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COLOR LAPLACIAN ENERGY OF SOME CLUSTER GRAPHS

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Abstract. The color energy of a graph G is defined as the sum of the absolute values of the color eigenvalues of G. The graphs with large number of edges are referred as cluster graphs. Cluster graphs are obtained from complete graphs by deleting few edges according to some criteria. Bipartite cluster graphs are obtained by deleting few edges from complete bipartite graphs according to some rule. In this paper, we study the color Laplacian energy of cluster graphs and bipartite cluster graphs obtained by deleting the edges of complete and complete bipartite graph respectively.

Keywords: color Laplacian matrix, color Laplacian eigenvalues, color Laplacian energy

AMS Subject Classification: 05C15, 05C50

1. Introduction

The energy $E(G)$ of a graph G, defined as the sum of the absolute values of its eigenvalues, belongs to the most popular graph invariants in chemical graph theory. It originates from the π-electron energy in the Hückel molecular orbital model, but has also gained purely mathematical interest. Gutman introduced this definition of the energy of a simple graph in his paper "The energy of a graph"[9]. In the past decade, interest in graph energy has increased and many different versions have been introduced. In 2006, Gutman and Zhou defined the Laplacian energy of a graph as the sum of the absolute deviations (i.e., distance from the mean) of the eigenvalues of its Laplacian matrix [10]. Similar variants of graph energy were developed for the signless Laplacian matrix, the distance matrix, the incidence matrix, and even for a general matrix not associated with a graph [1].

Let G be a simple undirected graph with n vertices. Let $A(G)$ be the adjacency matrix of G and D(G) be the diagonal matrix of vertex degrees. Then the Laplacian matrix of G is $L(G) = D(G) - A(G)$. The Laplacian energy of graph G [10] is defined as $LE(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$, where m is the number of edges in the graph G.

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A coloring of a graph G [12] is a coloring of its vertices such that no two adjacent vertices share the same color. The minimum number of colors needed for the coloring of a graph G is called the chromatic number of G and is denoted by $\chi(G)$.

In 2013, C. Adiga, E. Sampathkumar, M. A. Sriraj and A. S. Shrikanth [2], have introduced the energy of colored graph. The entries of the color adjacency matrix $A_c(G)$ are as follows: If $c(v_i)$ is the color of vertex v_i, then

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent with } c(v_i) \neq c(v_j), \\ -1, & \text{if } v_i \text{ and } v_j \text{ are non-adjacent with } c(v_i)=c(v_j), \\ 0, & \text{otherwise.} \end{cases}$$

The color energy of a graph G is the sum of absolute values of the color eigenvalues of G, i. e., $E_c(G) = \sum_{i=1}^{n} |\lambda_i|$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the color eigenvalues of graph G.

In 2015, P. G. Bhat and S. D'Souza [7] have introduced the color Laplacian matrix of G as $L_c(G) = D(G) - A_c(G)$. The eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ are called color Laplacian eigenvalues of the graph G. The characteristic polynomial of color Laplacian matrix of G is denoted by $\phi(L_c(G), \mu)$. The color Laplacian energy of graph is defined as $LE_c(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|$, where m is the number of edges in the graph G. For more information on energy, Laplacian energy and color energy of a graph we refer [1, 3, 4, 5, 6, 8].

I. Gutman and L. Pavlović [11] introduced cluster graphs obtained by deleting the edges of complete graphs and found energies of cluster graphs. H. B. Walikar and H. S. Ramane [14] introduced bipartite cluster graphs obtained by deleting edges of complete bipartite graph. In section 2, we establish color Laplacian energy of some cluster graphs. In section 3, we establish color Laplacian energy of bipartite cluster graphs.

2. Color Laplacian energy of some cluster graphs

Definition 2.1. [13] *Let $(K_m)i$, $i = 1, 2, \ldots, k$, $1 \leq k \leq \left\lfloor \frac{n}{m} \right\rfloor$, $1 \leq m \leq n$, be independent complete subgraphs with m vertices of the complete graph K_n, $n \geq 3$. The cluster graph $Ka_n(m, k)$ is obtained from K_n, by deleting all edges of $(K_m)_i$, $i = 1, 2, \ldots, k$. In addition $Ka_n(m, 0) = Ka_n(0, k) = Ka_n(0, 0) = K_n$.*

Example 2.1. *Consider the cluster graph*

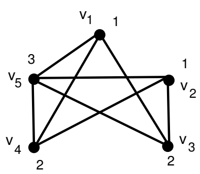

Figure 1. Graph $K_{a_5}(2, 2)$

Let $c(v_1) = c(v_2) = 1$, $c(v_3) = c(v_4) = 2$, $c(v_5) = 3$. The color Laplacian matrix is,

$$L_c\left(Ka_5(2,2)\right) = \left[\begin{array}{cccc|c} 3 & 1 & -1 & -1 & -1 \\ 1 & 3 & -1 & -1 & -1 \\ -1 & -1 & 3 & 1 & -1 \\ -1 & -1 & 1 & 3 & -1 \\ \hline -1 & -1 & -1 & -1 & 4 \end{array}\right] = \left[\begin{array}{c|c} X_{4\times 4} & -J_{4\times 1} \\ \hline -J_{1\times 4} & 5I_1 - J_{1\times 1} \end{array}\right].$$

Where $X = \left[\begin{array}{c|c} 2I_2+J_2 & -J_2 \\ \hline -J_2 & 2I_2-J_2 \end{array}\right]$, *I is identity matrix and J is the matrix with all entries one.*

Theorem 2.1. *For* $n \geq 3$, $0 \leq k \leq \left\lfloor \frac{n}{m} \right\rfloor$ *and* $1 \leq m \leq n$,

$$LE_c(Ka_n(m,k)) = \frac{1}{n}\left[2km(m-1)(n-1) - 2mk^2(m-1)^2 + n\left(n-m-1+ \sqrt{m^2(1-4k)+(n-1)^2+2m(n-1+2k)}\right)\right].$$

Proof. Consider the cluster graph $Ka_n(m,k)$ obtained from complete graph K_n by deleting all edges of $(K_m)_i$, $i = 1,2,\ldots,k$. Since $\chi(Ka_n(m,k)) = n-(m-1)k$, we have

$$L_c\left(Ka_n(m,k)\right) = \left[\begin{array}{c|c} X_{mk\times mk} & -J_{mk\times(n-mk)} \\ \hline -J_{(n-mk)\times mk} & (nI-J)_{(n-mk)\times(n-mk)} \end{array}\right]$$

$$X = \left[\begin{array}{c|c|c|c} (n-m-1)I_m+J_m & -J_m & \cdots & -J_m \\ \hline -J_m & (n-m-1)I_m+J_m & \cdots & -J_m \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline -J_m & -J_m & \cdots & (n-m-1)I_m+J_m \end{array}\right]_{mk\times mk}$$

Consider $\det\left(\mu I - L_c\left(Ka_n(m,k)\right)\right)$.

Step 1: Replace R_i by $R_i' = R_i - R_{i+1}$, for $i = 1,2,\ldots m-1, m+1, m+2,\ldots,2m-1,2m+1,\ldots,mk-2,mk-1,mk+1,mk+2,\ldots,n-2,n-1$. Then, $\det\left(\mu I - L_c\left(Ka_n(m,k)\right)\right)$ will reduce to a new determinant, say $\det(C)$.

Step 2: In $\det(C)$, replacing C_i by $C_i' = C_i + C_{i-1}$, for $i = 2,3,\ldots,m,m+2,m+3,\ldots,2m,2m+2,\ldots,mk-1,mk,mk+2,mk+3,\ldots,n-1,n$, a new determinant $\det(D)$ is obtained.

Step 3: On expanding $\det(D)$ along the rows R_i, for $i = 1,2,3,\ldots,k-1,k+1,k+2,\ldots,2k-1,2k+1,2k+2,\ldots,mk-2,mk-1,mk+1,mk+2,\ldots,n-2,n-1$, it becomes $(\mu-n+m+1)^{(m-1)k}(\mu-n)^{n-mk-1}\det(E)$. Where

$$\det(E) = \begin{vmatrix} \mu-n+1 & m & m & \cdots & m & n-mk \\ m & \mu-n+1 & m & \cdots & m & n-mk \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ m & m & m & \cdots & \mu-n+1 & n-mk \\ m & m & m & \cdots & m & \mu-mk \end{vmatrix}_{(k+1)\times(k+1)}$$

Step 4: In $\det(E)$, replacing R_i by $R_i' = R_i - R_{i+1}$, for $i = 1,2,\ldots,k$, it reduces to

$$\det(E) = \begin{vmatrix} \mu-n-m+1 & -(\mu-n-m+1) & 0 & \cdots & 0 & 0 \\ 0 & \mu-n-m+1 & -(\mu-n-m+1) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mu-n-m+1 & n-\mu \\ 0 & 0 & 0 & \cdots & m & \mu-mk \end{vmatrix}$$

Step 5: In $\det(E)$, replacing C_i by $C_i' = C_i + C_{i-1} + \cdots + C_1$, for $i = k, k-1, \ldots, 2$ it simplifies to

$$\begin{aligned}\det(E) &= (\mu - n - m + 1)^{k-1} \begin{vmatrix} \mu - n - m + 1 & n - \mu \\ mk & \mu + mk \end{vmatrix} \\ &= (\mu - n - m + 1)^{k-1}(\mu^2 - (m + n - 1)\mu - (m^2 - m)k).\end{aligned}$$

Thus,

$$\begin{aligned}\phi(L_c(Ka_n(m,k),\mu)) = (\mu - n + m + 1)^{(m-1)k}(\mu - n)^{n-mk-1}(\mu - n - m + 1)^{k-1} \\ (\mu^2 - (m + n - 1)\mu - (m^2 - m)k).\end{aligned}$$

So, the color Laplacian spectrum of $Ka_n(m,k)$ is $\{n + m - 1\ (k - 1 \text{ times}),$
$n - m - 1((m - 1)k \text{ times}), n\ (n - mk - 1 \text{ times}),$
$\dfrac{m + n - 1 + \sqrt{(m + n - 1)^2 - 4(m^2 - m)k}}{2}, \dfrac{m + n - 1 - \sqrt{(m + n - 1)^2 - 4(m^2 - m)k}}{2}\Bigg\}.$

Average degree of $Ka_n(m,k)$ is $\dfrac{n(n-1) - km(m-1)}{n}$.

Hence,

$$\begin{aligned}LE_c(Ka_n(m,k)) &= (m-1)k \left|(n - m - 1) - \frac{n(n-1) - km(m-1)}{n}\right| + (n - mk - 1) \\ &\left|n - \frac{n(n-1) - km(m-1)}{n}\right| + (k-1)\left|(n + m - 1) - \right. \\ &\left. \frac{n(n-1) - km(m-1)}{n}\right| + \\ &\left|\frac{(m + n - 1) + \sqrt{(m + n - 1)^2 - 4k(m^2 - m)}}{2} - \right. \\ &\left. \frac{n(n-1) - km(m-1)}{n}\right| \\ &+ \left|\frac{(m + n - 1) - \sqrt{(m + n - 1)^2 - 4k(m^2 - m)}}{2} - \right. \\ &\left. \frac{n(n-1) - km(m-1)}{n}\right| \\ &= \frac{1}{n}\left[2km(m-1)(n-1) - 2mk^2(m-1)^2 + n\left(n - m - 1 + \right.\right. \\ &\left.\left. \sqrt{m^2(1 - 4k) + (n-1)^2 + 2m(n - 1 + 2k)}\right)\right].\end{aligned}$$

□

Definition 2.2. [11] *For fixed integers $n \geq 3$ and $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, the cluster graph $K_{b_n}(k)$ is obtained from K_n by the deletion of k independent edges.*

Corollary 2.1. *For $n \geq 3$ and $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$,*

$$LE_c(K_{b_n}(k)) = \frac{1}{n}\left[4k(n - 4k - 1) + n\left(n - 3 + \sqrt{(n-1)^2 + 4(n - 2k)}\right)\right].$$

Proof. Observe that $K_{b_n}(k)$ is a special case of $Ka_n(m,k)$, when m=2. Thus, by substituting $m = 2$ in Theorem 2.1, the result follows. □

Definition 2.3. [11] *Let $n \geq 3$ and k, where $1 \leq k \leq n-1$, be fixed integers. The cluster graph $K_{c_n}(k)$ is obtained from K_n by deleting a k-clique.*

Corollary 2.2. *For $n \geq 3$ and $1 \leq k \leq n-1$,*

$$LE_c(K_{c_n}(k)) = \frac{1}{n}\left[2k(k-1)(n-k) + n\left(n-k-1+ \sqrt{(n-1)^2 - 3k^2 + 2k(n+1)}\right)\right].$$

Proof. The proof follows by noting that $K_{c_n}(k) = Ka_n(m,1)$ in Theorem 2.1. □

Definition 2.4. [11] *Let $n \geq 3$ and $1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$ be fixed integers. The cluster graph $K_{f_n}(k)$ is obtained from K_n by the deletion of k disjoint triangles.*

Corollary 2.3. *For $n \geq 3$ and $1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$,*

$$LE_c(K_{f_n}(k)) = \frac{1}{n}\left[12k(n-2k-1) + n\left(n-4+\sqrt{(n-1)^2 + 3(2n-8k+1)}\right)\right].$$

Proof. The proof follows by noting that $K_{f_n}(k) = Ka_n(3,k)$ in Theorem 2.1. □

Definition 2.5. [11] *For fixed integers n and k, $n \geq 3$ and $0 \leq k \leq n-1$, the graph $K_{a_n}(k)$ is obtained from K_n by the deletion of k edges with a common end vertex.*

Theorem 2.2. *For $n \geq 3$ and $0 \leq k \leq n-1$,*

$$\phi(L_c(K_{a_n}(k),\mu)) = (\mu-n)^{n-k-2}(\mu-n+1)^{k-2}(\mu^4 - (3n-k-2)\mu^3 + ((3n^2-4n)- (2n-1)k)\mu^2 - ((n^3-2n^2+2n-3) - (n^2-n+2)k)\mu + (2n^2-6n+3) - (2n-3)k).$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of complete graph K_n. The graph $K_{a_n}(k)$ is obtained from K_n by the deletion of k edges with common end vertex v_i, where $i = 1,2,3,\ldots,n$. and $\chi(K_{a_n}(k)) = n-1$. Then, we have

$$L_c\left(K_{a_n}(k)\right) = \left[\begin{array}{c|c|c} (n-k-1)_1 & C_{1\times k} & -J_{1\times n-k-1} \\ \hline C^T_{k\times 1} & (n-1)I_k - J_k & -J_{k\times n-k-1} \\ \hline -J_{n-k-1\times 1} & -J_{n-k-1\times k} & (nI-J)_{n-k-1} \end{array}\right]$$

Where $C = [1 \quad 0 \quad 0\ldots0]_{1\times k}$. Consider $\det\left(\mu I - L_c\left(K_{a_n}(k)\right)\right)$.

Step 1: Replace R_i by $R_i' = R_i - R_{i+1}$, for $i = 2,3,\ldots,k,k+1,\ldots,n-2,n-1$. Then, $\det\left(\mu I - L_c\left(K_{a_n}(k)\right)\right)$ will reduce to a new determinant, say $(\mu-n+1)^{k-2}(\mu-n)^{n-k-2}\det(C)$.

Step 2: In $\det(C)$, replacing C_i by $C_i' = C_i + C_{i-1}$, for $i = 4,5,\ldots,k+1,k+3,\ldots,n$, a new determinant say $\det(D)$ is obtained.

Step 3: On expanding $\det(D)$ along the rows R_i, for $i = 3,4,\ldots,k,k+2,\ldots,n-2,n-1$, it reduces to

$$\det(D) = \begin{vmatrix} \mu-n+k+1 & -1 & 0 & n-k-1 \\ -1 & \mu-n+1 & n-\mu-1 & 0 \\ 0 & 1 & \mu-n+k & n-k-1 \\ 1 & 1 & k-1 & \mu-k-1 \end{vmatrix}$$
$$= (\mu^4 - (3n-k-2)\mu^3 + ((3n^2-4n)-(2n-1)k)\mu^2 - ((n^3-2n^2+2n-3)-(n^2-n+2)k)\mu + (2n^2-6n+3) - (2n-3)k).$$

Thus,

$$\phi(L_c(K_{a_n}(k),\mu)) = (\mu-n)^{n-k-2}(\mu-n+1)^{k-2}(\mu^4-(3n-k-2)\mu^3+((3n^2-4n)-(2n-1)k)\mu^2-((n^3-2n^2+2n-3)-(n^2-n+2)k)\mu+(2n^2-6n+3)-(2n-3)k).$$

□

Definition 2.6. *For $n \geq 3$ and $0 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$, the graph $K_{e_n}(k)$ is obtained from K_n by the deletion of k independent paths P_3.*

Example 2.2. *Consider the cluster graph obtained from K_7 deleting two independent paths $\{v_1, v_3, v_5\}$ and $\{v_2, v_4, v_6\}$ respectively. Let $c(v_1) = c(v_3) = 1$, $c(v_2) = c(v_4) = 3$, $c(v_5) = 2$, $c(v_6) = 4$ and $c(v_7) = 5$.*

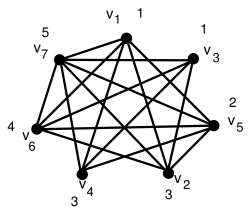

FIGURE 2. Graph $K_{e_7}(2)$

Theorem 2.3. *For $n \geq 3$ and $1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$, $\phi(L_c(K_{e_n}(k),\mu)) = (\mu-n+1)^{k-1}(\mu-n)^{n-3k-1}(\mu^2-(2n-3)\mu+n^2-3n-3)^{k-1}(\mu^4-(3n-4)\mu^3+(3n^2-8n+2k)\mu^2+(k(10-4n)-(n^3-4n^2+3))\mu+(2n^2-10n+9)k)$.*

Proof. Consider the graph $K_{e_n}(k)$. Since $\chi(K_{e_n}(k)) = n-k$, we have

$$L_c\left(K_{e_n}(k)\right) = \left[\begin{array}{c|c|c|c} (n-1)I_k - J_k & (-J+2I)_k & -J_k & -J_{k\times(n-3k)} \\ \hline (-J+2I)_k & (n-2)I_k - J_k & (-J+I)_k & -J_{k\times(n-3k)} \\ \hline -J_k & (-J+I)_k & (n-1)I_k - J_k & -J_{k\times(n-3k)} \\ \hline -J_{(n-3k)\times k} & -J_{(n-3k)\times k} & -J_{(n-3k)\times k} & (nI-J)_{(n-3k)} \end{array}\right]$$

Consider $\det\left(\mu I - L_c\left(K_{e_n}(k)\right)\right)$.

Step 1: Replace R_i by $R_i' = \begin{cases} R_i - R_{i+1}, & \text{for i= 1, 2,..., k-1, k+1, k+2, ..., 2k-1, 2k+1, 2k+2,...,3k-1,} \\ R_i - R_{i-1}, & \text{for i= n, n-1, ..., 3k+2.} \end{cases}$

Then, $\det\left(\mu I - L_c\left(K_{e_n}(k)\right)\right)$ will reduce to a new determinant, say $\det(C)$.

Step 2: In $\det(C)$, replacing

$$C_i \text{ by } C_i' = \begin{cases} C_i + C_{i+1}, & \text{for i=n-1, n-2,..., 3k+1,} \\ C_i + C_{i-1} + \ldots + C_1, & \text{for i=k, k-1,...,2,} \\ C_i + C_{i-1} + \ldots + C_{k+1}, & \text{for i= 2k, 2k-1,...,k+2,} \\ C_i + C_{i-1} + \ldots + C_{2k+1}, & \text{for i= 3k, 3k-1,...,2k+2.} \end{cases}$$

a new determinant say $\det(D)$ is obtained.

Step 3: In $\det(D)$, replacing C_i by $C_i' = (\mu-n+1)C_i + 2C_j$, for $i = k+1, k+2, \ldots, 2k-1$ and $j = 1, 2, \ldots, k-1$ a new determinant say $\det(E)$ is obtained.

Step 4: On expanding $\det(E)$ along the rows R_i, for $i = 1, 2, \ldots, k-1, 3k+2, 3k+3, \ldots, n-1, n$, it reduces to $(\mu - n)^{n-3k-1} \det(F)$ which is of order $2k + 2$. Where

$$\det(F) = \begin{vmatrix} P & \mu-n+3 & \cdots & N & k-2 & 1 & \cdots & k-1 & k & n-3k \\ 0 & M & \cdots & 0 & 0 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & M & 0 & -0 & \cdots & -1 & 0 & 0 \\ k-2 & \mu-n+3 & \cdots & N & \mu-n+k+2 & 1 & \cdots & k-1 & k-1 & n-3k \\ 0 & -\mu+n-1 & \cdots & 0 & 0 & \mu-n+1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -\mu+n-1 & 0 & 0 & \cdots & \mu-n+1 & 0 & 0 \\ k & \mu-n+3 & \cdots & N & k-1 & 1 & \cdots & k-1 & P & n-3k \\ k & \mu-n+3 & \cdots & N & k & 1 & \cdots & k-1 & k & \mu-3k \end{vmatrix}$$

where $M = \mu^2 - (2n-3)\mu + n^2 - 3n - 2$, $P = \mu - n - k + 1$ and $N = (k-1)(\mu - n + 3)$.

Step 5: In $\det(F))$, replacing C_i by $C_i' = C_i + C_{k+i}$, for $i = 2, 3, \ldots, k-1$ and then expanding $\det(E)$ along the rows R_i, for $i = 2, 3, \ldots, k, k+2, k+3 \ldots, 2k-1, 2k$, it reduces to

$$\det(F) = (\mu - n)^{n-3k-1}(\mu - n + 1)^{k-1}(\mu^2 - (2n-3)\mu + n^2 - 3n - 3)^{k-1}$$

$$\begin{vmatrix} \mu-n+k+1 & k-2 & k & n-3k \\ k-2 & \mu-n+k+2 & k-1 & n-3k \\ k & k-1 & \mu-n+k+1 & n-3k \\ k & k & k & \mu-3k \end{vmatrix}$$

$$= (\mu - n + 1)^{k-1}(\mu^2 - (2n-3)\mu + n^2 - 3n - 3)^{k-1}(\mu^4 - (3n-4)\mu^3 +$$
$$(3n^2 - 8n + 2k)\mu^2 + (k(10 - 4n) - (n^3 - 4n^2 + 3))\mu + (2n^2 - 10n + 9)k).$$

Thus,

$$\phi(L_c(K_{e_n}(k), \mu)) = (\mu - n + 1)^{k-1}(\mu - n)^{n-3k-1}(\mu^2 - (2n-3)\mu + n^2 - 3n - 3)^{k-1}$$
$$(\mu^4 - (3n-4)\mu^3 + (3n^2 - 8n + 2k)\mu^2 + (k(10 - 4n) - (n^3 - 4n^2 + 3))\mu +$$
$$(2n^2 - 10n + 9)k).$$

□

Definition 2.7. *For fixed integers $n \geq 4$ and $0 \leq k \leq \left\lfloor \frac{n}{4} \right\rfloor$, the graph $K_{d_n}(k)$ is obtained from K_n by the deletion of k independent paths P_4.*

Example 2.3. *Consider the cluster graph on 7 vertices obtained by deleting one independent path $\{v_1, v_3, v_4, v_2\}$. Let $c(v_1) = c(v_3) = 1$, $c(v_2) = c(v_4) = 2$, $c(v_5) = 3$, $c(v_6) = 4$ and $c(v_7) = 5$.*

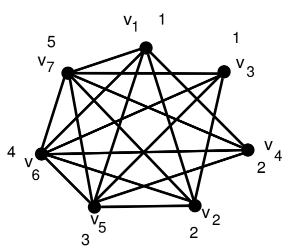

FIGURE 3. Graph $K_{d_7}(1)$

The color Laplacian matrix is, $L_c\left(K_{d_n}(k)\right)=\left[\begin{array}{c|c|c} 6I_2-J_2 & (-J+2I)_2 & -J_{2\times 3} \\ \hline (-J+2I)_2 & 4I_2 & -J_{2\times 3} \\ \hline -J_{3\times 2} & -J_{3\times 2} & (7I-J)_3 \end{array}\right]$

Theorem 2.4. *For* $n\geq 4$ *and* $1\leq k\leq \left\lfloor \frac{n}{4}\right\rfloor$,

$$LE_c(K_{d_n}(k))=\frac{1}{n}\left[2k((3+\sqrt{5})n-6)-24k^2+n(n-3+\sqrt{(n+1)^2-16k})\right].$$

Proof. Consider the cluster graph $K_{d_n}(k)$. As $\chi(K_{d_n}(k))=n-2k$, we have

$$L_c\left(K_{d_n}(k)\right)=\left[\begin{array}{c|c|c} (n-1)I_{2k}-J_{2k} & (-J+2I)_{2k} & -J_{2k\times(n-4k)} \\ \hline (-J+2I)_{2k} & X_{2k} & -J_{2k\times(n-4k)} \\ \hline -J_{(n-4k)\times 2k} & -J_{(n-4k)\times 2k} & (nI-J)_{(n-4k)} \end{array}\right]$$

$$X=\left[\begin{array}{c|c|c|c} (n-3)I_{2\times 2} & -J_{2\times 2} & \cdots & -J_{2\times 2} \\ \hline -J_{2\times 2} & (n-3)I_{2\times 2} & \cdots & -J_{2\times 2} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline -J_{2\times 2} & -J_{2\times 2} & \cdots & (n-3)I_{2\times 2} \end{array}\right]$$

Consider $\det\left(\mu I-L_c\left(K_{d_n}(k)\right)\right)$.

Step 1: Replace R_i by $R_i^{'}=\begin{cases} R_i-R_{i+1}, & \text{for i= 1, 2,..., 2k-1, 2k+1, 2k+3, ..., 4k-3, 4k-1,} \\ R_i-R_{i-1}, & \text{for i= n, n-1, ..., 4k+2.} \end{cases}$

Then, $\det\left(\mu I-L_c\left(K_{d_n}(k)\right)\right)$ will reduce to a new determinant, say $\det(C)$.

Step 2: In $\det(C)$, replacing

$$C_i \text{ by } C_i^{'}=\begin{cases} C_i+C_{i+1}, & \text{for i=n-1, n-2,..., 4k+2, 4k+1,} \\ C_i+C_{i-1}+\ldots+C_1, & \text{for i=2k, 2k-1,...,2,} \\ C_i+C_{i-1}+\ldots+C_{2k}, & \text{for i=4k, 4k-1,...,2k+2.} \end{cases}$$

a new determinant say $\det(D)$ is obtained.

Step 3: In $\det(D)$, replace C_i by $C_i^{'}=(\mu-n+1)C_i+2C_j$, for $i=2k+1,2k+2,\ldots,4k$ and $j=1,2,\ldots,2k-1$. It reduces to $\det(E)$.

Step 4: On expanding $\det(E)$ along the rows R_i, for $i=1,2,\ldots,2k-1,2k+1,2k+3,\ldots,4k-3,4k-1,4k+2,4k+3,\ldots,n-1,n$, it becomes $(\mu-n)^{n-4k+1}(\mu^2-(2n-4)\mu+n^2-4n-1)^k\det(F)$. Where

$$\det(F)=\begin{vmatrix} P & 2(\mu-n+3) & 4(\mu-n+3) & \cdots & (2k-2)(\mu-n+3) & 2k-2 & n-4k \\ 2k-2 & Q & R & \cdots & S & P & n-4k \\ 2k-2 & 2(\mu-n+3) & R & \cdots & S & P & n-4k \\ 2k-2 & 2(\mu-n+3) & 4(\mu-n+3) & \cdots & S & P & n-4k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2k-2 & 2(\mu-n+3) & 4(\mu-n+3) & \cdots & S & P & n-4k \\ 2k-2 & 2(\mu-n+3) & 4(\mu-n+3) & \cdots & (2k-2)(\mu-n+3) & P & n-4k \\ 2k-2 & 2(\mu-n+3) & 4(\mu-n+3) & \cdots & (2k-2)(\mu-n+3) & 2k & \mu-4k \end{vmatrix}$$

where $P=\mu-n+2k+1$, $Q=\mu^2+2\mu(2-n)+n^2-4n+3$, $R=\mu^2+2\mu(3-n)+n^2-6n+9$ and $S=\mu^2+2\mu(k-n)+n^2-k(2n-6)-9$

Step 5: In $\det(F)$, replacing R_i by $R_i^{'}=R_i-R_{i+1}$, for $i=2,3,\ldots,k,k+1$, it reduces to

$$\det(F)=(\mu^2-(n-1)2\mu+n^2-2n-3)^{k-1}\begin{vmatrix} \mu-n+2k+1 & 2k-2 & n-4k \\ -2 & \mu-n+1 & n-\mu \\ 2k & 2k & \mu-4k \end{vmatrix}$$

$$= (\mu^2 - (n-1)2\mu + n(n-2) - 3)^{k-1}(\mu^3 + (2-2n)\mu^2 - (n^2 - 2n + 4k - 3)\mu + 4k(3-n)).$$

Hence, by back substitution, we obtain characteristic polynomial of cluster graph $K_{d_n}(k)$,

$$\phi(L_c(K_{d_n}(k),\mu)) = (\mu - n + 3)^k(\mu - n - 1)^{k-1}(\mu - n)^{n-4k-1}(\mu^2 - (n+1)\mu + 4k)(\mu^2 - (2n-4)\mu + (n^2 - 4n - 1))^k.$$

So, the color Laplacian spectrum of $K_{d_n}(k)$ is $\{n-3$ (k times), $n+1$ ($k-1$ times), n ($n-4k-1$ times), $n-2+\sqrt{5}$ (k times), $n-2-\sqrt{5}$ (k times), $\dfrac{n+1+\sqrt{(n+1)^2-16k}}{2}, \dfrac{n+1-\sqrt{(n+1)^2-16k}}{2}\Big\}$.

Average degree of $K_{d_n}(k)$ is $\dfrac{n^2-n-6k}{n}$. Hence,

$$\begin{aligned} LE_c(K_{d_n}(k)) &= k\left|n-3-\frac{n^2-n-6k}{n}\right| + (k-1)\left|n+1-\frac{n^2-n-6k}{n}\right| + \\ &\quad (n-4k-1)\left|n-\frac{n^2-n-6k}{n}\right| + \left|\frac{n+1\pm\sqrt{(n+1)^2-16k}}{2} - \frac{n^2-n-6k}{n}\right| + k\left|n-2\pm\sqrt{5}-\frac{n^2-n-6k}{n}\right| \\ &= \frac{1}{n}\left[2k((3+\sqrt{5})n-6) - 24k^2 + n(n-3+\sqrt{(n+1)^2-16k})\right]. \end{aligned}$$

□

3. Color Laplacian energy of bipartite cluster graphs

Definition 3.1. [14] *Let $K_{m,n}$ be the complete bipartite graph, $1 \le r \le m$, $1 \le s \le n$ and $m, n \ge 1$ be the integers. The bi-cluster graph $Kb_{m,n}(r,s)$ is obtained by deleting the edges of $K_{r,s}$ from $K_{m,n}$.*

Example 3.1. *Consider the bi-cluster graph. Let $c(v_1) = c(v_2) = c(v_3) = c(v_4) = 1$ and $c(v_5) = c(v_6) = c(v_7) = c(v_8) = 2$.*

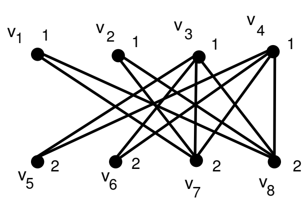

Figure 4. Graph $Kb_{4,4}(2,2)$

Theorem 3.1. *For $m, n \ge 1$ and $0 \le r \le m$, $0 \le s \le n$,*

$\phi(L_c(Kb_{m,n}(r,s),\mu)) = (\mu-n+s+1)^{r-1}(\mu-m+r+1)^{s-1}(\mu-n+1)^{m-r-1}(\mu-m+1)^{n-s-1}(\mu^4-(3(n+m)-r-s-4)\mu^3+(3n^2+(6m-9)n+3(m^2-3m+2)-(3n+2m-3)r-(2n+3m-4r-3)s)\mu^2-(n^3+(4m-6)n^2+(4m^2-12m+9)n+m^3-6m^2+9m-4-(3n^2+(3m-6)n+m^2-4m+3)r-(n^2+(3m-4)n+3m^2-6m+3-(4r(n+m-2))s)\mu+((m-1)n^3+(m^2-4m+3)n^2+(m^3-4m^2+6m-3)n-(m^3-3m^2+3m-1)-(n^3+(m-3)n^2+(m^2-3m+3)n-m^2+2m-1)r-((m-1)n^2+(m^2-3m+2)n+(m^3-3m^2+3m-1)-(n^2+(2m-4)n+(m^2-4m+4))r)s).$

Proof. Let $U=\{u_1,u_2,\ldots,u_r,u_{r+1},\ldots,u_m\}$ and $V=\{v_1,v_2,\ldots,v_s,v_{s+1},\ldots,v_n\}$ be the partites of complete bipartite graph $K_{m,n}$. The graph $Kb_{m,n}(r,s)$ is obtained by deleting the edges of $K_{r,s}$ from $K_{m,n}$. Note that, $\chi(Kb_{m,n}(r,s))=2$. Then, we have

$$L_c\left(K_{b_{m,n}}(r,s)\right)=\left[\begin{array}{c|c|c|c} J_r+(n-s-1)I_r & J_{r\times(m-r)} & 0_{r\times s} & -J_{r\times(n-s)} \\ \hline J_{(m-r)\times r} & J_r+(n-1)I_{(m-r)} & -J_{(m-r)\times s} & -J_{(m-r)\times(n-s)} \\ \hline 0_{s\times r} & -J_{s\times(m-r)} & J_r+(m-r-1)I_s & J_{s\times(n-s)} \\ \hline -J_{n-s\times r} & -J_{(n-s)\times(m-r)} & J_{(n-s)\times s} & (J+(m-1)I)_{(n-s)} \end{array}\right]$$

Consider $\det\left(\mu I-L_c\left(Kb_{m,n}(r,s)\right)\right)$.

Step 1: Replace R_i by $R_i^{'}=R_i-R_{i+1}$, for $i=1,2,\ldots,r-1,r+1,r+2,\ldots,m-1,1,2,\ldots,s-1,s+1,s+2,\ldots,n-1$. Then, $\det\left(\mu I-L_c\left(Kb_{m,n}(r,s)\right)\right)$ will reduce to a new determinant, say $(\mu-n+s+1)^{r-1}(\mu-m+r+1)^{s-1}(\mu-n+1)^{m-r-1}(\mu-m+1)^{n-s-1}\det(C)$.

Step 2: In $\det(C)$, replacing

$$C_i \text{ by } C_i^{'}=\begin{cases} C_i+C_{i-1}+\ldots+C_1, & \text{for i=r, r-1,\ldots,2} \\ C_i+C_{i-1}+\ldots+C_1, & \text{for i=s, s-1,\ldots,2,} \\ C_i+C_{i-1}+\ldots+C_{r+1}, & \text{for i=m, m-1,\ldots,r+2,} \\ C_i+C_{i-1}+\ldots+C_{s+1}, & \text{for i=n, n-1,\ldots,s+2.} \end{cases}$$

a new determinant say $\det(D)$ is obtained.

Step 3: On expanding $\det(D)$ along the rows R_i, for $i=1,2,\ldots,r-1,r+1,r+2,\ldots,m-1,1,2,\ldots,s-1,s+1,s+2,\ldots,n-1$, it reduces to

$$\det(D)=\begin{vmatrix} \mu-n+s-r+1 & r-m & 0 & n-s \\ -r & \mu-m-n+r+1 & s & n-s \\ 0 & m-r & \mu-m+r-s+1 & s-n \\ r & m-r & -s & \mu-m-n+s+1 \end{vmatrix}$$

$=(\mu^4-(3(n+m)-r-s-4)\mu^3+(3n^2+(6m-9)n+3(m^2-3m+2)-(3n+2m-3)r-(2n+3m-4r-3)s)\mu^2-(n^3+(4m-6)n^2+(4m^2-12m+9)n+m^3-6m^2+9m-4-(3n^2+(3m-6)n+m^2-4m+3)r-(n^2+(3m-4)n+3m^2-6m+3-(4r(n+m-2))s)\mu+((m-1)n^3+(m^2-4m+3)n^2+(m^3-4m^2+6m-3)n-(m^3-3m^2+3m-1)-(n^3+(m-3)n^2+(m^2-3m+3)n-m^2+2m-1)r-((m-1)n^2+(m^2-3m+2)n+(m^3-3m^2+3m-1)-(n^2+(2m-4)n+(m^2-4m+4))r)s).$

Thus,

$$\phi(L_c(Kb_{m,n}(r,s),\mu)) = (\mu-n+s+1)^{r-1}(\mu-m+r+1)^{s-1}(\mu-n+1)^{m-r-1}(\mu- m+1)^{n-s-1}(\mu^4-(3(n+m)-r-s-4)\mu^3+(3n^2+(6m-9)n+3(m^2-3m+2)- (3n+2m-3)r-(2n+3m-4r-3)s)\mu^2-(n^3+(4m-6)n^2+(4m^2-12m+9)n +m^3-6m^2+9m-4-(3n^2+(3m-6)n+m^2-4m+3)r-(n^2+(3m-4)n+ 3m^2-6m+3-(4r(n+m-2))s)\mu+((m-1)n^3+(m^2-4m+3)n^2+(m^3-4m^2 +6m-3)n-(m^3-3m^2+3m-1)-(n^3+(m-3)n^2+(m^2-3m+3)n-m^2+ 2m-1)r-((m-1)n^2+(m^2-3m+2)n+(m^3-3m^2+3m-1)-(n^2+(2m-4) n+(m^2-4m+4))r)s)$$

□

Definition 3.2. [14] *Let e_i, $i=1,2,\dots,k$, $1\le k\le \min\{m,n\}$, be independent edges of the complete bipartite graph $K_{m,n}$, $m,n\ge 1$. The graph $Ka_{m,n}(k)$ is obtained by deleting e_i, $i=1,2,\dots,k$ from $K_{m,n}$.*

Theorem 3.2. *For $m,n\ge 1$ and $0\le k\le \min\{m,n\}$*

$$\phi(L_c(Ka_{m,n}(k),\mu)) = (\mu-m+1)^{n-k-1}(\mu-n+1)^{m-k-1}(\mu^2-(n+m-4)\mu+ (n(m-2)-(2m-3)))^{k-1}(\mu^4-3(m+n-2)\mu^3+(3(n^2+m^2)+(6m-14)n- 14m+4k+12)\mu^2-(n^3+(4m-10)n^2+(4m^2-18m+19)n+4k(n+m-2) +m^3-10m^2-19m-10)\mu+((m-2)n^3+(m^2-6m+7)n^2+(m^3-6m^2+ 12m-8)n-(2m^3-7m^2+8m-3)+(n^2+(2m-4)n+(m^2-4m+4))k)).$$

Proof. Let $U=\{u_1,u_2,\dots,u_k,u_{k+1},\dots,u_m\}$ and $V=\{v_1,v_2,\dots,v_k,v_{k+1},\dots,v_n\}$ be the partites of complete bipartite graph $K_{m,n}$. The graph $Ka_{m,n}(k)$ is obtained by deleting independent edges e_i of the complete bipartite graph $K_{m,n}$, $i=1,2,\dots,k$. Observe that $\chi(Ka_{m,n}(k))=2$. Then, we have

$$L_c\left(Ka_{m,n}(k)\right)=\left[\begin{array}{c|c|c|c} J_k+(n-2)I_k & J_{k\times(m-k)} & (-J+I)_k & -J_{k\times(n-k)} \\ \hline J_{(m-k)\times k} & (J+(n-1)I)_{(m-k)} & -J_{(m-k)\times k} & -J_{(m-k)} \\ \hline (-J+I)_k & -J_{k\times(m-k)} & J_k+(m-2)I_k & J_{k\times(n-k)} \\ \hline -J_{(n-k)\times k} & -J_{(n-k)\times(m-k)} & J_{(n-k)\times k} & (J+(m-1)I)_{(n-k)} \end{array}\right]$$

Consider $\det\left(\mu I-L_c\left(Ka_{m,n}(k)\right)\right)$.

Step 1: Replace R_i by $R_i'=R_i-R_{i+1}$, for $i=u_1,u_2,\dots,u_{k-1},u_{k+1},u_{k+2},\dots,u_{m-1}$, $v_1,v_2,\dots,v_{k-1},v_{k+1},v_{k+2},\dots,v_{n-1}$. Then, $\det\left(\mu I-L_c\left(Ka_{m,n}(k)\right)\right)$ will reduce to a new determinant, say $(\mu-n+1)^{m-k-1}(\mu-m+1)^{n-k-1}\det(C)$.

Step 2: In $\det(C)$, replacing

$$C_i \text{ by } C_i' = \begin{cases} C_i+C_{i-1}+\ldots+C_1, & \text{for } i=u_k,u_{k-1},\dots,u_2,\\ C_i+C_{i-1}+\ldots+C_1, & \text{for } i=v_k,u_{k-1},\dots,v_2,\\ C_i+C_{i-1}+\ldots+C_{k+1}, & \text{for } i=u_m,u_{m-1},\dots,u_{k+2},\\ C_i+C_{i-1}+\ldots+C_{k+1}, & \text{for } i=v_n,u_{n-1},\dots,v_{k+2}.\end{cases}$$

a new determinant say $\det(D)$ is obtained.

Step 3: In $\det(D)$, replacing C_i by $C_i'=(\mu-n+2)C_i+C_j$, for $i=v_1,v_2,\dots,v_{k-1}$ and $j=u_1,u_2,\dots,u_{k-1}$, it simplifies to a determinant say $\det(E)$.

Step 4: On expanding $\det(E)$ along the rows R_i, for $i=u_1,u_2,\dots,u_{k-1},u_{k+1}$, $u_{k+2},\dots,u_{m-1},v_1,v_2,\dots,v_{k-1},v_{k+1},v_{k+2},\dots,v_{n-1}$, it reduces to

$$\det(F) = (\mu^2 - (n+m-4)\mu + (n(m-2) - (2m-3)))^{k-1}$$
$$\begin{vmatrix} \mu-n-k+2 & k-m & k-1 & n-k \\ -k & \mu-n-m+k+1 & k & n-k \\ k-1 & m-k & \mu-m-k+2 & k-n \\ k & m-k & -k & \mu-m-n+k+1 \end{vmatrix}$$
$$= (\mu^2-(n+m-4)\mu+(n(m-2)-(2m-3)))^{k-1}(\mu^4-3(m+n-2)\mu^3+ (3(n^2+m^2)+(6m-14)n-14m+4k+12)\mu^2-(n^3+(4m-10)n^2+(4m^2-18m+ 19)n+4k(n+m-2)+m^3-10m^2-19m-10)\mu+((m-2)n^3+(m^2-6m+7)n^2+ (m^3-6m^2+12m-8)n-(2m^3-7m^2+8m-3)+(n^2+(2m-4)n+(m^2-4m+4))k)).$$

Thus,

$$\phi(L_c(Ka_{m,n}(k),\mu)) = (\mu-m+1)^{n-k-1}(\mu-n+1)^{m-k-1}(\mu^2-(n+m-4)\mu+ (n(m-2)-(2m-3)))^{k-1}(\mu^4-3(m+n-2)\mu^3+(3(n^2+m^2)+(6m-14)n- 14m+4k+12)\mu^2-(n^3+(4m-10)n^2+(4m^2-18m+19)n+4k(n+m-2)+ m^3-10m^2-19m-10)\mu+((m-2)n^3+(m^2-6m+7)n^2+(m^3-6m^2+12m -8)n-(2m^3-7m^2+8m-3)+(n^2+(2m-4)n+(m^2-4m+4))k)).$$

□

4. Conclusion

The energy, color energy and color Laplacian energy of a graph are the emerging concepts within graph theory. In this paper we have evaluated color Laplacian energy of Cluster and bi-cluster graphs. The color Laplacian spectrum of cluster and bi-cluster graphs is expressed in terms of its parameters.

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