
</tr>
<tr>
<td>$f(u_{2r+1}) = 3r + 1 - 1 = 3r,$</td>
<td>$f(u_{2r}u_{2r+2}) = 3r + 1,$</td>
</tr>
<tr>
<td>$f(u_{2r+2}) = 3r + 1,$</td>
<td>$f(u_{2i}v_{i}) = i + 2r + 1, 1 \leq i \leq r - 1,$</td>
</tr>
<tr>
<td>$f(v_{i}) = 1, 1 \leq i \leq r - 1,$</td>
<td>$f(u_{2i+1}v_{i}) = 3r + 1, 1 \leq i \leq r - 1,$</td>
</tr>
<tr>
<td>$f(x_{i}) = 3r + 1, 1 \leq i \leq r,$</td>
<td>$f(v_{i}x_{i}) = 2r - 1, 1 \leq i \leq r - 1,$</td>
</tr>
<tr>
<td>$f(y_{i}^{1}) = 2r - 1, 1 \leq i \leq r - 1,$</td>
<td>$f(v_{i}x_{i+1}) = 3r + 1, 1 \leq i \leq r - 1,$</td>
</tr>
<tr>
<td>$f(y_{i}^{1}) = 3r + 1,$</td>
<td>$f(y_{1}^{1}) = 3r + 1,$</td>
</tr>
<tr>
<td>$f(y_{i}^{1}) = 2r + 1, 1 \leq i \leq r,$</td>
<td>$f(u_{2r+1}x_{1}) = 3r + 1,$</td>
</tr>
<tr>
<td>$f(y_{i}^{1}) = 3r + 1, 1 \leq i \leq r,$</td>
<td>$f(u_{2r+2}x_{r}) = 3r + 1,$</td>
</tr>
<tr>
<td>$f(y_{i}^{1}) = 2r + 1, 1 \leq i \leq r,$</td>
<td>$f(x_{i}y_{i}^{1}) = i + r + 2, 1 \leq i \leq r - 1,$</td>
</tr>
<tr>
<td>$f(y_{i}^{1}) = 2r, 1 \leq i \leq r,$</td>
<td>$f(x_{r}y_{r}^{1}) = r; f(y_{i}^{1}y_{i}^{2}) = 3r, 1 \leq i \leq r,$</td>
</tr>
<tr>
<td>$f(y_{i}^{1}) = r, 1 \leq i \leq r,$</td>
<td>$f(y_{i}^{2}y_{i}^{2}) = i,$</td>
</tr>
<tr>
<td>$f(y_{i}^{1}) = 1, 1 \leq i \leq r.$</td>
<td>$f(y_{i}^{2}y_{i}^{1}) = r,$</td>
</tr>
</tbody>
</table>

We observe that each vertex has a distinct weight. Also, vertex and edge labels used are less than or equal to $2r + 1$. Hence, $tvs(T_r(5, 4)) \leq \left\lceil \frac{6r + 3}{3} \right\rceil = 2r + 1$. Combining with the lower bound, we have $tvs(T_r(5, 4)) = 2r + 1$.

**Case 2.** For $n = 7$.

We construct a function $f : V \cup E \rightarrow \{1, 2, \ldots, k\}$ which is a total $k$-labeling with $k = \left\lceil \frac{9r + 3}{3} \right\rceil = 3r + 1$. We establish labels of vertices and edges as in Table 3.
By means of labeling $f$, we have the weights of vertices as follows:

$$ wt_f(u_i) = i + 7r + 1, \quad 1 \leq i \leq 2r, $$
$$ wt_f(u_{2r+1}) = 3(3r+1) - 1, $$
$$ wt_f(u_{r+2}) = 3(3r+1), $$
$$ wt_f(v_i) = i + 2(3r+1) + 4r + 1 = i + 10r + 3, \quad 1 \leq i \leq r - 1, $$
$$ wt_f(x_i) = i + 2(3r+1) + 3r + 1 = i + 9r + 3, \quad 1 \leq i \leq r, $$
$$ wt_f(y_i^l) = i + (7-l)r + 1, \quad 1 \leq i \leq r; 1 \leq l \leq 7. $$

The vertex weights are all distinct and the labels used are at most $\left\lceil \frac{9r+3}{3} \right\rceil = 3r + 1$. Thus, $tvs(T_r(5,7)) = 3r + 1$.

**Case 3.** For $n = 3j + 10$ with $j \geq 0$.

We construct a total $k$-labeling $f : V \cup E \rightarrow \{1, 2, \ldots , k\}$ with $k = \left\lceil \frac{(n+2)r+3}{3} \right\rceil$. We create vertex labels in the following:

$$ f(u_{2i-1}) = i + \left\lceil \frac{n+1}{3} \right\rceil r - 2, \quad 1 \leq i \leq r, $$
$$ f(u_{2i}) = \left\lceil \frac{n+2}{3} \right\rceil r - 2, \quad 1 \leq i \leq r, $$
$$ f(u_{2r+1}) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil r - 1, $$
$$ f(u_{r+2}) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil, $$
$$ f(v_i) = 1, \quad 1 \leq i \leq r - 1, $$
$$ f(x_i) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil, \quad 1 \leq i \leq r - 1, $$
$$ f(x_r) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil - \left\lceil \frac{n-7}{3} \right\rceil r - 1, $$
$$ f(y_i^1) = \left\lceil \frac{n+2}{3} \right\rceil r - 2, \quad 1 \leq i \leq r - 1, $$
$$ f(y_i^r) = \left\lceil \frac{n+2}{3} \right\rceil r + 1, $$
$$ f(y_i^j_l) = f(y_i^j) \ldots = f\left(y_i^{2j+1-l}\right) = \left\lceil \frac{n+2}{3} \right\rceil r + 1, \quad 1 \leq i \leq r; j \geq 1, $$
$$ f(y_i^{2+2j}) = \left\lceil \frac{n+2}{3} \right\rceil r, \quad 1 \leq i \leq r; j \geq 0, $$
$$ f(y_i^{2+2j+1}) = \left\lceil \frac{n+2}{3} \right\rceil r + 1, \quad 1 \leq i \leq r; j \geq 0, $$
$$ f(y_l^{2+2j+2}) = \left\lceil \frac{n+2}{3} \right\rceil r - 1, \quad 1 \leq i \leq r; j \geq 0, $$
$$ f(y_l^{2+2j+3}) = \left\lceil \frac{n+2}{3} \right\rceil r + 1, \quad 1 \leq i \leq r; j \geq 0, $$
$$ f(y_l^{2+2j+4}) = \left\lceil \frac{n+2}{3} \right\rceil r + 1, \quad 1 \leq i \leq r; j \geq 0, $$
$$ f(y_l^{2+2j+5}) = \left\lceil \frac{n+2}{3} \right\rceil r, \quad 1 \leq i \leq r; j \geq 0, $$
$$ f(y_l^{2+2j+1}) = \left\lceil \frac{n+2}{3} \right\rceil - (l - 5) r, \quad 1 \leq i \leq r; 6 \leq l \leq \left\lceil \frac{n+2}{3} \right\rceil + 2; j \geq 0, $$
$$ f(y_l^{2+2j+1}) = r, \quad 1 \leq i \leq r; l = \left\lceil \frac{n+2}{3} \right\rceil + 3; j \geq 0, $$
$$ f(y_l^n) = i, \quad 1 \leq i \leq r; n is even, $$
$$ f(y_l^n) = 1, \quad 1 \leq i \leq r; n is odd. $$

Meanwhile, each edge label is constructed below:
\[ f(u_{2i-1}u_{2i}) = i + \left(\frac{n-1}{3}\right) r + 1, \quad 1 \leq i \leq r, \]
\[ f(u_{1}u_{2r+1}) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil, \quad f(u_{2r}u_{2r+2}) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil, \]
\[ f(u_{2i}v_i) = i + \left(\frac{n-1}{3}\right) r + 1, \quad 1 \leq i \leq r-1, \]
\[ f(u_{2i+1}v_i) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil, \quad 1 \leq i \leq r-1, \]
\[ f(v_ix_i) = 2r - 1, \quad 1 \leq i \leq r-1, \]
\[ f(v_ix_{i+1}) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil, \quad 1 \leq i \leq r-1, \]
\[ f(u_{2r+1}x_1) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil, \]
\[ f(u_{2r+2}x_r) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil, \]
\[ f(x_iy_i^1) = i + \left(\frac{n-4}{3}\right) r + 2, \quad 1 \leq i \leq r-1, \]
\[ f(x_iy_i^2) = \left(\frac{n-4}{3}\right) r - 1, \]
\[ f(y_i^{2l+1}y_i^{2l+2}) = \frac{(n+2)j+2}{3} - (l+1) r + 1, \quad 1 \leq i \leq r; 1 \leq l \leq j; j \geq 0, \]
\[ f(y_i^{2j+1}y_i^{2j+2}) = i; f(y_i^{2j+3}y_i^{2j+4}) = 3r, \quad 1 \leq i \leq r; j \geq 0, \]
\[ f(y_i^{2j+5}y_i^{2j+6}) = i; f(y_i^{2j+7}y_i^{2j+8}) = r, \quad 1 \leq i \leq r; j \geq 0, \]
\[ f(y_i^{2j+8}y_i^{2j+9}) = i; f(y_i^{2j+9}y_i^{2j+10}) = 1, \quad 1 \leq i \leq r; j \geq 0, \]
\[ f(y_i^{2n-1}y_i^{2n}) = 1, \quad 1 \leq i \leq r; n \text{ is even}, \]
\[ f(y_i^{2n-1}y_i^{2n}) = i, \quad 1 \leq i \leq r; n \text{ is odd}. \]

Under labeling \( f \), we derive the vertex weights in the following:

\[ wt_f(u_i) = i + nr + 1, \quad 1 \leq i \leq 2r, \]
\[ wt_f(u_{2r+1}) = 3\left\lceil \frac{(n+2)r+3}{3} \right\rceil - 1, \]
\[ wt_f(u_{r+2}) = 3\left\lceil \frac{(n+2)r+3}{3} \right\rceil, \]
\[ wt_f(v_i) = i + 2\left(\frac{n+5}{3}\right)r + 1, \quad 1 \leq i \leq r-1, \]
\[ wt_f(x_i) = i + 2\left(\frac{n+2}{3}\right)r + 1, \quad 1 \leq i \leq r, \]
\[ wt_f(y_i^l) = i + (n-l)r + 1, \quad 1 \leq i \leq r; 1 \leq l \leq n. \]

It is obvious that each vertex has a different weight and the labels used are not more than \( \left\lceil \frac{(n+2)r+3}{3} \right\rceil \). Therefore, \( tvs(T_r(5, n)) \leq \left\lceil \frac{(n+2)r+3}{3} \right\rceil \). Combining with Lower bound (3), we get \( tvs(T_r(5, n)) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil \).

\[ \square \]

Figure 1 describes the pattern of vertex and edge labels to get \( tvs(T_5(5, 4)) = 11 \).
2.2. Total vertex irregularity strength of $T_r(4, n)$. The graph $T_r(4, n)$ contains:

- a set of vertices with degrees 1, i.e. $\{y_1^n, y_2^n, \ldots, y^n_r\}$;
- a set of vertices with degrees 2, i.e. $\{y_1^{n-1}, y_2^{n-1}, \ldots, y_r^{n-1}, y_{r+1}^{n-2}, y_{r+2}^{n-2}, \ldots, y_s^n, y_r^2, \ldots, y_s^2, y_r^1, y_{r+1}^1, \ldots, y_{r+1}^{r-1}\} \cup \{u_1, u_2, \ldots, u_{r+1}, u_{r+2}\}$;
- a set of vertices with degrees 3, i.e. $\{x_1, x_2, \ldots, x_r\}$; and
- a set of vertices with degrees 14, i.e. $\{v_1, v_2, \ldots, v_{r-1}\}$.

**Lemma 2.1.** If $T_r(4, 2)$ is a tadpole chain graph with length $r$ ($r \geq 2$), then
\[
tvs(T_r(4, 2)) = r + 1.
\]

**Proof.** Based on (1), we get the lower bound as follows:
\[
tvs(T_r(4, 2)) \geq \max \left\{ \left\lceil \frac{r + 1}{2} \right\rceil, \left\lceil \frac{3r + 3}{3} \right\rceil, \left\lceil \frac{4r + 3}{4} \right\rceil, \left\lceil \frac{5r + 2}{5} \right\rceil \right\} = \left\lceil \frac{3r + 3}{3} \right\rceil = r + 1. \tag{4}
\]

To establish that $tvs(T_r(4, 2)) \leq r + 1$, we define a function $f$ from $V \cup E$ into a set of integers $\{1, 2, \ldots, k\}$ which is a total $k$-labeling with $k = \left\lceil \frac{3r + 3}{3} \right\rceil = r + 1$.

We construct labels of vertices and edges as in Table 4.

We have the weights of vertices under the labeling $f$ as follows:
\[
\begin{align*}
wt_f(y_i^n) & = i + 1, & 1 \leq i \leq r, \\
wt_f(y_i^1) & = i + r + 1, & 1 \leq i \leq r, \\
wt_f(u_i) & = i + 2r + 1, & 1 \leq i \leq r + 2, \\
wt_f(v_i) & = i + 3(r + 1) + r, & 1 \leq i \leq r - 1, \\
wt_f(x_i) & = i + 3(r + 1), & 1 \leq i \leq r.
\end{align*}
\]
Further, we prove the upper bound by means of 2 cases. Theorem 2.3.

If 

\[ \text{tvs} \leq 140 \] 

\[ \text{tvs}(T_r(4, 2)) \leq r + 1 \] 

We observe that each vertex has distinct label. Also, vertex and edge labels are less than or equal to 

\[ \frac{3r+3}{3} = r + 1 \] 

Hence, we get upper bound \( \text{tvs}(T_r(4, 2)) \leq r + 1 \). Thus, \( \text{tvs}(T_r(4, 2)) = r + 1 \).

\[ \square \]

**Theorem 2.3.** If \( T_r(4, n) \) are tadpole chain graphs of length \( r \) where \( r \geq 2, n = 5 \mod 3, \) and \( n \geq 5 \), then \( \text{tvs}(T_r(4, n)) = \frac{(n+1)r+3}{3} \).

**Proof.** According to (1), we obtain the lower bound as follows:

\[ \text{tvs}(T_r(4, n)) \geq \max \left\{ \frac{r+1}{2}, \frac{(n+1)r+3}{3}, \frac{(n+2)r+3}{4}, \frac{(n+3)r+2}{5} \right\} = \frac{(n+1)r+3}{3} \]

We create a total \( k \)-labeling \( f \) from \( V \cup E \) into \( \{1, 2, \ldots, k\} \) with 

\[ k = \frac{(n+1)r+3}{3} \]

Further, we prove the upper bound by means of 2 cases.

**Case 1.** For \( n = 5 \) and \( n = 8 \).

Firstly, we define Labels of vertices as follows:

\[ f(u_i) = \begin{cases} 
2r - 1, & 1 \leq i \leq r, n = 5, \\
3r - 1, & 1 \leq i \leq r, n = 8. 
\end{cases} \]

\[ f(u_{r+1}) = \left\lfloor \frac{(n+1)r+3}{3} \right\rfloor - 1, \ \ n = 5, 8 \]

\[ f(u_{r+2}) = \left\lfloor \frac{3r+3}{3} \right\rfloor, \ \ n = 5, 8 \]

\[ f(v_i) = 1, \ \ 1 \leq i \leq r - 1, n = 5, 8 \]

\[ f(x_i) = \left\lfloor \frac{(n+1)r+2}{3} \right\rfloor, \ \ 1 \leq i \leq r, n = 5, 8, \]

\[ f(y^1_i) = \begin{cases} 
2r - 1, & 1 \leq i \leq r - 1, n = 5, \\
2r - 2, & 1 \leq i \leq r - 1, n = 8. 
\end{cases} \]

\[ f(y^2_i) = \begin{cases} 
2r, & n = 5, \\
3r, & n = 8. 
\end{cases} \]

\[ f(y^3_i) = \begin{cases} 
1, & 1 \leq i \leq r, n = 5, \\
1, & 1 \leq i \leq r, n = 8. 
\end{cases} \]
\[ f(y_i^3) = \begin{cases} 
2r, & 1 \leq i \leq r, n = 5, \\
3r + 1, & 1 \leq i \leq r, n = 8. 
\end{cases} \]

\[ f(y_i^4) = \begin{cases} 
r, & 1 \leq i \leq r, n = 5, \\
2r + 1, & 1 \leq i \leq r, n = 8. 
\end{cases} \]

\[ f(y_i^5) = \begin{cases} 
1, & 1 \leq i \leq r, n = 5, \\
3r, & 1 \leq i \leq r, n = 8. 
\end{cases} \]

\[ f(y_i^6) = 2r; f(y_i^7) = r; f(y_i^8) = i, \quad 1 \leq i \leq r, n = 8. \]

Secondly, we construct labels of edges as follows:

\[ f(u_{r+1}v_i) = \begin{cases} 
\left(\frac{(n+1)r+3}{3}\right), & 1 \leq i \leq r - 1, n = 5, 8 
\end{cases}, \]

\[ f(v_iy_i^1) = \begin{cases} 
i + 2, & 1 \leq i \leq r - 1, n = 5, \\
i + 2r + 2, & 1 \leq i \leq r - 1, n = 8. 
\end{cases} \]

\[ f(x_iy_i^1) = \begin{cases} 
r, & n = 5, \\
2r, & n = 8. 
\end{cases} \]

\[ f(y_i^1y_i^2) = \begin{cases} 
2r + 1, & 1 \leq i \leq r, n = 5, \\
3r + 2, & 1 \leq i \leq r, n = 8. 
\end{cases} \]

\[ f(y_i^2y_i^3) = i; f(y_i^4y_i^5) = i, \quad 1 \leq i \leq r, n = 5, 8, \]

\[ f(y_i^3y_i^4) = \begin{cases} 
1, & 1 \leq i \leq r, n = 5, \\
2r, & 1 \leq i \leq r, n = 8. 
\end{cases} \]

\[ f(y_i^5y_i^6) = 1; f(y_i^6y_i^7) = i; f(y_i^7y_i^8) = 1, \quad 1 \leq i \leq r, n = 8. \]

We have the vertex weights below:

\[ wt_f(u_i) = \begin{cases} 
i + 2 \left(\frac{(n+1)r+3}{3}\right) + (r - 1), & 1 \leq i \leq r + 2, n = 5, \\
i + 2 \left(\frac{(n+1)r+3}{3}\right) + (2r - 1), & 1 \leq i \leq r + 2, n = 8. 
\end{cases} \]

\[ wt_f(v_i) = \begin{cases} 
i + 2 \left(\frac{(n+1)r+3}{3}\right) + (3r + 1), & 1 \leq i \leq r - 1, n = 5, \\
i + 2 \left(\frac{(n+1)r+3}{3}\right) + (4r + 1), & 1 \leq i \leq r - 1, n = 8. 
\end{cases} \]

\[ wt_f(x_i) = \begin{cases} 
i + 2 \left(\frac{(n+1)r+3}{3}\right) + (2r + 1), & 1 \leq i \leq r, n = 5, \\
i + 2 \left(\frac{(n+1)r+3}{3}\right) + (3r + 1), & 1 \leq i \leq r, n = 8. 
\end{cases} \]
\[ wt_f(y_i^l) = i + (n-l)r + 1, \quad 1 \leq i \leq r, 1 \leq l \leq n, n = 5, 8. \]

It is obvious that each vertex has a different weight. Also, the labels are not more than \( \left\lceil \frac{(n+1)r+3}{3} \right\rceil \). Hence, we establish the upper bound \( tvs(T_r(4, n)) \leq \left\lceil \frac{(n+1)r+3}{3} \right\rceil \). According to Lower bound (5), we have \( tvs(T_r(4, n)) = \left\lceil \frac{(n+1)r+3}{3} \right\rceil \).

**Case 2.** For \( n \geq 11. \)

Analog to the proof of Theorem 2.2 in Case 3, we have \( tvs(T_r(4, n)) = \left\lceil \frac{(n+1)r+3}{3} \right\rceil \). \( \square \)

Figure 2 shows the pattern of vertex and edge labels to obtain \( tvs(T_5(4, 5)) = 11. \)

**Figure 2.** The pattern of vertex irregular total 11-labeling of \( T_5(4, 5) \)

3. Conclusions

We have verified \( tvs \) of \( T_r(4, n) \) and \( T_r(5, n) \) in this paper. The patterns to get \( tvs \) of the graphs were presented in the theorems. We have proved the upper bounds and got \( tvs(T_r(4, n)) = \left\lceil \frac{(n+1)r+3}{3} \right\rceil \) and \( tvs(T_r(5, n)) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil \). In further research, we are going to investigate \( tvs \) of general tadpole chain graph \( T_r(m, n) \) for \( m \geq 6. \) Also, we will verify the formulas using algorithmic approaches.

**References**


Isnaini Rosyida is a mathematics lecturer from Indonesia. She received the BS degree from Diponegoro University-Indonesia in 1996 and MS degree from Bandung Institute of Technology-Indonesia in 2003. In 2016, she received her Ph.D degree from Department of Mathematics, Gadjah Mada University, Indonesia. Her research interests include fuzzy graph theory and combinatorics.

Diari Indriati is a mathematics lecturer and researcher in the field of graph theory from Indonesia. Bachelor, master and doctoral courses are all taken at the Department of Mathematics, Faculty of Mathematics and Natural Sciences, Gadjah Mada University, Yogyakarta, Indonesia. She also became a reviewer for several international journals in the field of graph theory.