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APPROXIMATE FIXED POINT PROPERTY IN IFNS

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Abstract. In this paper, we define concept of approximate fixed point property of a function and a set in intuitionistic fuzzy normed space. Furthermore, we give intuitionistic fuzzy version of some class of maps used in fixed point theory and investigate approximate fixed point property of these maps.

Keywords: Approximate fixed point property, intuitionistic fuzzy normed space, classes of contractions

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1. introduction and preliminaries

Fuzzy theory was introduced by Zadeh [23] and was generalized by Atanassov [3] as intuitionistic fuzzy theory. This theory is used in many branches of science. Using idea of intuitionistic fuzzy, Park [18] defined intuitionistic fuzzy metric space, later Saadati and Park [20] introduced intuitionistic fuzzy normed space. Intuitionistic fuzzy analogous concepts used in functional analysis were studied via intuitionistic fuzzy metric and norm ([11], [16],[13], [15], [8],[21], [14]). Fixed point theory is one of fields studied in intuitionistic fuzzy version. Some of works related to intuitionistic fuzzy fixed point theory can be found in [1], [19], [4], [10], [9].

On the other hand, there are many problems which can be solved with fixed point theory. But in most cases, it is enough finding an approximate solution. So, the existence of fixed point may not be necessary for solution of a problem. A reason of being attractive of this approximate approach is addition of strong conditions for the exact solution of problem. To find approximate solution of problem may be easier putting less requirement. Hence, it is natural to define approximate fixed point of a function and to produce theory

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related to this concept. It is meant that x is close to $f(x)$ with approximate fixed point of $f(x)$. There are several studies related to this concept([6], [17], [2], [5], [12], [7]).

In this study, we define and study the concept of approximate fixed point property of a function and a set which is used fixed point theory in intuitionistic fuzzy normed space by inspiring studies of Berinde [5] and Anoop [2]. We give examples related to this concept in intuitionistic fuzzy normed space.

Firstly, we mention some concepts used in our article.

Definition 1.1 (see [22]). *A binary operation $*:[0,1]\times[0,1]$ is a continuous t-norm if it satisfies the following conditions: (i) $*$ is associative and commutative; (ii) $*$ is continuous; (iii) $a*1=a$ for all $a\in[0,1]$; (iv) $a*b\le c*d$ whenever $a\le c$ and $b\le d$ for each $a,b,c,d\in[0,1]$.*

Definition 1.2 (see [22]). *A binary operation $\Diamond:[0,1]\times[0,1]$ is a continuous t-conorm if it satisfies the following conditions: (i) $\Diamond$ is associative and commutative; (ii) $\Diamond$ is continuous; (iii) $a\Diamond 0=a$ for all $a\in[0,1]$; (iv) $a\Diamond b\le c\Diamond d$ whenever $a\le c$ and $b\le d$ for each $a,b,c,d\in[0,1]$.*

Definition 1.3 (see [20]). *Let $*$ be a continuous t-norm, $\Diamond$ be a continuous t-conorm and X be a linear space over the field IF($\mathbb{R}$ or $\mathbb{C}$). If μ and ν are fuzzy sets on $X\times(0,\infty)$ satisfying the following conditions, the five-tuple $(X,\mu,\nu,*,\Diamond)$ is said to be an intuitionistic fuzzy normed space and (μ,ν) is called an intuitionistic fuzzy norm on X. For every $x,y\in X$ and $s,t>0$,*

(i) $\mu(x,t)+\nu(x,t)\le 1$,
(ii) $\mu(x,t)>0$,
(iii) $\mu(x,t)=1 \Longleftrightarrow x=0$,
(iv) $\mu(ax,t)=\mu\left(x,\frac{t}{|a|}\right)$ *for each* $a\neq 0$,
(v) $\mu(x,t)*\mu(y,s)\le\mu(x+y,t+s)$,
(vi) $\mu(x,.):(0,\infty)\to[0,1]$ *is continuous,*
(vii) $\lim\limits_{t\to\infty}\mu(x,t)=1$ *and* $\lim\limits_{t\to 0}\mu(x,t)=0$,
(viii) $\nu(x,t)<1$,
(ix) $\nu(x,t)=0\Longleftrightarrow x=0$,
(x) $\nu(ax,t)=\nu\left(x,\frac{t}{|a|}\right)$ *for each* $a\neq 0$,
(xi) $\nu(x,t)\Diamond\nu(y,s)\ge\nu(x+y,t+s)$,
(xii) $\nu(x,.):(0,\infty)\to[0,1]$ *is continuous,*
(xiii) $\lim\limits_{t\to\infty}\nu(x,t)=0$ *and* $\lim\limits_{t\to 0}\nu(x,t)=1$,

we further assume that $(X,\mu,\nu,*,\Diamond)$ satisfies the following axiom:

(xiv) $\left.\begin{array}{c} a\Diamond a=a \\ a*a=a \end{array}\right\}$ *for all* $a\in[0,1]$.

We use IFNS instead of intuitionistic fuzzy normed space for the sake of abbreviation.

Lemma 1.1 (see [20]). *Let (μ,ν) be intuitionistic fuzzy norm on X. The followings hold:*

(i) $\mu(x,.)$ *and* $\nu(x,.)$ *are nondecreasing and nonincreasing for all* $x\in X$*, respectively.*
(ii) $\mu(x-y,t)=\mu(y-x,t)$ *and* $\nu(x-y,t)=\nu(y-x,t)$ *for any* $t>0$.

Definition 1.4 (see [20]). *A sequence (x_k) in $(X,\mu,\nu,*,\Diamond)$ converges to x if and only if*

$$\mu(x_k-x,t)\to 1 \text{ and } \nu(x_k-x,t)\to 0$$

as $k\to\infty$, for each $t>0$. We denote the convergence of (x_k) to x by $x_k\overset{(\mu,\nu)}{\to}x$.

Definition 1.5 (see [20]). *Let $(X, \mu, \nu, *, \Diamond)$ be an IFNS. $(X, \mu, \nu, *, \Diamond)$ is said to be complete if every Cauchy sequence in $(X, \mu, \nu, *, \Diamond)$ is convergent.*

Definition 1.6 (see [16]). *Let X and Y be two IFNSs. $f : X \to Y$ is continuous at $x_0 \in X$ if $(f(x_k))$ in Y convergences to $f(x_0)$ for any (x_k) in X converging to x_0. If $f : X \to Y$ is continuous at each element of X, then $f : X \to Y$ is said to be continuous on X.*

Definition 1.7 (see [13]). *Let $(X, \mu, \nu, *, \Diamond)$ be an IFNS. $A \subset X$ is dense in $(X, \mu, \nu, *, \Diamond)$ if there exists a sequence (x_k) in A such that $x_k \overset{(\mu,\nu)}{\to} x$ for all $x \in X$.*

Definition 1.8 (see [1]). *$(X, \mu, \nu, *, \Diamond)$ be an intuitionistic fuzzy metric space. We call the mapping $f : X \to X$ intuitionistic fuzzy contraction map, if there exists $a \in (0, 1)$ such that*

$$\mu(f(x), f(y), at) \geq \mu(x, y, t) \text{ and } \nu(f(x), f(y), at) \leq \nu(x, y, t)$$

for all $x, y \in X$ and $t > 0$.

Definition 1.9 (see [4]). *$(X, \mu, \nu, *, \Diamond)$ be an intuitionistic fuzzy metric space. We call the mapping $f : X \to X$ intuitionistic fuzzy nonexpansive, if*

$$\mu(f(x), f(y), t) \geq \mu(x, y, t) \text{ and } \nu(f(x), f(y), t) \leq \mu(x, y, t)$$

for all $x, y \in X$ and $t > 0$.

Lemma 1.2 (see [1]). *Let $(X, \mu, \nu, *, \Diamond)$ be an intuitionistic fuzzy metric space.*

(i) If $x_k \overset{(\mu,\nu)}{\to} x$ and $y_k \overset{(\mu,\nu)}{\to} y$

$$\mu(x, y, t) \leq \lim_{k\to\infty} \inf \mu(x_k, y_k, t) \text{ and } \nu(x, y, t) \geq \lim_{k\to\infty} \sup \nu(x_k, y_k, t)$$

for all $t > 0$.

(ii) If $x_k \overset{(\mu,\nu)}{\to} x$ and $y_k \overset{(\mu,\nu)}{\to} y$

$$\mu(x, y, t) \geq \lim_{k\to\infty} \sup \mu(x_k, y_k, t) \text{ and } \nu(x, y, t) \leq \lim_{k\to\infty} \inf \nu(x_k, y_k, t)$$

for all $t > 0$.

2. Main Results

Firstly, we define approximate fixed point property, diameter of a set in intitionistic fuzzy normed spaces and give examples.

Definition 2.1. *Let $(X, \mu, \nu, *, \Diamond)$ be an IFNS and $f : X \to X$ be a function. Given $\epsilon > 0$. It is said that $x_0 \in X$ is an intuitionistic fuzzy $\epsilon-$fixed point or approximate fixed point (IFAFP) of f if*

$$\mu(f(x_0) - x_0, t) > 1 - \epsilon \text{ and } \nu(f(x_0) - x_0, t) < \epsilon$$

for all $t > 0$. We denote the set of intuitionistic fuzzy $\epsilon-$fixed points of f with $F_\epsilon^{(\mu,\nu)}(f)$.

Definition 2.2. *It is said that f has the intuitionistic fuzzy approximate fixed point property (IFAFPP) if $F_\epsilon^{(\mu,\nu)}(f)$ is not empty for every $\epsilon > 0$.*

Example 2.1. *Consider $f(x) = x^2$ defined on $(0, 1)$. $((0, 1), \mu, \nu, *, \Diamond)$ is an intuitionistic fuzzy normed space with*

$$\mu(x, t) = \frac{t}{t + |x|} \text{ and } \nu(x, t) = \frac{|x|}{t + |x|}$$

where $|.|$ *is usual norm on* $(0,1)$, $a*b = a.b$ *and* $a\Diamond b = \min\{a+b,1\}$ *for all* $a,b \in [a,b]$. *As known,* f *has not any fixed point on* $(0,1)$. *We investigate intuitionistic fuzzy approximate fixed point of* f. *For every* $\epsilon > 0$ *and* $t > 0$, *there exists* $x \in (0,1)$ *such that* x *satisfies*

$$\mu(f(x)-x,t) = \frac{t}{t+|f(x)-x|} > 1-\epsilon \text{ and } \nu(f(x)-x,t) = \frac{|f(x)-x|}{t+|f(x)-x|}$$

that is

$$\left|x^2-x\right| < \frac{\epsilon t}{1-\epsilon}.$$

So, f *has the intuitionistic fuzzy approximate fixed point property, since* $F_\epsilon^{(\mu,\nu)}(f)$ *is not empty for every* $\epsilon > 0$.

Definition 2.3. *Let* K *be nonempty subset of* $(X,\mu,\nu,*,\Diamond)$. *We say that* $(\delta_\mu(K),\delta_\nu(K))$ *is intuitionistic fuzzy diameter of* K *with respect to* t *where*

$$\delta_\mu(K) = \inf\{\mu(x-y,t) : x,y \in K\} \text{ and } \delta_\nu(K) = \sup\{\nu(x-y,t) : x,y \in K\}$$

for $t > 0$.

Theorem 2.1. *Let* X *be a intuitionistic fuzzy normed space, and* $f : X \to X$ *be a function. We suppose that*

(i) $F_\epsilon^{(\mu,\nu)}(f) \neq \varnothing$,

(ii) *There exist* $\vartheta(\epsilon_1), \vartheta(\epsilon_2)$ *such that*

$$\begin{aligned} \mu(x-y,t) &\geq \epsilon_1 * \mu(f(x)-f(y),t_1) \Rightarrow \mu(x-y,t) \geq \vartheta(\epsilon_1), \forall x,y \in F_\epsilon^{(\mu,\nu)}(f) \\ \nu(x-y,t) &\leq \epsilon_2 \Diamond \nu(f(x)-f(y),t_2) \Rightarrow \nu(x-y,t) \leq \vartheta(\epsilon_2), \forall x,y \in F_\epsilon^{(\mu,\nu)}(f) \end{aligned}$$

for $x,y \in F_\epsilon^{(\mu,\nu)}(f)$ *and* $\epsilon_1,\epsilon_2 \in (0,1)$.

Then $\left(\delta_\mu\left(F_\epsilon^{(\mu,\nu)}(f)\right), \delta_\nu\left(F_\epsilon^{(\mu,\nu)}(f)\right)\right) = (\vartheta(1-\epsilon), \vartheta(\epsilon))$.

Proof. Let $\varepsilon > 0$ and $x,y \in F_\epsilon^{(\mu,\nu)}(f)$. Then

$$\mu(f(x)-x,t) > 1-\epsilon \text{ and } \nu(f(x)-x,t) < \epsilon$$

and

$$\mu(f(y)-y,t) > 1-\epsilon \text{ and } \nu(f(y)-y,t) < \epsilon.$$

It can be written

$$\begin{aligned} \mu(x-y,t) &\geq \mu\left(f(x)-x,\frac{t}{3}\right) * \mu\left(f(x)-f(y),\frac{t}{3}\right) * \mu\left(f(y)-y,\frac{t}{3}\right) \\ &\geq (1-\epsilon)*(1-\epsilon)*\mu\left(f(x)-f(y),\frac{t}{3}\right) \\ &= (1-\epsilon)*\mu\left(f(x)-f(y),\frac{t}{3}\right) \end{aligned}$$

and

$$\begin{aligned} \nu(x-y,t) &\leq \nu\left(f(x)-x,\frac{t}{3}\right) \Diamond \nu\left(f(x)-f(y),\frac{t}{3}\right) \Diamond \nu\left(f(y)-y,\frac{t}{3}\right) \\ &\leq \epsilon \Diamond \epsilon \Diamond \nu\left(f(x)-f(y),\frac{t}{3}\right) \\ &= \epsilon \Diamond \nu\left(f(x)-f(y),\frac{t}{3}\right). \end{aligned}$$

By (ii), for every $x, y \in F_{\epsilon}^{(\mu,\nu)}(f)$, we get

$$\mu(x-y,t) \geq \vartheta(1-\epsilon) \text{ and } \nu(x-y,t) \leq \vartheta(\epsilon).$$

Hence,

$$\left(\delta_{\mu}\left(F_{\epsilon}^{(\mu,\nu)}(f)\right), \delta_{\nu}\left(F_{\epsilon}^{(\mu,\nu)}(f)\right)\right) = (\vartheta(1-\epsilon), \vartheta(\epsilon)).$$

□

Now, we introduce intuitionistic fuzzy asymptotic regularity to investigate intuitionistic fuzzy approximate fixed point property of some operators.

Definition 2.4. *Let $(X, \mu, \nu, *, \Diamond)$ be an IFNS and $f : X \to X$ be a function. It is said that f is intuitionistic fuzzy asymptotic regular if*

$$\lim_{k\to\infty} \mu\left(f^{k+1}(x) - f^{k}(x), t\right) = 1 \text{ and } \lim_{k\to\infty} \nu\left(f^{k+1}(x) - f^{k}(x), t\right) = 0$$

for every $x \in X$ and $t > 0$.

Theorem 2.2. *Let $(X, \mu, \nu, *, \Diamond)$ be an IFNS and $f : X \to X$ be a function. If f has intuitionistic fuzzy asymptotic regular, then f has IFAFPP.*

Proof. Let x_0 be arbitrary element of X. Since f is intuitionistic fuzzy asymptotic regular,

$$\lim_{k\to\infty} \mu\left(f^{k+1}(x_0) - f^{k}(x_0), t\right) = 1 \text{ and } \lim_{k\to\infty} \nu\left(f^{k+1}(x_0) - f^{k}(x_0), t\right) = 0.$$

In this case, for every $\epsilon > 0$ there exists $k_0(\epsilon, t) \in \mathbb{N}$ such that

$$\mu\left(f^{k+1}(x_0) - f^{k}(x_0), t\right) > 1-\epsilon \text{ and } \nu\left(f^{k+1}(x_0) - f^{k}(x_0), t\right) < \epsilon$$

for every $k \geq k_0(\epsilon, t)$. If we denote $f^{k}(x_0)$ by y_0, we have

$$\mu\left(f^{k+1}(x_0) - f^{k}(x_0), t\right) = \mu\left(f\left(f^{k}(x_0)\right) - f^{k}(x_0), t\right) = \mu(f(y_0)) - y_0, t) > 1-\epsilon$$

and

$$\nu\left(f^{k+1}(x_0) - f^{k}(x_0), t\right) = \nu\left(f\left(f^{k}(x_0)\right) - f^{k}(x_0), t\right) = \nu(f(y_0)) - y_0, t) < \epsilon.$$

This shows that y_0 is intuitionistic fuzzy approximate fixed point of f. □

Now, we introduce intuitionistic fuzzy analogous of mappings such as Kannan, Chatterjea, Zamfrescu and weak contraction and investigate that these maps have approximate fixed point under certain conditions. Firstly, we show that intuitionistic fuzzy contraction map has approximate fixed point property.

Theorem 2.3. *Let $(X, \mu, \nu, *, \Diamond)$ be an IFNS, and $f : X \to X$ be a intuiuionistic fuzzy contraction. Then $F_{\epsilon}^{(\mu,\nu)}(f) \neq \varnothing$ for every $\epsilon \in (0,1)$.*

Proof. Let $x \in X$ and $\epsilon \in (0,1)$, $t > 0$.

$$\begin{aligned}
\mu\left(f^{k}(x) - f^{k+1}(x), t\right) &= \mu\left(f\left(f^{k-1}(x)\right) - f\left(f^{k}(x)\right), t\right) \\
&\geq \mu\left(f^{k-1}(x) - f^{k}(x), \frac{t}{a}\right) \\
&\geq \mu\left(f^{k-2}(x) - f^{k-1}(x), \frac{t}{a^2}\right) \\
&\geq \ldots \\
&\geq \mu\left(x - f(x), \frac{t}{a^k}\right)
\end{aligned}$$

and

$$\begin{aligned}\nu\left(f^k(x)-f^{k+1}(x),t\right) &= \nu\left(f\left(f^{k-1}(x)\right)-f\left(f^k(x)\right),t\right)\\ &\leq \nu\left(f^{k-1}(x)-f^k(x),\frac{t}{a}\right)\\ &\leq \nu\left(f^{k-2}(x)-f^{k-1}(x),\frac{t}{a^2}\right)\\ &\leq ...\\ &\leq \nu\left(x-f(x),\frac{t}{a^k}\right)\end{aligned}$$

For $a\in(0,1), k\to\infty\Rightarrow\frac{t}{a^k}\to\infty$, by properties (vii) and (xiii) of intuitionistic fuzzy norm

$$\begin{aligned}\mu\left(f^k(x)-f^{k+1}(x),t\right) &\to 1\\ \nu\left(f^{k+1}(x)-f^k(x),t\right) &\to 0.\end{aligned}$$

By Theorem 2.2, it follows that $F_\epsilon^{(\mu,\nu)}(f)\neq\varnothing$ for every $\epsilon\in(0,1)$. □

Example 2.2. *The open interval* $(0,1)$ *is an intuitionistic fuzzy normed space with intuitionistic fuzzy norm, t-norm and t-conorm given in Example 2.1. Consider* $f:(0,1)\to(0,1)$ *given by* $f(x)=\frac{1}{2}x$. *This map has not any fixed point in* $(0,1)$. *Furthermore,* f *is an intuitionistic fuzzy contraction map since*

$$\begin{aligned}\mu\left(f(x)-f(y),\frac{t}{2}\right) &= \frac{\frac{t}{2}}{\frac{t}{2}+\left|\frac{x}{2}-\frac{y}{2}\right|}=\mu(x-y,t),\\ \nu\left(f(x)-f(y),\frac{t}{2}\right) &= \frac{\left|\frac{x}{2}-\frac{y}{2}\right|}{\frac{t}{2}+\left|\frac{x}{2}-\frac{y}{2}\right|}=\nu(x-y,t)\end{aligned}$$

for every $x,y\in(0,1)$ *and* $t>0$. *We write* $\frac{1}{2}x<\frac{\epsilon t}{1-\epsilon}$ *from*

$$\begin{aligned}\mu(x-f(x),t) &= \mu\left(x-\frac{x}{2},t\right)\\ &= \mu\left(\frac{x}{2},t\right)=\frac{t}{t+\frac{x}{2}}>1-\epsilon\end{aligned}$$

and

$$\begin{aligned}\nu(x-f(x),t) &= \nu\left(x-\frac{x}{2},t\right)\\ &= \nu\left(\frac{x}{2},t\right)=\frac{\frac{x}{2}}{t+\frac{x}{2}}<\epsilon.\end{aligned}$$

For every $\epsilon\in(0,1)$ *and* $t>0$ *there exists* $x\in(0,1)$ *such that* $\frac{1}{2}x<\frac{\epsilon t}{1-\epsilon}$. *So,* f *has intuitionistic fuzzy approximate fixed point property.*

Definition 2.5. *Let* X *be an intuitionistic fuzzy normed space. If there exists* $a\in\left(0,\frac{1}{2}\right)$ *such that*

$$\begin{aligned}\mu(f(x)-f(y),at) &\geq \mu(x-f(x),t)*\mu(y-f(y),t)\\ \nu(f(x)-f(y),at) &\leq \nu(x-f(x),t)\lozenge\nu(y-f(y),t)\end{aligned}$$

for every $x,y\in X$ *and* $t>0$, *then* $f:X\to X$ *is called intuitionistic fuzzy Kannan operator.*

Theorem 2.4. *Let $(X, \mu, \nu, *, \Diamond)$ be an IFNS havig partial order relation denoted by $\preceq$, where $a * b = \min\{a, b\}$ and $a \Diamond b = \max\{a, b\}$, and $f : X \to X$ be an intuitionistic fuzzy Kannan operator satisfying $x \preceq f(x)$ for every $x \in X$. Assume that $\preceq \subset X \times X$ holds one of the following conditions:*

(i) $\preceq$ is subvector space.
(ii) X is a totally ordered space.

If $\mu(., t)$ is non-decreasing, $\nu(., t)$ is non-increasing for all $t \in (0, \infty)$, $x \succeq \theta$ (θ is unit element in vector space X), then $F_\epsilon^{(\mu,\nu)}(f) \neq \varnothing$ for every $\epsilon \in (0, 1)$.

Proof. Let $x \in X$ and $\epsilon \in (0, 1)$, $t > 0$. We can write from $x \preceq f(x)$ for every $x \in X$,

$$x \preceq f(x) \preceq f^2(x) \preceq f^3(x) \preceq ... \preceq f^k(x) \preceq \dots.$$

Considering assumptions, we have

$$\begin{aligned}
\mu\left(f^{k+1}(x) - f^k(x), t\right) &= \mu(f\left(f^k(x)\right) - f\left(f^{k-1}(x)\right), t) \\
&\geq \mu(f^k(x) - f^{k+1}(x), \frac{t}{a}) * \mu(f^{k-1}(x) - f^k(x), \frac{t}{a}) \\
&\geq \mu(f^k(x) - f^{k-1}(x), \frac{t}{2a}) * \mu(f^{k-1}(x) - f^{k+1}(x), \frac{t}{2a}) \\
&\quad * \mu(f^{k-1}(x) - f^k(x), \frac{t}{a})
\end{aligned}$$

$$\begin{aligned}
&\geq \mu(f^k(x) - f^{k-1}(x), \frac{t}{2a}) * \mu(f^{k-1}(x) - f^{k+1}(x), \frac{t}{2a}) \\
&\quad * \mu(f^{k-1}(x) - f^k(x), \frac{t}{2a}) \\
&= \mu(f^k(x) - f^{k-1}(x), \frac{t}{2a}) * \mu(f^{k-1}(x) - f^{k+1}(x), \frac{t}{2a})
\end{aligned}$$

$$\begin{aligned}
&= \min\left\{\mu(f^k(x) - f^{k-1}(x), \frac{t}{2a}), \mu(f^{k+1}(x) - f^{k-1}(x), \frac{t}{2a})\right\} \\
&= \mu(f^k(x) - f^{k-1}(x), \frac{t}{2a})
\end{aligned}$$

$$\begin{aligned}
&\geq \mu\left(f^{k-1}(x) - f^k(x), \frac{t}{2a^2}\right) * \mu\left(f^{k-2}(x) - f^{k-1}(x), \frac{t}{2a^2}\right) \\
&\geq \mu\left(f^{k-1}(x) - f^{k-2}(x), \frac{t}{4a^2}\right) \\
&\quad * \mu\left(f^{k-2}(x) - f^k(x), \frac{t}{4a^2}\right) * \mu\left(f^{k-2}(x) - f^{k-1}(x), \frac{t}{2a^2}\right)
\end{aligned}$$

$$\begin{aligned}
&\geq \mu\left(f^{k-1}(x) - f^{k-2}(x), \frac{t}{4a^2}\right) \\
&\quad * \mu\left(f^{k-2}(x) - f^k(x), \frac{t}{4a^2}\right) * \mu\left(f^{k-2}(x) - f^{k-1}(x), \frac{t}{4a^2}\right) \\
&= \mu\left(f^{k-1}(x) - f^{k-2}(x), \frac{t}{4a^2}\right) * \mu\left(f^{k-2}(x) - f^k(x), \frac{t}{4a^2}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \min\left\{\mu\left(f^{k-1}(x)-f^{k-2}(x),\frac{t}{4a^2}\right),\mu\left(f^{k}(x)-f^{k-2}(x),\frac{t}{4a^2}\right)\right\}\\
&= \mu\left(f^{k-1}(x)-f^{k-2}(x),\frac{t}{4a^2}\right)\\
&\vdots\\
&\geq \mu\left(f^{k-(k-2)}(x)-f^{k-(k-1)}(x),\frac{t}{2^{k-1}a^{k-1}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \mu\left(f^{2}(x)-f(x),\frac{t}{2^{k-1}a^{k-1}}\right)\\
&\geq \mu\left(f^{2}(x)-f(x),\frac{t}{2^{k-1}a^{k}}\right)*\mu\left(x-f(x),\frac{t}{2^{k-1}a^{k}}\right)\\
&\geq \mu\left(f^{2}(x)-x,\frac{t}{2^{k}a^{k}}\right)*\mu\left(x-f(x),\frac{t}{2^{k}a^{k}}\right)*\mu\left(x-f(x),\frac{t}{2^{k-1}a^{k}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\geq \mu\left(f^{2}(x)-x,\frac{t}{2^{k}a^{k}}\right)*\mu\left(x-f(x),\frac{t}{2^{k}a^{k}}\right)*\mu\left(x-f(x),\frac{t}{2^{k}a^{k}}\right)\\
&= \mu\left(f^{2}(x)-x,\frac{t}{2^{k}a^{k}}\right)*\mu\left(x-f(x),\frac{t}{2^{k}a^{k}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \min\left\{\mu\left(x-f^{2}(x),\frac{t}{2^{k}a^{k}}\right),\mu\left(x-f(x),\frac{t}{2^{k}a^{k}}\right)\right\}\\
&= \mu\left(x-f(x),\frac{t}{2^{k}a^{k}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\nu\left(f^{k+1}(x)-f^{k}(x),t\right) &= \nu(f\left(f^{k}(x)\right)-f\left(f^{k-1}(x)\right),t)\\
&\leq \nu(f^{k}(x)-f^{k+1}(x),\frac{t}{a})\Diamond\nu(f^{k-1}(x)-f^{k}(x),\frac{t}{a})\\
&\leq \nu(f^{k}(x)-f^{k-1}(x),\frac{t}{2a})\Diamond\nu(f^{k-1}(x)-f^{k+1}(x),\frac{t}{2a})\\
&\quad\Diamond\nu(f^{k-1}(x)-f^{k}(x),\frac{t}{a})
\end{aligned}
$$

$$
\begin{aligned}
&\leq \nu(f^{k}(x)-f^{k-1}(x),\frac{t}{2a})\Diamond\nu(f^{k-1}(x)-f^{k+1}(x),\frac{t}{2a})\\
&\quad\Diamond\nu(f^{k-1}(x)-f^{k}(x),\frac{t}{2a})\\
&= \nu(f^{k}(x)-f^{k-1}(x),\frac{t}{2a})\Diamond\nu(f^{k-1}(x)-f^{k+1}(x),\frac{t}{2a})
\end{aligned}
$$

$$
\begin{aligned}
&= \max\left\{\nu(f^{k}(x)-f^{k-1}(x),\frac{t}{2a}),\nu(f^{k+1}(x)-f^{k-1}(x),\frac{t}{2a})\right\}\\
&= \nu(f^{k}(x)-f^{k-1}(x),\frac{t}{2a})\\
&\leq \nu\left(f^{k-1}(x)-f^{k}(x),\frac{t}{2a^2}\right)\Diamond\nu\left(f^{k-2}(x)-f^{k-1}(x),\frac{t}{2a^2}\right)
\end{aligned}
$$

$$\begin{aligned}
&\leq \nu\left(f^{k-1}(x)-f^{k-2}(x),\frac{t}{4a^2}\right)\\
&\quad\Diamond\nu\left(f^{k-2}(x)-f^{k}(x),\frac{t}{4a^2}\right)\Diamond\nu\left(f^{k-2}(x)-f^{k-1}(x),\frac{t}{2a^2}\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \nu\left(f^{k-1}(x)-f^{k-2}(x),\frac{t}{4a^2}\right)\\
&\quad\Diamond\nu\left(f^{k-2}(x)-f^{k}(x),\frac{t}{4a^2}\right)\Diamond\nu\left(f^{k-2}(x)-f^{k-1}(x),\frac{t}{4a^2}\right)\\
&= \nu\left(f^{k-1}(x)-f^{k-2}(x),\frac{t}{4a^2}\right)\Diamond\nu\left(f^{k-2}(x)-f^{k}(x),\frac{t}{4a^2}\right)
\end{aligned}$$

$$\begin{aligned}
&= \max\left\{\nu\left(f^{k-1}(x)-f^{k-2}(x),\frac{t}{4a^2}\right),\nu\left(f^{k}(x)-f^{k-2}(x),\frac{t}{4a^2}\right)\right\}\\
&= \nu\left(f^{k-1}(x)-f^{k-2}(x),\frac{t}{4a^2}\right)\\
&\ \ \vdots\\
&\leq \nu\left(f^{k-(k-2)}(x)-f^{k-(k-1)}(x),\frac{t}{2^{k-1}a^{k-1}}\right)
\end{aligned}$$

$$\begin{aligned}
&= \nu\left(f^{2}(x)-f(x),\frac{t}{2^{k-1}a^{k-1}}\right)\\
&\leq \nu\left(f^{2}(x)-f(x),\frac{t}{2^{k-1}a^{k}}\right)\Diamond\nu\left(x-f(x),\frac{t}{2^{k-1}a^{k}}\right)\\
&\leq \nu\left(f^{2}(x)-x,\frac{t}{2^{k}a^{k}}\right)\Diamond\nu\left(x-f(x),\frac{t}{2^{k}a^{k}}\right)\\
&\quad\Diamond\nu\left(x-f(x),\frac{t}{2^{k-1}a^{k}}\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \nu\left(f^{2}(x)-x,\frac{t}{2^{k}a^{k}}\right)\Diamond\nu\left(x-f(x),\frac{t}{2^{k}a^{k}}\right)\\
&\quad\Diamond\nu\left(x-f(x),\frac{t}{2^{k}a^{k}}\right)\\
&= \nu\left(f^{2}(x)-x,\frac{t}{2^{k}a^{k}}\right)\Diamond\nu\left(x-f(x),\frac{t}{2^{k}a^{k}}\right)
\end{aligned}$$

$$\begin{aligned}
&= \max\left\{\nu\left(x-f^{2}(x),\frac{t}{2^{k}a^{k}}\right),\nu\left(x-f(x),\frac{t}{2^{k}a^{k}}\right)\right\}\\
&= \nu\left(x-f(x),\frac{t}{2^{k}a^{k}}\right)
\end{aligned}$$

Now, if we take limit for $k \to \infty$, $\frac{t}{(2a)^k}$ tends to infinity for $a \in \left(0, \frac{1}{2}\right)$. Using properties (vii) and (xiii) of intuitionistic fuzzy norm,

$$\begin{aligned}\lim_{k\to\infty} \mu\left(f^k(x) - f^{k+1}(x), t\right) &\geq \lim_{k\to\infty} \mu\left(x - f(x), \frac{t}{(2a)^k}\right) = 1\\ \lim_{k\to\infty} \nu\left(f^k(x) - f^{k+1}(x), t\right) &\leq \lim_{k\to\infty} \nu\left(x - f(x), \frac{t}{(2a)^k}\right) = 0.\end{aligned}$$

This means that intuitionistic fuzzy Kannan operator is intuitionistic fuzzy asymptotic regular. That is, by Theorem 2.2 intuitionistic fuzzy Kannan operator has IFAFPP. □

Corollary 2.1. *In the Theorem 2.4 , if $x \succeq f(x)$ for intuitionistic fuzzy Kannan operator f and $\mu(., t)$ is non-increasing, $\nu(., t)$ is non-decreasing for every $t \in (0, \infty)$, $x \succeq \theta, f$ has still IFAFPP.*

Definition 2.6. *Let X be an intuitionistic fuzzy normed space. If there exists $a \in \left(0, \frac{1}{2}\right)$ such that*

$$\begin{aligned}\mu(f(x) - f(y), at) &\geq \mu(x - f(y), t) * \mu(y - f(x), t)\\ \nu(f(x) - f(y), at) &\leq \nu(x - f(y), t) \lozenge \nu(y - f(x), t)\end{aligned}$$

for every $x, y \in X$ and $t > 0$, then $f : X \to X$ is called intuitionistic fuzzy Chatterjea operator.

Theorem 2.5. *Let $(X, \mu, \nu, *, \lozenge)$ be an IFNS havig partial order relation denoted by $\preceq$, where $a * b = \min\{a, b\}$ and $a \lozenge b = \max\{a, b\}$, and $f : X \to X$ be an intuitionistic fuzzy Chatterjea operator satisfying $x \preceq f(x)$ for every $x \in X$. Assume that $\preceq \subset X \times X$ holds satisfies one of the following conditions:*

(i) $\preceq$ is subvector space.
(ii) X is a totally ordered space.

If $\mu(., t)$ is non-decreasing, $\nu(., t)$ is non-increasing for every $t \in (0, \infty)$, $x \succeq \theta$ (θ is unit element in vector space X), then $F_\epsilon^{(\mu,\nu)}(f) \neq \varnothing$ for every $\epsilon \in (0, 1)$.

Proof. By taking into consideration assumption of theorem, we get

$$\begin{aligned}\mu\left(f^{k+1}(x) - f^k(x), t\right) &\geq \mu(f^k(x) - f^k(x), \frac{t}{a}) * \mu(f^{k-1}(x) - f^{k+1}(x), \frac{t}{a})\\ &= 1 * \mu(f^{k-1}(x) - f^{k+1}(x), \frac{t}{a})\\ &= \mu(f^{k-1}(x) - f^{k+1}(x), \frac{t}{a})\end{aligned}$$

$$\begin{aligned}&\geq \mu(f^{k-1}(x) - f^k(x), \frac{t}{2a}) * \mu(f^k(x) - f^{k+1}(x), \frac{t}{2a})\\ &= \min\left\{\mu(f^{k-1}(x) - f^k(x), \frac{t}{2a}), \mu(f^{k+1}(x) - f^k(x), \frac{t}{2a})\right\}\\ &= \mu(f^{k-1}(x) - f^k(x), \frac{t}{2a})\end{aligned}$$

$$\begin{aligned}&\geq \mu\left(f^{k-2}(x) - f^k(x), \frac{t}{2a^2}\right) * \mu\left(f^{k-1}(x) - f^{k-1}(x), \frac{t}{2a^2}\right)\\ &= \mu\left(f^{k-2}(x) - f^k(x), \frac{t}{2a^2}\right) * 1 = \mu\left(f^{k-2}(x) - f^k(x), \frac{t}{2a^2}\right)\end{aligned}$$

$$
\begin{aligned}
&\geq \mu\left(f^{k-2}(x)-f^{k-1}(x),\frac{t}{4a^2}\right)*\mu\left(f^{k-1}(x)-f^{k}(x),\frac{t}{4a^2}\right)\\
&= \min\left\{\mu\left(f^{k-2}(x)-f^{k-1}(x),\frac{t}{4a^2}\right),\mu\left(f^{k}(x)-f^{k-1}(x),\frac{t}{4a^2}\right)\right\}\\
&= \mu\left(f^{k-2}(x)-f^{k-1}(x),\frac{t}{4a^2}\right)\\
&\ \vdots\\
&\geq \mu\left(f^{k-(k-2)}(x)-f^{k-(k-3)}(x),\frac{t}{2^{k-2}a^{k-2}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \mu\left(f^{2}(x)-f^{3}(x),\frac{t}{2^{k-2}a^{k-2}}\right)\\
&\geq \mu\left(f\,(x)-f^{3}(x),\frac{t}{2^{k-2}a^{k-1}}\right)*\mu\left(f^{2}(x)-f^{2}(x),\frac{t}{2^{k-2}a^{k-1}}\right)\\
&= \mu\left(f\,(x)-f^{3}(x),\frac{t}{2^{k-2}a^{k-1}}\right)*1
\end{aligned}
$$

$$
\begin{aligned}
&= \mu\left(f\,(x)-f^{3}(x),\frac{t}{2^{k-2}a^{k-1}}\right)\\
&\geq \mu\left(f\,(x)-f^{2}(x),\frac{t}{2^{k-1}a^{k-1}}\right)*\mu\left(f^{2}(x)-f^{3}\,(x)\,,\frac{t}{2^{k-1}a^{k-1}}\right)\\
&= \min\left\{\mu\left(f\,(x)-f^{2}(x),\frac{t}{2^{k-1}a^{k-1}}\right),\mu\left(f^{3}\,(x)-f^{2}(x),\frac{t}{2^{k-1}a^{k-1}}\right)\right\}\\
&= \mu\left(f\,(x)-f^{2}(x),\frac{t}{2^{k-1}a^{k-1}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\geq \mu\left(x-f^{2}(x),\frac{t}{2^{k-1}a^{k}}\right)*\mu\left(f\,(x)-f(x),\frac{t}{2^{k-1}a^{k}}\right)\\
&= \mu\left(x-f^{2}(x),\frac{t}{2^{k-1}a^{k}}\right)*1\\
&= \mu\left(x-f^{2}(x),\frac{t}{2^{k-1}a^{k}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\geq \mu\left(x-f(x),\frac{t}{2^{k}a^{k}}\right)*\mu\left(f(x)-f^{2}(x),\frac{t}{2^{k}a^{k}}\right)\\
&= \min\left\{\mu\left(x-f(x),\frac{t}{2^{k}a^{k}}\right),\mu\left(f^{2}(x)-f(x),\frac{t}{2^{k}a^{k}}\right)\right\}\\
&= \mu\left(x-f(x),\frac{t}{2^{k}a^{k}}\right)
\end{aligned}
$$

and

$$\begin{aligned}
\nu\left(f^{k+1}(x)-f^{k}(x),t\right) &\leq \nu(f^{k}(x)-f^{k}(x),\frac{t}{a})\Diamond\nu(f^{k-1}(x)-f^{k+1}(x),\frac{t}{a}) \\
&= 0\Diamond\nu(f^{k-1}(x)-f^{k+1}(x),\frac{t}{a}) \\
&= \nu(f^{k-1}(x)-f^{k+1}(x),\frac{t}{a}) \\
&\leq \nu(f^{k-1}(x)-f^{k}(x),\frac{t}{2a})\Diamond\nu(f^{k}(x)-f^{k+1}(x),\frac{t}{2a}) \\
&= \max\left\{\nu(f^{k-1}(x)-f^{k}(x),\frac{t}{2a}),\nu(f^{k+1}(x)-f^{k}(x),\frac{t}{2a})\right\}
\end{aligned}$$

$$\begin{aligned}
&= \nu(f^{k-1}(x)-f^{k}(x),\frac{t}{2a}) \\
&\leq \nu\left(f^{k-2}(x)-f^{k}(x),\frac{t}{2a^2}\right)\Diamond\nu\left(f^{k-1}(x)-f^{k-1}(x),\frac{t}{2a^2}\right) \\
&= \nu\left(f^{k-2}(x)-f^{k}(x),\frac{t}{2a^2}\right)\Diamond 0 \\
&= \nu\left(f^{k-2}(x)-f^{k}(x),\frac{t}{2a^2}\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \nu\left(f^{k-2}(x)-f^{k-1}(x),\frac{t}{4a^2}\right)\Diamond\nu\left(f^{k-1}(x)-f^{k}(x),\frac{t}{4a^2}\right) \\
&= \max\left\{\nu\left(f^{k-2}(x)-f^{k-1}(x),\frac{t}{4a^2}\right),\nu\left(f^{k}(x)-f^{k-1}(x),\frac{t}{4a^2}\right)\right\} \\
&= \nu\left(f^{k-2}(x)-f^{k-1}(x),\frac{t}{4a^2}\right) \\
&\vdots
\end{aligned}$$

$$\begin{aligned}
&\leq \nu\left(f^{k-(k-2)}(x)-f^{k-(k-3)}(x),\frac{t}{2^{k-2}a^{k-2}}\right) \\
&= \nu\left(f^{2}(x)-f^{3}(x),\frac{t}{2^{k-2}a^{k-2}}\right) \\
&\leq \nu\left(f(x)-f^{3}(x),\frac{t}{2^{k-2}a^{k-1}}\right)\Diamond\nu\left(f^{2}(x)-f^{2}(x),\frac{t}{2^{k-2}a^{k-1}}\right) \\
&= \nu\left(f(x)-f^{3}(x),\frac{t}{2^{k-2}a^{k-1}}\right)\Diamond 0
\end{aligned}$$

$$\begin{aligned}
&= \nu\left(f(x)-f^{3}(x),\frac{t}{2^{k-2}a^{k-1}}\right) \\
&\leq \nu\left(f(x)-f^{2}(x),\frac{t}{2^{k-1}a^{k-1}}\right)\Diamond\nu\left(f^{2}(x)-f^{3}(x),\frac{t}{2^{k-1}a^{k-1}}\right) \\
&= \max\left\{\nu\left(f(x)-f^{2}(x),\frac{t}{2^{k-1}a^{k-1}}\right),\nu\left(f^{3}(x)-f^{2}(x),\frac{t}{2^{k-1}a^{k-1}}\right)\right\} \\
&= \nu\left(f(x)-f^{2}(x),\frac{t}{2^{k-1}a^{k-1}}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \nu\left(x-f^{2}(x),\frac{t}{2^{k-1}a^{k}}\right)\Diamond\nu\left(f\left(x\right)-f(x),\frac{t}{2^{k-1}a^{k}}\right)\\
&= \nu\left(x-f^{2}(x),\frac{t}{2^{k-1}a^{k}}\right)\Diamond 0=\nu\left(x-f^{2}(x),\frac{t}{2^{k-1}a^{k}}\right)\\
&\leq \nu\left(x-f(x),\frac{t}{2^{k}a^{k}}\right)\Diamond\nu\left(f(x)-f^{2}(x),\frac{t}{2^{k}a^{k}}\right)\\
&= \max\left\{\nu\left(x-f(x),\frac{t}{2^{k}a^{k}}\right),\nu\left(f^{2}(x)-f(x),\frac{t}{2^{k}a^{k}}\right)\right\}\\
&= \nu\left(x-f(x),\frac{t}{2^{k}a^{k}}\right).
\end{aligned}
$$

Since $\frac{t}{(2a)^{k}}\to\infty$ for $k\to\infty$, we have

$$
\begin{aligned}
\lim_{k\to\infty}\mu\left(f^{k}(x)-f^{k+1}(x),t\right) &\geq \lim_{k\to\infty}\mu\left(x-f(x),\frac{t}{(2a)^{k}}\right)=1,\\
\lim_{k\to\infty}\nu\left(f^{k}(x)-f^{k+1}(x),t\right) &\leq \lim_{k\to\infty}\nu\left(x-f(x),\frac{t}{(2a)^{k}}\right)=0
\end{aligned}
$$

by means of (vii) and (xiii) properties of intuitionistic fuzzy norm. We see that intuitionistic fuzzy Chatterjea operator has approximate fixed point property by Theorem 2.2. □

Corollary 2.2. *In the Theorem 2.5, if* $x\succeq f(x)$ *for intuitionistic fuzzy Chatterjea operator* f *and* $\mu\left(.,t\right)$ *is non-increasing,* $\nu\left(.,t\right)$ *is non-decreasing for every* $t\in(0,\infty)$ *,*$x\succeq\theta, f$ *has still IFAFPP.*

Definition 2.7. *Let* X *be an intuitionistic fuzzy normed space. A mapping* $f:X\to X$ *is called intuitionistic fuzzy Zamfirecsu operator if there exists at least* $a\in(0,1), k\in\left(0,\frac{1}{2}\right), c\in\left(0,\frac{1}{2}\right)$ *such that at least one of the followings is true for every* $x,y\in X$ *and* $t>0:$

(i)

$$
\begin{aligned}
\mu\left(f(x)-f(y),at\right) &\geq \mu\left(x-y),t\right)\\
\nu\left(f(x)-f(y),at\right) &\leq \nu\left(x)-y),t\right).
\end{aligned}
$$

(ii)

$$
\begin{aligned}
\mu\left(f(x)-f(y),kt\right) &\geq \mu\left(x-f(x),t\right)*\mu\left(y-f(y),t\right)\\
\nu\left(f(x)-f(y),kt\right) &\leq \nu\left(x-f(x),t\right)\Diamond\nu\left(y-f(y),t\right).
\end{aligned}
$$

(iii)

$$
\begin{aligned}
\mu\left(f(x)-f(y),ct\right) &\geq \mu\left(x-f(y),t\right)*\mu\left(y-f(x),t\right)\\
\nu\left(f(x)-f(y),ct\right) &\leq \nu\left(x-f(y),t\right)\Diamond\nu\left(y-f(x),t\right).
\end{aligned}
$$

Theorem 2.6. *Let* $(X,\mu,\nu,*,\Diamond)$ *be an IFNS havig partial order relation denoted by* $\preceq$*, where* $a*b=\min\{a,b\}$ *and* $a\Diamond b=\max\{a,b\}$*, and* $f:X\to X$ *be an intuitionistic fuzzy Zamfirescu operator satisfying* $x\preceq f(x)$ *for every* $x\in X$*. Assume that* $\preceq\subset X\times X$ *holds one of the following conditions:*

(i) $\preceq$ *is subvector space.*

(ii) X *is a totally ordered space.*

If $\mu(.,t)$ *is non-decreasing,* $\nu(.,t)$ *is non-increasing for every* $t \in (0,\infty)$, $x \succeq \theta$ *(*θ *is unit element in vector space* X*), then* $F_{\epsilon}^{(\mu,\nu)}(f) \neq \varnothing$ *for every* $\epsilon \in (0,1)$.

Proof. The proof is clear from Theorem 2.4 and Theorem 2.5. □

Definition 2.8. *Let* X *be an IFNS. If there exist* $a \in (0,1)$ *and* $L \geq 0$ *such that*

$$\begin{aligned}
\mu(f(x)-f(y),t) &\geq \mu\left(x-y,\frac{t}{a}\right) * \mu\left(y-f(x),\frac{t}{L}\right) \\
\nu(f(x)-f(y),t) &\leq \nu\left(x-y,\frac{t}{a}\right) \Diamond \nu\left(y-f(x),\frac{t}{L}\right)
\end{aligned}$$

for every $x,y \in X$ *and* $t > 0$, *then* $f : X \to X$ *is called intuitionistic fuzzy weak contraction operator.*

Theorem 2.7. *Let* X *be an IFNS, and* $f : X \to X$ *be intuitionistic fuzzy weak contraction. Then* $F_{\epsilon}^{(\mu,\nu)}(f) \neq \varnothing$, *for every* $\epsilon \in (0,1)$.

Proof. Let $x \in X$ and $\epsilon \in (0,1)$.

$$\begin{aligned}
\mu\left(f^k(x)-f^{k+1}(x),t\right) &= \mu\left(f\left(f^{k-1}(x)\right)-f\left(f^k(x)\right),t\right) \\
&\geq \mu\left(f^{k-1}(x)-f^k(x),\frac{t}{a}\right) * \mu\left(f^k(x)-f^k(x),\frac{t}{L}\right) \\
&= \mu\left(f^{k-1}(x)-f^k(x),\frac{t}{a}\right) * 1 \\
&= \mu\left(f^{k-1}(x)-f^k(x),\frac{t}{a}\right)
\end{aligned}$$

$$\begin{aligned}
&\geq \mu\left(f^{k-2}(x)-f^{k-1}(x),\frac{t}{a^2}\right) * \mu\left(f^{k-1}(x)-f^{k-1}(x),\frac{t}{L}\right) \\
&= \mu\left(f^{k-2}(x)-f^{k-1}(x),\frac{t}{a^2}\right) * 1 \\
&\geq \mu\left(f^{k-2}(x)-f^{k-1}(x),\frac{t}{a^2}\right) \\
&\geq \ldots
\end{aligned}$$

$$\begin{aligned}
&= \mu\left(f^{k-(k-1)}(x)-f^{k-(k-2)}(x),\frac{t}{a^{k-1}}\right) = \mu\left(f(x)-f^2(x),\frac{t}{a^{k-1}}\right) \\
&\geq \mu\left(x-f(x),\frac{t}{a^k}\right) * \mu\left(f(x)-f(x),\frac{t}{L}\right) \\
&\geq \mu\left(x-f(x),\frac{t}{a^k}\right) * 1 \\
&= \mu\left(x-f(x),\frac{t}{a^k}\right)
\end{aligned}$$

and

$$\begin{aligned}
\nu\left(f^k(x)-f^{k+1}(x),t\right) &= \nu\left(f\left(f^{k-1}(x)\right)-f\left(f^k(x)\right),t\right)\\
&\leq \nu\left(f^{k-1}(x)-f^k(x),\frac{t}{a}\right)\Diamond\nu\left(f^k(x)-f^k(x),\frac{t}{L}\right)\\
&= \nu\left(f^{k-1}(x)-f^k(x),\frac{t}{a}\right)\Diamond 0
\end{aligned}$$

$$\begin{aligned}
&= \nu\left(f^{k-1}(x)-f^k(x),\frac{t}{a}\right)\\
&\leq \nu\left(f^{k-2}(x)-f^{k-1}(x),\frac{t}{a^2}\right)\Diamond\nu\left(f^{k-1}(x)-f^{k-1}(x),\frac{t}{L}\right)\\
&= \nu\left(f^{k-2}(x)-f^{k-1}(x),\frac{t}{a^2}\right)\Diamond 0
\end{aligned}$$

$$\begin{aligned}
&\leq \nu\left(f^{k-2}(x)-f^{k-1}(x),\frac{t}{a^2}\right)\\
&\leq ...\\
&= \nu\left(f^{k-(k-1)}(x)-f^{k-(k-2)}(x),\frac{t}{a^{k-1}}\right)
\end{aligned}$$

$$\begin{aligned}
&= \nu\left(f(x)-f^2(x),\frac{t}{a^{k-1}}\right)\\
&\leq \nu\left(x-f(x),\frac{t}{a^k}\right)\Diamond\nu\left(f(x)-f(x),\frac{t}{L}\right)\\
&\leq \nu\left(x-f(x),\frac{t}{a^k}\right)\Diamond 0\\
&= \nu\left(x-f(x),\frac{t}{a^k}\right)
\end{aligned}$$

Since $\frac{t}{a^k}\to\infty$ for $k\to\infty$, by means of (vii) and (xiii) properties of intuitionistic fuzzy norm, we see intuitionistic fuzzy weak contraction map has approximate fixed point property by Theorem 2.2. $\square$

In the following, we give definition of approximate fixed point property of a set. Furthermore, we prove that a dense set of intuitionistic fuzzy Banach space has approximate fixed point property.

Definition 2.9. *Let X be IFNS and let K be subset of X. Then K is said to have intuitionistic fuzzy approximate fixed point property (IFAFPP) if every intuitionistic fuzzy nonexpansive map $f:K\to K$ satisfies the property that* $\sup\{\mu(x-f(x),t):x\in K\}=1$ *and* $\inf\{\nu(x-f(x),t):x\in K\}=0$.

Theorem 2.8. *Let X be an intuitionistic fuzzy normed space having IFAFPP, K be dense subset of X. Then K has IFAFPP.*

Proof. Let $f:X\to X$ be an intuitionistic fuzzy nonexpansive mapping. Firstly we prove that

$$\sup\{\mu(x-f(x),t):x\in K\}=\sup\{\mu(y-f(y),s):y\in X\}$$

and

$$\inf\left\{\nu\left(x-f\left(x\right),t\right):x\in K\right\}=\inf\nu\left\{\left(y-f\left(y\right),s\right):y\in X\right\}$$

for $t,s>0$. Since $K\subset X$,

$$\sup\left\{\mu\left(y-f\left(y\right),s\right):y\in X\right\}\geq\sup\left\{\mu\left(x-f\left(x\right),t\right):x\in K\right\}$$

and

$$\inf\left\{\nu\left(y-f\left(y\right),s\right):y\in X\right\}\leq\inf\left\{\nu\left(x-f\left(x\right),t\right):x\in K\right\}.$$

Let $y\in X$. There exists a sequence (y_k) in K such that $y_k\overset{(\mu,\nu)}{\to}y$ for all $y\in X$ because of K is dense. We know that for each $k\in\mathbb{N}$ and $t,s>0$,

$$\begin{aligned}\sup\left\{\mu\left(x-f\left(x\right),t\right):x\in K\right\} &\geq \mu\left(y_k-f\left(y_k\right),t\right)\\ &\geq \mu\left(y_k-y+y-f\left(y\right)+f\left(y\right)-f\left(y_k\right),t\right)\\ &\geq \mu\left(y_k-y,\frac{t}{3}\right)*\mu\left(y-f\left(y\right),\frac{t}{3}\right)\\ &\quad *\mu\left(y_k-f\left(y_k\right),\frac{t}{3}\right)\end{aligned}$$

and

$$\begin{aligned}\inf\left\{\nu\left(x-f\left(x\right),t\right):x\in K\right\} &\leq \nu\left(y_k-f\left(y_k\right),t\right)\\ &\leq \nu\left(y_k-y+y-f\left(y\right)+f\left(y\right)-f\left(y_k\right),t\right)\\ &\leq \nu\left(y_k-y,\frac{t}{3}\right)\Diamond\nu\left(y-f\left(y\right),\frac{t}{3}\right)\\ &\quad \Diamond\nu\left(y_k-f\left(y_k\right),\frac{t}{3}\right).\end{aligned}$$

Since f is intuitionistic fuzzy nonexpansive mapping, it is intuitionistic fuzzy continuous. Because, if $y_k\overset{(\mu,\nu)}{\to}y$, then

$$\begin{aligned}\mu\left(f\left(y_k\right)-f\left(y\right),t\right) &\geq \mu\left(y_k-y,t\right)\to 1\\ \nu\left(f\left(y_k\right)-f\left(y\right),t\right) &\leq \nu\left(y_k-y,t\right)\to 0.\end{aligned}$$

So $f\left(y_k\right)\overset{(\mu,\nu)}{\to}f\left(y\right)$ when $y_k\overset{(\mu,\nu)}{\to}y$. If we take limit above inequalities, we get

$$\sup\left\{\mu\left(x-f\left(x\right),t\right):x\in K\right\}\geq\mu\left(y-f\left(y\right),\frac{t}{3}\right)$$

and

$$\inf\left\{\nu\left(x-f\left(x\right),t\right):x\in K\right\}\leq\nu\left(y-f\left(y\right),\frac{t}{3}\right)$$

for all $y\in X$ and $t>0$. Thus, if we take $\frac{t}{3}=s'$,

$$\sup\left\{\mu\left(x-f\left(x\right),t\right):x\in K\right\}\geq\sup\left\{\mu\left(y-f\left(y\right),s'\right):y\in X\right\}$$

and

$$\inf\left\{\nu\left(x-f\left(x\right),t\right):x\in K\right\}\geq\inf\left\{\nu\left(y-f\left(y\right),s'\right):y\in X\right\}.$$

Therefore our claim is proved. Now consider any intuitionistic fuzzy nonexpansive mapping $f_K:K\to K$. Since K is dense, there exists a sequence (y_k) in K such that $y_k\overset{(\mu,\nu)}{\to}y$ for any $y\in X$. Since an intuitionistic fuzzy nonexpansive mapping is continuous, $f_K:K\to K$ is intuitionistic fuzzy continuous and it can be extending by defining $f\left(x\right)=\lim\left(\mu,\nu\right)-f\left(x_k\right)$ on X. Hence we can consider f as an intuitionistic fuzzy nonexpansive mapping on X. Because, using Lemma 1.2, we get

$$\begin{aligned}\mu\left(f\left(x\right)-f\left(y\right),t\right) &= \lim_{k\to\infty}\sup\mu\left(f\left(x_k\right)-f\left(y_k\right),t\right)\geq\lim_{k\to\infty}\sup\mu\left(x_k-y_k,t\right)\\ &= \mu\left(x,y,t\right)\end{aligned}$$

and

$$\begin{aligned}\nu\left(f\left(x\right)-f\left(y\right),t\right) &= \liminf_{k\to\infty}\nu\left(f\left(x_k\right)-f\left(y_k\right),t\right)\leq\lim_{k\to\infty}\inf\nu\left(x_k-y_k,t\right)\\ &= \nu\left(x,y,t\right)\end{aligned}$$

for all $x,y\in X$ and $t>0$. Then, since X has IFAFPP,

$$\sup\left\{\mu\left(x-f\left(x\right),t\right):x\in K\right\}=\sup\left\{\mu\left(y-f\left(y\right),t\right):y\in X\right\}=1$$

and

$$\inf\left\{\nu\left(x-f\left(x\right),t\right):x\in K\right\}=\inf\nu\left\{\left(y-f\left(y\right),t\right):y\in X\right\}=0.$$

That is, for given any intuitionistic fuzzy nonexpansive mapping f on K we have $\sup\left\{\mu\left(x-f\left(x\right),t\right):x\in K\right\}=1$ and $\inf\left\{\nu\left(x-f\left(x\right),t\right):x\in K\right\}=0$ and K has IFAFPP. □

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