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UNIQUENESS AND STABILITY OF SOLUTIONS FOR A COUPLED SYSTEM OF NABLA FRACTIONAL DIFFERENCE BOUNDARY VALUE PROBLEMS

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Abstract. In this article, we obtain sufficient conditions on existence, uniqueness and Ulam–Hyers stability of solutions for a coupled system of two-point nabla fractional difference boundary value problems, using Banach, Brouwer fixed point theorems and Urs's approach. Further, we illustrate the applicability of established results through an example.

Keywords: Nabla fractional Riemann–Liouville difference, boundary value problem, existence, uniqueness, Ulam–Hyers stability.

AMS Subject Classification: 39A12, 39A70.

1. Introduction

In this article, we consider the following coupled system of nabla fractional difference equations with conjugate boundary conditions

$$\begin{cases} \Big(\nabla^{\alpha_1-1}_{\rho(a)}\big(\nabla u_1\big)\Big)(t) + f_1(t, u_1(t), u_2(t)) = 0, \quad t \in \mathbb{N}^b_{a+2}, \\ \Big(\nabla^{\alpha_2-1}_{\rho(a)}\big(\nabla u_2\big)\Big)(t) + f_2(t, u_1(t), u_2(t)) = 0, \quad t \in \mathbb{N}^b_{a+2}, \\ u_1(a) = 0, \ u_1(b) = 0, \\ u_2(a) = 0, \ u_2(b) = 0, \end{cases} \tag{1}$$

where a, $b \in \mathbb{R}$ with $b - a \in \mathbb{N}_2$; $1 < \alpha_1, \alpha_2 < 2$; f_1, $f_2 : \mathbb{N}^b_{a+1} \times \mathbb{R}^2 \to \mathbb{R}$ and $\nabla^{\nu}_{\rho(a)}$ denotes the ν^{th}-th order Riemann–Liouville backward (nabla) difference operator with $\nu \in \{\alpha_1 - 1, \alpha_2 - 1\}$.

In 1940, Ulam [41] posed a problem on the stability of functional equations and Hyers [19] solved it in the next year for additive functions defined on Banach spaces. In 1978, Rassias [38] provided a generalization of the Hyers theorem for linear mappings. Since then, several mathematicians investigated Ulam's problem in different directions for various classes of functional equations [23, 30], differential equations [24, 25, 26, 33, 34, 39, 40],

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difference equations [27, 28, 29, 35], fractional differential equations [4, 6, 11, 12, 13, 14, 15, 43], and fractional difference equations [9, 10, 21].

In particular, Urs [42] presented some Ulam–Hyers stability results for the coupled fixed point of a pair of contractive type operators on complete metric spaces. Motivated by this work, in this article, we study the Ulam–Hyers stability of (1).

The present paper is organized as follows: Section 2 contains preliminaries on nabla fractional calculus. In sections 3 and 4, we establish sufficient conditions on existence, uniqueness and Ulam–Hyers stability of solutions for the discrete fractional boundary value problem (1), respectively. We present an example in section 4.

2. Preliminaries

2.1. **Nabla Fractional Calculus.** We use the following notations, definitions and known results of nabla fractional calculus throughout the article. Denote by $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$ and $\mathbb{N}_a^b = \{a, a+1, a+2, \ldots, b\}$ for any a, $b \in \mathbb{R}$ such that $b - a \in \mathbb{N}_1$.

Definition 2.1 (See [7]). *The backward jump operator $\rho : \mathbb{N}_a \to \mathbb{N}_a$ is defined by*

$$\rho(t) = \begin{cases} a, & t = a, \\ t-1, & t \in \mathbb{N}_{a+1}. \end{cases}$$

Definition 2.2 (See [32, 36]). *The Euler gamma function is defined by*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0.$$

Using its reduction formula, the Euler gamma function can also be extended to the half-plane $\Re(z) \le 0$ except for $z \in \{\ldots, -2, -1, 0\}$.

Definition 2.3 (See [17]). *For $t \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$ and $r \in \mathbb{R}$ such that $(t + r) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$, the generalized rising function is defined by*

$$t^{\overline{r}} = \frac{\Gamma(t+r)}{\Gamma(t)}.$$

Also, if $t \in \{\ldots, -2, -1, 0\}$ and $r \in \mathbb{R}$ such that $(t+r) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$, then we use the convention that $t^{\overline{r}} = 0$.

Definition 2.4 (See [17]). *Let $\mu \in \mathbb{R} \setminus \{\ldots, -2, -1\}$. Define the μ^{th}-order nabla fractional Taylor monomial by*

$$H_\mu(t, a) = \frac{(t-a)^{\overline{\mu}}}{\Gamma(\mu + 1)},$$

provided the right-hand side exists. Observe that $H_\mu(a, a) = 0$ and $H_\mu(t, a) = 0$ for all $\mu \in \{\ldots, -2, -1\}$ and $t \in \mathbb{N}_a$.

Definition 2.5 (See [7]). *Let $u : \mathbb{N}_a \to \mathbb{R}$ and $N \in \mathbb{N}_1$. The first order backward (nabla) difference of u is defined by*

$$\big(\nabla u\big)(t) = u(t) - u(t-1), \quad t \in \mathbb{N}_{a+1},$$

and the N^{th}-order nabla difference of u is defined recursively by

$$\big(\nabla^N u\big)(t) = \Big(\nabla\big(\nabla^{N-1} u\big)\Big)(t), \quad t \in \mathbb{N}_{a+N}.$$

Definition 2.6 (See [17]). *Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $N \in \mathbb{N}_1$. The N^{th}-order nabla sum of u based at a is given by*

$$\left(\nabla_a^{-N} u\right)(t) = \sum_{s=a+1}^{t} H_{N-1}(t, \rho(s))u(s), \quad t \in \mathbb{N}_a,$$

where by convention $\left(\nabla_a^{-N} u\right)(a) = 0$. We define $\left(\nabla_a^{-0} u\right)(t) = u(t)$ for all $t \in \mathbb{N}_{a+1}$.

Definition 2.7 (See [17]). *Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu > 0$. The ν^{th}-order nabla sum of u based at a is given by*

$$\left(\nabla_a^{-\nu} u\right)(t) = \sum_{s=a+1}^{t} H_{\nu-1}(t, \rho(s))u(s), \quad t \in \mathbb{N}_a,$$

where by convention $\left(\nabla_a^{-\nu} u\right)(a) = 0$.

Definition 2.8 (See [17]). *Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nu > 0$ and choose $N \in \mathbb{N}_1$ such that $N - 1 < \nu \le N$. The ν^{th}-order Riemann–Liouville nabla difference of u is given by*

$$\left(\nabla_a^{\nu} u\right)(t) = \left(\nabla^N \left(\nabla_a^{-(N-\nu)} u\right)\right)(t), \quad t \in \mathbb{N}_{a+N}.$$

2.2. **Boundary Value Problem.** Let a, $b \in \mathbb{R}$ with $b - a \in \mathbb{N}_2$. Assume $1 < \alpha < 2$ and $h : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$. Consider the boundary value problem

$$\begin{cases} \left(\nabla_{\rho(a)}^{\alpha} u\right)(t) + h(t) = 0, & t \in \mathbb{N}_{a+2}^{b}, \\ u(a) = 0, \ u(b) = 0. \end{cases} \tag{2}$$

Brackins [8], Gholami et al. [16] and the author [22] have obtained the following expression for the unique solution of (2), independently.

Theorem 2.1. [8, 16, 22] *The nabla fractional boundary value problem* (2) *has the unique solution*

$$u(t) = \sum_{s=a+1}^{b} G(t, s)h(s), \quad t \in \mathbb{N}_a^{b}, \tag{3}$$

where

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{(b-s+1)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}}(t - a)^{\overline{\alpha-1}}, & t \in \mathbb{N}_a^{\rho(s)}, \\ \frac{(b-s+1)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}}(t - a)^{\overline{\alpha-1}} - (t - s + 1)^{\overline{\alpha-1}}, & t \in \mathbb{N}_s^{b}. \end{cases} \tag{4}$$

Theorem 2.2. [8] *The Green's function $G(t, s)$ defined in* (4) *satisfies the following properties:*

(1) $G(a, s) = G(b, s) = 0$ *for all* $s \in \mathbb{N}_{a+1}^{b}$;
(2) $G(t, a + 1) = 0$ *for all* $t \in \mathbb{N}_a^{b}$;
(3) $G(t, s) > 0$ *for all* $(t, s) \in \mathbb{N}_{a+1}^{b-1} \times \mathbb{N}_{a+2}^{b}$;
(4) $\max_{t \in \mathbb{N}_{a+1}^{b-1}} G(t, s) = G(s - 1, s)$ *for all* $s \in \mathbb{N}_{a+2}^{b}$;
(5) $\sum_{s=a+1}^{b} G(t, s) \le \lambda$ *for all* $(t, s) \in \mathbb{N}_a^{b} \times \mathbb{N}_{a+1}^{b}$, *where*

$$\lambda = \left(\frac{b - a - 1}{\alpha\Gamma(\alpha + 1)}\right) \left(\frac{(\alpha - 1)(b - a) + 1}{\alpha}\right)^{\overline{\alpha-1}}.$$

3. Existence & Uniqueness of Solutions of (1)

Let $X = \mathbb{R}^{b-a+1}$ be the Banach space of all real $(b-a+1)$-tuples equipped with the maximum norm

$$\|u\|_X = \max_{t\in\mathbb{N}_a^b} |u(t)|.$$

Obviously, the product space $\big(X \times X, \|\cdot\|_{X\times X}\big)$ is also a Banach space with the norm

$$\|(u_1, u_2)\|_{X\times X} = \|u_1\|_X + \|u_2\|_X.$$

A closed ball with radius R centred on the zero function in $X \times X$ is defined by

$$\mathcal{B}_R = \{(u_1, u_2) \in X \times X : \|(u_1, u_2)\|_{X\times X} \le R\}.$$

Define the operator $T : X \times X \to X \times X$ by

$$T(u_1, u_2)(t) = \begin{pmatrix} T_1(u_1, u_2)(t) \\ T_2(u_1, u_2)(t) \end{pmatrix}, \quad t \in \mathbb{N}_a^b, \tag{5}$$

where

$$T_1(u_1, u_2)(t) = \sum_{s=a+1}^{b} G_1(t,s) f_1(s, u_1(s), u_2(s)), \quad t \in \mathbb{N}_a^b, \tag{6}$$

and

$$T_2(u_1, u_2)(t) = \sum_{s=a+1}^{b} G_2(t,s) f_2(s, u_1(s), u_2(s)), \quad t \in \mathbb{N}_a^b. \tag{7}$$

The Green's functions $G_1(t,s)$ and $G_2(t,s)$ are given by

$$G_1(t,s) = \frac{1}{\Gamma(\alpha_1)} \begin{cases} \frac{(b-s+1)^{\overline{\alpha_1-1}}}{(b-a)^{\overline{\alpha_1-1}}}(t-a)^{\overline{\alpha_1-1}}, & t \in \mathbb{N}_a^{\rho(s)}, \\ \frac{(b-s+1)^{\overline{\alpha_1-1}}}{(b-a)^{\overline{\alpha_1-1}}}(t-a)^{\overline{\alpha_1-1}} - (t-s+1)^{\overline{\alpha_1-1}}, & t \in \mathbb{N}_s^b, \end{cases} \tag{8}$$

and

$$G_2(t,s) = \frac{1}{\Gamma(\alpha_2)} \begin{cases} \frac{(b-s+1)^{\overline{\alpha_2-1}}}{(b-a)^{\overline{\alpha_2-1}}}(t-a)^{\overline{\alpha_2-1}}, & t \in \mathbb{N}_a^{\rho(s)}, \\ \frac{(b-s+1)^{\overline{\alpha_2-1}}}{(b-a)^{\overline{\alpha_2-1}}}(t-a)^{\overline{\alpha_2-1}} - (t-s+1)^{\overline{\alpha_2-1}}, & t \in \mathbb{N}_s^b. \end{cases} \tag{9}$$

From Theorem 2.2, we have $\displaystyle\sum_{s=a+1}^{b} G_1(t,s) \le \lambda_1$ for all $(t,s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+1}^b$, where

$$\lambda_1 = \left(\frac{b-a-1}{\alpha_1\Gamma(\alpha_1+1)}\right)\left(\frac{(\alpha_1-1)(b-a)+1}{\alpha_1}\right)^{\overline{\alpha_1-1}}, \tag{10}$$

and $\displaystyle\sum_{s=a+1}^{b} G_2(t,s) \le \lambda_2$ for all $(t,s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+1}^b$, where

$$\lambda_2 = \left(\frac{b-a-1}{\alpha_2\Gamma(\alpha_2+1)}\right)\left(\frac{(\alpha_2-1)(b-a)+1}{\alpha_2}\right)^{\overline{\alpha_2-1}}. \tag{11}$$

Clearly, (u_1, u_2) is a fixed point of T if and only if (u_1, u_2) is a solution of (1). Assume

(H1) $f_1,\ f_2 : \mathbb{N}_{a+1}^b \times \mathbb{R}^2 \to \mathbb{R}$ are continuous.

(H2) There exist nonnegative constants L_1, L_2, L_3 and L_4 such that

$$|f_1(t,u_1,u_2) - f_2(t,v_1,v_2)| \leq L_1|u_1 - v_1| + L_2|u_2 - v_2|,$$

and

$$|f_2(t,u_1,u_2) - f_2(t,v_1,v_2)| \leq L_3|u_1 - v_1| + L_4|u_2 - v_2|,$$

for all (t,u_1,u_2), $(t,v_1,v_2) \in \mathbb{N}_{a+1}^b \times \mathbb{R}^2$.

(H3) Take

$$\max_{t\in\mathbb{N}_{a+1}^b} |f_1(t,0,0)| = M_1, \quad \max_{t\in\mathbb{N}_{a+1}^b} |f_2(t,0,0)| = M_2.$$

(H4) There exist nonnegative constants N_1, N_2, N_3, N_4, N_5, and N_6 such that

$$|f_1(t,u_1,u_2)| \leq N_1|u_1| + N_2|u_2| + N_3,$$

and

$$|f_2(t,u_1,u_2)| \leq N_4|u_1| + N_5|u_2| + N_6,$$

for all $(t,u_1,u_2) \in \mathbb{N}_{a+1}^b \times \mathbb{R}^2$.

(H5) $L = \lambda_1(L_1 + L_2) + \lambda_2(L_3 + L_4) \in (0,1)$.

(H6) $N = \lambda_1(N_1 + N_2) + \lambda_2(N_4 + N_5) \in (0,1)$.

We apply Banach fixed point theorem to establish existence and uniqueness of solutions of (1).

Theorem 3.1. [37] *Let $\mathcal{B}_r$ be the closed ball of radius $r > 0$, centred at zero, in a Banach space X with $\Upsilon : \mathcal{B}_r \to X$ a contraction and $\Upsilon(\partial\mathcal{B}_r) \subseteq \mathcal{B}_r$. Then, Υ has a unique fixed point in $\mathcal{B}_r$.*

Theorem 3.2. *Assume (H1), (H2), (H3) and (H5) hold. If we choose*

$$R \geq \frac{(\lambda_1 M_1 + \lambda_2 M_2)}{1 - [\lambda_1(L_1 + L_2) + \lambda_2(L_3 + L_4)]},$$

then the system (1) *has a unique solution $(u_1,u_2) \in \mathcal{B}_R$. Here λ_1 and λ_2 are given by* (10) *and* (11)*, respectively.*

Proof. Clearly, $T : \mathcal{B}_R \to X \times X$. First, we show that T is a contraction mapping. To see this, let (u_1,u_2), $(v_1,v_2) \in \mathcal{B}_R$, $t \in \mathbb{N}_a^b$, and consider

$$\begin{aligned}\big|T_1(u_1,u_2)(t) - T_1(v_1,v_2)(t)\big| &\leq \sum_{s=a+1}^{b} G_1(t,s)\big|f_1(s,u_1(s),u_2(s)) - f_2(s,v_1(s),v_2(s))\big| \\ &\leq \sum_{s=a+1}^{b} G_1(t,s)\big[L_1|u_1(s) - v_1(s)| + L_2|u_2(s) - v_2(s)|\big] \\ &\leq \lambda_1\big[L_1\|u_1 - v_1\|_X + L_2\|u_2 - v_2\|_X\big],\end{aligned}$$

implying that

$$\big\|T_1(u_1,u_2) - T_1(v_1,v_2)\big\|_X \leq \lambda_1\big[L_1\|u_1 - v_1\|_X + L_2\|u_2 - v_2\|_X\big]. \tag{12}$$

Similarly, we obtain

$$\big\|T_2(u_1,u_2) - T_2(v_1,v_2)\big\|_X \leq \lambda_2\big[L_3\|u_1 - v_1\|_X + L_4\|u_2 - v_2\|_X\big]. \tag{13}$$

Thus, from (12) and (13), we have

$$\begin{aligned}\|T(u_1,u_2)-T(v_1,v_2)\|_{X\times X} &= \big\|T_1(u_1,u_2)-T_1(v_1,v_2)\big\|_X+\big\|T_2(u_1,u_2)-T_2(v_1,v_2)\big\|_X\\ &\le \big[(\lambda_1L_1+\lambda_2L_3)\|u_1-v_1\|_X+(\lambda_1L_2+\lambda_2L_4)\|u_2-v_2\|_X\big]\\ &< L\big[(\|u_1-v_1\|_X+\|u_2-v_2\|_X\big]\\ &= L\|(u_1,u_2)-(v_1,v_2)\|_{X\times X}.\end{aligned}$$

Since $L<1$, T is a contraction mapping with contraction constant L. Next, we show that

$$T(\partial\mathcal{B}_R)\subseteq\mathcal{B}_R. \tag{14}$$

To see this, let $(u_1,u_2)\in\partial\mathcal{B}_R$, $t\in\mathbb{N}_a^b$, and consider

$$\begin{aligned}\big|T_1(u_1,u_2)(t)\big| &\le \sum_{s=a+1}^{b}G_1(t,s)|f_1(s,u_1(s),u_2(s))|\\ &\le \sum_{s=a+1}^{b}G_1(t,s)|f_1(s,u_1(s),u_2(s))-f_1(s,0,0)|\\ &\quad+\sum_{s=a+1}^{b}G_1(t,s)|f_1(s,0,0)|\\ &\le \sum_{s=a+1}^{b}G_1(t,s)\big[L_1|u_1(s)|+L_2|u_2(s)|\big]+M_1\sum_{s=a+1}^{b}G_1(t,s)\\ &\le \lambda_1\big[(L_1+L_2)R+M_1\big],\end{aligned}$$

implying that

$$\big\|T_1(u_1,u_2)\big\|_X\le\lambda_1\big[(L_1+L_2)R+M_1\big]. \tag{15}$$

Similarly, we obtain

$$\big\|T_2(u_1,u_2)\big\|_X\le\lambda_2\big[(L_3+L_4)R+M_2\big]. \tag{16}$$

Thus, from (15) and (16), we have

$$\begin{aligned}\|T(u_1,u_2)\|_{X\times X} &= \big\|T_1(u_1,u_2)\big\|_X+\big\|T_2(u_1,u_2)\big\|_X\\ &\le \big[\lambda_1(L_1+L_2)+\lambda_2(L_3+L_4)\big]R+(\lambda_1M_1+\lambda_2M_2)\le R,\end{aligned}$$

implying that (14) holds. Therefore, by Banach fixed point theorem, T has a unique fixed point $(u_1,u_2)\in\mathcal{B}_R$. The proof is complete. □

We apply Brouwer fixed point theorem to establish existence of solutions of (1).

Theorem 3.3. [37] *Let C be a compact convex subset of $\mathbb{R}^n$, and $T:C\to C$ be a continuous mapping. Then, f has a fixed point in C.*

Theorem 3.4. *Assume (H1), (H4) and (H6) hold. If we choose*

$$R\ge\frac{(\lambda_1N_3+\lambda_2N_6)}{1-[\lambda_1(N_1+N_2)+\lambda_2(N_4+N_5)]},$$

then the system (1) *has at least one solution* $(u_1,u_2)\in\mathcal{B}_R$. *Here* λ_1 *and* λ_2 *are given by* (10) *and* (11), *respectively.*

Proof. Clearly, $\mathcal{B}_R$ is a compact convex subset of $X\times X$. First, we show that $T:\mathcal{B}_R\to\mathcal{B}_R$. To see this, let $(u_1,u_2)\in\mathcal{B}_R$, $t\in\mathbb{N}_a^b$, and consider

$$\begin{aligned}\big|T_1(u_1,u_2)(t)\big| &\leq \sum_{s=a+1}^{b} G_1(t,s)\big|f_1(s,u_1(s),u_2(s))\big| \\ &\leq \sum_{s=a+1}^{b} G_1(t,s)\big[N_1|u_1(s)|+N_2|u_2(s)|+N_3\big] \\ &\leq \lambda_1\big[(N_1+N_2)R+N_3\big],\end{aligned}$$

implying that

$$\big\|T_1(u_1,u_2)\big\|_X \leq \lambda_1\big[(N_1+N_2)R+N_3\big]. \tag{17}$$

Similarly, we obtain

$$\big\|T_2(u_1,u_2)\big\|_X \leq \lambda_2\big[(N_4+N_5)R+N_6\big]. \tag{18}$$

Thus, from (17) and (18), we have

$$\begin{aligned}\|T(u_1,u_2)\|_{X\times X} &= \big\|T_1(u_1,u_2)\big\|_X+\big\|T_2(u_1,u_2)\big\|_X \\ &\leq \lambda_1(N_1+N_2)R+\lambda_2(N_4+N_5)R+(\lambda_1N_3+\lambda_2N_6)\big] \leq R,\end{aligned}$$

implying that $T:\mathcal{B}_R\to\mathcal{B}_R$. Since T is a summation operator on a discrete finite set, T is trivially continuous on $\mathcal{B}_R$. Therefore, by Brouwer fixed point theorem, T has at least one fixed point $(u_1,u_2)\in\mathcal{B}_R$. The proof is complete. □

4. Stability of Solutions of (1)

We use Urs's [42] approach to establish Ulam–Hyers stability of solutions of (1).

Theorem 4.1. [42] *Let X be a Banach space and T_1, $T_2:X\times X\to X$ be two operators. Then, the operational equations system*

$$\begin{cases}u_1=T_1(u_1,u_2),\\ u_2=T_2(u_1,u_2),\end{cases} \tag{19}$$

is said to be Ulam–Hyers stable if there exist C_1, C_2, C_3, $C_4>0$ such that for each ε_1, $\varepsilon_2>0$ and each solution-pair $(u_1^*,u_2^*)\in X\times X$ of the inequalities:

$$\begin{cases}\|u_1-T_1(u_1,u_2)\|_X\leq\varepsilon_1,\\ \|u_2-T_2(u_1,u_2)\|_X\leq\varepsilon_2,\end{cases} \tag{20}$$

there exists a solution $(v_1^*,v_2^*)\in X\times X$ of (19) such that

$$\begin{cases}\|u_1^*-v_1^*\|_X\leq C_1\varepsilon_1+C_2\varepsilon_2,\\ \|u_2^*-v_2^*\|_X\leq C_3\varepsilon_1+C_4\varepsilon_2.\end{cases} \tag{21}$$

Theorem 4.2. [42] *Let X be a Banach space, T_1, $T_2:X\times X\to X$ be two operators such that*

$$\begin{cases}\|T_1(u_1,u_2)-T_1(v_1,v_2)\|_X\leq k_1\|u_1-v_1\|_X+k_2\|u_2-v_2\|_X,\\ \|T_2(u_1,u_2)-T_2(v_1,v_2)\|_X\leq k_3\|u_1-v_1\|_X+k_4\|u_2-v_2\|_X,\end{cases} \tag{22}$$

for all (u_1,u_2), $(v_1,v_2)\in X\times X$. Suppose

$$H=\begin{pmatrix}k_1 & k_2\\ k_3 & k_4\end{pmatrix}$$

converges to zero. Then, the operational equations system (19) is Ulam–Hyers stable.

Set

$$H = \begin{pmatrix} \lambda_1 L_1 & \lambda_1 L_2 \\ \lambda_2 L_3 & \lambda_2 L_4 \end{pmatrix}. \tag{23}$$

Theorem 4.3. *Assume (H1), (H2), (H3), and (H4) hold. Choose*

$$R \geq \frac{(\lambda_1 M_1 + \lambda_2 M_2)}{1 - [\lambda_1(L_1 + L_2) + \lambda_2(L_3 + L_4)]}.$$

Further, assume the spectral radius of H is less than one. Then, the solution of system (1) *is Ulam–Hyers stable. Here λ_1 and λ_2 are given by* (10) *and* (11), *respectively.*

Proof. In view of Theorem 3.2, we have

$$\begin{cases} \big\|T_1(u_1, u_2) - T_1(v_1, v_2)\big\|_X \leq \lambda_1\big[L_1\|u_1 - v_1\|_X + L_2\|u_2 - v_2\|_X\big], \\ \big\|T_2(u_1, u_2) - T_2(v_1, v_2)\big\|_X \leq \lambda_2\big[L_3\|u_1 - v_1\|_X + L_4\|u_2 - v_2\|_X\big], \end{cases} \tag{24}$$

which implies that

$$\|T(u_1, u_2) - T(v_1, v_2)\|_{X\times X} \leq H \begin{pmatrix} \|u_1 - v_1\|_X \\ \|v_2 - v_2\|_X \end{pmatrix}. \tag{25}$$

Since the spectral radius of H is less than one, the solution of (1) is Ulam–Hyers stable. The proof is complete. □

5. Example

Consider the following coupled system of two-point nabla fractional difference boundary value problems

$$\begin{cases} \Big(\nabla^{0.5}_{\rho(0)}\big(\nabla u_1\big)\Big)(t) + (0.01)e^{-t}\left[1 + \tan^{-1} u_1(t) + \tan^{-1} u_2(t)\right] = 0, \quad t \in \mathbb{N}_2^9, \\ \Big(\nabla^{0.5}_{\rho(0)}\big(\nabla u_2\big)\Big)(t) + (0.02)\left[e^{-t} + \sin u_1(t) + \sin u_2(t)\right] = 0, \quad t \in \mathbb{N}_2^9, \\ u_1(0) = 0, \ u_1(9) = 0, \\ u_2(0) = 0, \ u_2(9) = 0. \end{cases} \tag{26}$$

Comparing (1) and (26), we have $a = 0$, $b = 9$, $\alpha_1 = \alpha_2 = 1.5$,

$$f_1(t, u_1, u_2) = (0.01)e^{-t}\left[1 + \tan^{-1} u_1 + \tan^{-1} u_2\right],$$

and

$$f_2(t, u_1, u_2) = (0.02)\left[e^{-t} + \sin u_1 + \sin u_2\right],$$

for all $(t, u_1, u_2) \in \mathbb{N}_0^9 \times \mathbb{R}^2$. Clearly, f_1 and f_2 are continuous on $\mathbb{N}_0^9 \times \mathbb{R}^2$. Next, f_1 and f_2 satisfy assumption (H2) with $L_1 = 0.01$, $L_2 = 0.01$, $L_3 = 0.02$ and $L_4 = 0.02$. We have,

$$M_1 = \max_{t\in\mathbb{N}_1^9} |f_1(t, 0, 0)| = \frac{0.01}{e},$$

$$M_2 = \max_{t\in\mathbb{N}_1^9} |f_2(t, 0, 0)| = \frac{0.02}{e},$$

$$\lambda_1 = \left(\frac{b - a - 1}{\alpha_1\Gamma(\alpha_1 + 1)}\right)\left(\frac{(\alpha_1 - 1)(b - a) + 1}{\alpha_1}\right)^{\overline{\alpha_1 - 1}} \approx 7.4259,$$

and

$$\lambda_2 = \left(\frac{b - a - 1}{\alpha_2\Gamma(\alpha_2 + 1)}\right)\left(\frac{(\alpha_2 - 1)(b - a) + 1}{\alpha_2}\right)^{\overline{\alpha_2 - 1}} \approx 7.4259.$$

Also,

$$L = \lambda_1(L_1 + L_2) + \lambda_2(L_3 + L_4) \approx 0.4456 < 1,$$

implying that (H5) holds. Thus, by Theorem 3.2, the system (26) has a unique solution $(u_1, u_2) \in \mathcal{B}_R$, where

$$R \geq \frac{(\lambda_1 M_1 + \lambda_2 M_2)}{1 - [\lambda_1(L_1 + L_2) + \lambda_2(L_3 + L_4)]} = 0.1479.$$

Further,

$$H = \begin{pmatrix} \lambda_1 L_1 & \lambda_1 L_2 \\ \lambda_2 L_3 & \lambda_2 L_4 \end{pmatrix} = \begin{pmatrix} 0.0743 & 0.0743 \\ 0.1486 & 0.1486 \end{pmatrix}.$$

The spectral radius of H is 0.0223 which is less than one. Hence, by Theorem 4.3 the solution of (26) is Ulam–Hyers stable.

6. Conclusions

In this article, we obtained sufficient conditions on existence, uniqueness and Ulam–Hyers stability of solutions for a coupled system of two-point nabla fractional difference boundary value problems, using Banach, Brouwer fixed point theorems and Urs's approach. Finally, we illustrated the applicability of established results through an example.

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