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UNIT GRAPH OF TYPE - 2

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ABSTRACT. The unit graph of ring R was introduced in commutative rings by Vasantha Kandasamy[27]. In this short note, we introduce the concept namely "Unit graph of type-2" denoted by UG2(R) in associative rings R and announced a few important fundamental results. In section 3, we prove that the number of edges in the unit graph of type-2 of Z_p is $\frac{p-3}{2}$. In section 4, we prove that sum of the degrees of the vertices in UG2(R) is equal to ($|U(R)|-$number of self units). Also we have included some examples.

Keywords: Graph, Unit graph of a ring, Star graph.

AMS Subject Classification: 05C07, 05C20, 05C76, 05C99, 13E15.

1. Introduction

Let $G = (V, E)$ be a graph consist of a finite non-empty set V of vertices and finite set E of edges such that each edge e_k is identified as an unordered pair of vertices $\{\mathrm{v}_i, \mathrm{v}_j\}$, where v_i, v_j are called end points of e_k . The edge e_k is also denoted by either v_iv_j or $\overline{v_iv_j}$. We also write $G(V,E)$ for the graph. Vertex set and edge set of G are also denoted by $V(G)$ and $E(G)$ respectively. An edge associated with a vertex pair $\{\mathrm{v}_i, \mathrm{v}_i\}$ is called a self-loop. The number of edges associated with the vertex is the degree of the vertex, and $d(v)$ denotes the degree of the vertex v. If there is more than one edge associated with a given pair of vertices, then these edges are called parallel edges or multiple edges.
A graph that does not have self-loops or parallel edges is called a simple graph. We consider simple graphs only.

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2. Unit graph of type-2

Definition 2.1. *(i) A graph $G(V, E)$ is said to be a star graph if there exists a fixed vertex v (called the center of the star graph)such that $E = \{\overline{vu}\ /\ u \in V$ and $u \neq v\}$. A star graph is said to be an n-star graph if the number of vertices of the graph is n.*

(ii) In a graph G, a subset S of $V(G)$ is said to be a dominating set if every vertex not in S has a neighbour in S. The domination number, denoted by $\gamma(G)$ is defined as min $\{|S|\ /\ S$ is a dominating set in $G\}$.

Definition 2.2. [27] *Let R be a finite commutative ring with 1. We take U(R) the set of units in R (clearly $U(R) \neq \phi$ as $1 \in U(R)$). Now the elements of $U(R)$ form the vertices of simple graph. Two elements x and y in R are adjacent if and only if $x.y = 1$. We assume that 1 is adjacent with every unit in R. The graph associated with $U(R)$ is defined to be the unit graph of R. Unit Graph of type-1.*

Remark 2.1. *In the case of a zero divisor graph of R, the vertex set V is equal to R. In the case of the unit graph of R we take $V = U(R)$, the set of all units in R.*

Remark 2.2. *If R has no unit (no invertible) element other than 1, then $U(R)=\{1\}$.*

Now we introduce a new concept "Unit Graph of type-2" and observe some examples.

Definition 2.3. *Let R be a finite commutative ring with 1. A graph $G\ (V,\ E)$ is said to be unit graph of type-2 if $V = U(R)$ and $E = \{\overline{xy} \mid x,\ y \in U(R)$ such that $xy = 1$ and $x \neq y\}$. We denote the unit graph of type-2 of a ring R by $UG2(R)$.*

If $x^2 = 1$ then we say that x is a self unit, and if $x^2 \neq 1$ then we say that x is a non-self unit.
Notation: We write $SU(R) = \{x \in U(R) \mid x = x^{-1}\}$ = the set of all self units; and $NSU(R) = \{x \in U(R) \mid x \neq x^{-1}\}$ = the set of all non-self units

Note 2.1. *Unit graph of type-2 is a subgraph of the unit graph of Type-1*

Note 2.2. *Let n be a positive integer and ' p' be a prime number.*

(i) $(n-1)^2 \equiv 1 \pmod{n}$

(ii) $(p-1)$ is the inverse of $(p-1)$ in the multiplicative group $Z_p^* = Z_p$-$\{0\}$.

(iii) In $UG2(Z_p)$, the vertex $(p-1)$ is an isolated vertex.

Verification: For (i) consider $(n-1)(n-1) = (n-1)^2$

$$= n^2 + 1 - 2n \cong 1 \pmod{n}$$

So, $(n-1)$ is inverse of $(n-1)$. Hence $(n-1)$ is an isolated vertex

Example 2.1. *Let $Z_2=\{0,1\}$, $U(R)=\{1\}$. So $V(UG2(R))=\{1\}$. Since '1' is adjacent to all units in R, as there are no units other than 1, there are no edges and $E(UG2(R))=\emptyset$. Now $UG2(R)$ is given in Figure-1.0.1.*

Unit graph of type-2 of Z_2

Example 2.2. *Consider $Z_7 = \{0,1,2,3,4,5,6\}$ the ring of integers modulo 7. Now $U(R) = \{1,2,3,4,5,6\}$, 2 and 4 are inverses to each other, and so there is an edge between 2 and 4. Also 3 and 5 are inverses to each other, so there is an edge between 3 and 5. Also 1 and 6 are self units. The unit graph of type-2 of Z_7 is given in Fig 1.0.2.*

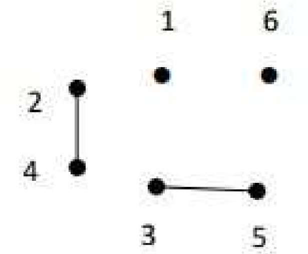

Fig 1.0.2

Unit graph of type-2 of Z_7

Lemma 2.1. *There is no triangle in UG2(Z_p) for any prime number p*

Verification: In a contrary way, suppose x, y, z $\in U(Z_p)$ are three distinct elements such that {x, y, z} is a triangle.
Now xy =1 and yz =1
$\Rightarrow$ x and z are two distinct inverses of y, which is a contradiction. Hence the unit graph of type-2 of Z_p contains no triangle.

Note 2.3. *(i) UG2(Zn) is a is disconnected graph for any positive integer n. (ii)UG2(R) $\subseteq$ UG1(R) for any ring R [that is, the unit graph of type-2 is a subgraph of unit graph of type-1 for all the rings R].*

Example 2.3. *Consider Z_5={0,1,2,3,4}, the ring of integers modulo 5. Now U(R)={1,2,3,4}, 2 and 3 are inverses to each other and so there is an edge between 2 and 3. Also 1 and* 4 *are self units. The unit Graph of type-2 of Z_5 is given in Fig. 1.0.3.*

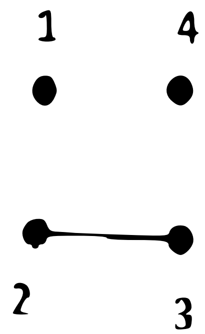

Fig. 1.0.3

Number of edges in unit Graph of type-2 of Z_5 is 1

Example 2.4. *Consider Z_{11}={0,1,2,3,4,5,6,7,8,9,10}, the ring of integers modulo 11. Now U(R)={1,2,3,4,5,6,7,8,9,10}, Since 1 and 10 are self inverses, there are no edge with end point 1 or 10. Since 2 and 6 are inverses to each other, we connect 2 and 6 by an edge. Since 3 and 4 are inverses to each other we connect 3 and 4 by an edge. We connect all such pairs of vertices and finally we get the Unit graph of type-2 of Z_{11} given in the Fig.1.0.*4.
The number of edges in unit Graph of type-2 of Z_{11} is 4.

Now we conclude the following:

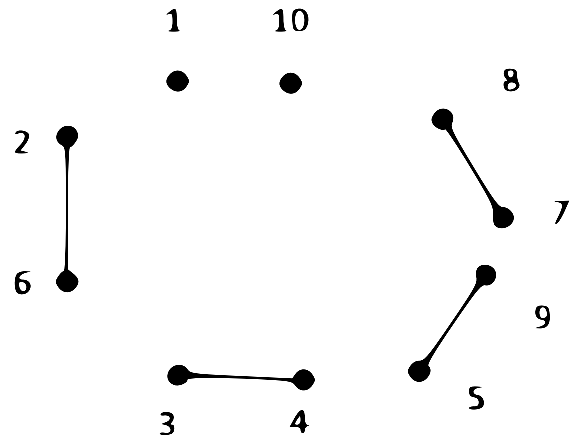

Fig 1.0.4

3. Number of Edges in $UG2(Z_p)$

Lemma 3.1. *$UG2(Z_p)$ contains at least two isolated vertices.*

Proof. We know that $(p-1)^2 \equiv 1(modp)$ and $1^2 \equiv 1(modp)$. Since 1 and $(p-1)$are self units, we conclude that the vertices 1 and $(p-1)$ in $UG2(Z_p)$ are isolated vertices.

□

The following Result is well known in number theory, but we provide proof for completeness.

Lemma 3.2. [26] *Let 'a' be an integer such that 1≤ $a < p$ and $a^2 \equiv 1(\mod p)$. Then either a=1 or a = p-1 .*

Proof: Suppose a $\neq$ 1 and $a^2 \equiv 1$(mod p), then $\frac{p}{a^2-1}$ and so $\frac{p}{a-1}$ or $\frac{p}{a+1}$

$\Rightarrow \frac{p}{a+1}$ (since p cannot divide (a-1) as a < p)

$\Rightarrow$ p $\leq$ (a+1)
$\Rightarrow$ p = (a+1), since (a+1)$\leq$p
$\Rightarrow$ a = (p-1)

Therefore in Z_p only self units are 1 and (p-1).

Corollary 3.1. *In $UG2(Z_p)$ there exit exactly two isolated vertices.*

Proof. By above lemma-3.1 we know that the two vertices 1 and (p-1) are isolated. Suppose $v \in V(UG2(Z_p))$ with $1 \neq v \neq (p-1)$. By lemma-3.2, v=1 or v=(p-1), a contradiction. Hence the set of isolated vertices in $UG2(Z_p)$is exactly {1, (p-1)}. □

Theorem 3.1. *Let Z_p={0, 1, 2, ..., $(p-1)$} be the ring of integers modulo 'p' . Then the number of edges in unit graph of type-2 of Z_p is $\frac{p-3}{2}$*

Proof. Since 1 and $p-1$ are self units, the degree of these two vertices is equal to zero. As p is prime, Z_p is a field. Since every element in $Z_p - \{0\}$ has inverse, for the remaining vertices, that is {2,3,...,(p-2)} are (p-3) in number and so we can group into $\frac{P-3}{2}$ pairs {x, y} as y=x^{-1} or that xy $\equiv$ 1 (mod p)

As xy $\equiv 1 \pmod{p}$, x and y are adjacent in the unit graph of type-2, we have that $\frac{P-3}{2}$ pairs exist with xy $\equiv 1 \pmod{p}$ and x $\neq$ y. Therefore the number of edges in the unit graph of type-2 of Z_p is $\frac{P-3}{2}$. □

Theorem 3.2. *Let R be an associative ring. Then R is a Division ring if and only if the number of edges in UG2(R) is equal to* $\dfrac{|R|-1-(number\ of\ self\ inverses\ in\ R)}{2}$.

Proof. Since R is a division ring all the elements of R^*=R-{0} are units. Write S = {u $\in$ R/u^2 = 1} (the set of all self inverses), N={x$\in$$R$/xy=1 and x$\neq$y}. Then R-{0}=R^*=$S \cup N$ $\Rightarrow$|R|-1=|S|+|N| $\Rightarrow$ $|N|$=$|R|$-1-|S| or $|S|$=$|R|$-1-$|N|$ (say (i)). Since self inverses units are not end points of any edge in $UG2(R)$, the edges that are having end points in S is zero. If $0 \neq x$$\in$$N$, then there is an edge between x and y $\in N$ with x$\neq$y and xy=1. Thus the number of edges that can be formed in $UG2(R)$ is equal to $\frac{t}{2}$ where t = $|N|$. Hence the number of edges in $UG2(R)$ is equal to $\frac{t}{2}$=$\frac{|N|}{2}$=$\frac{|R|-1-|S|}{2}$

Converse: Suppose R is a ring with the condition
$|E(UG2(R))| = \dfrac{|R|-1-(number\ of\ self\ units\ in\ R)}{2}$
From the given condition $2|E(UG2(R))|$=$|R|-1-(number\ of\ self\ units\ in\ R)$
Number of self units = $|R|$-1-2$|E(UG2(R))|$.
In a contrary way suppose that R is not a division ring. Then there exists $0 \neq\ x \in R$ such that x has no inverse. The number of elements in R have inverse <|R-{0,x}|=|R|-2
$\Rightarrow$ (number of self units+ number of invertible elements which are not self units)< |R|-2
$\Rightarrow$ |R|-1-2|$E(UG2(R))$| +(number of inverse elements which are not self units)< |R|-2

$\Rightarrow$ number of inverse elements which are not self units < 2|$E(UG2$(R))|-1

$\Rightarrow 2|E(UG2(R))< \dfrac{|R|-1-(number\ of\ self\ units\ in\ R)}{2}$ a contradiction.
Hence R is a division ring. □

Corollary 3.2. *The following conditions are equivalent*
(i) (0) is the maximal ideal of a ring
(ii) R is a division ring

$$(iii)|E(UG2(R))| = \frac{|R|-1-(number\ of\ self\ units\ in\ R)}{2}$$

Proof. (i)$\Leftrightarrow$ (ii): By proposition 1 of chapter 3 of [1].

(ii)$\Leftrightarrow$ (iii)By Theorem 3.2. □

4. Degrees of vertices in $UG2(R)$

Theorem 4.1. *(i) If v $\in$ V(UG2(R)) where R is the ring then degree of v is either 0 or 1.*
(ii) v $\in$ V(UG2(R)) is isolated vertex if and only if it is a self unit.

Proof. Let $v \in V(UG2(R))$. Since v is a unit there exits $u \in V(UG2(R))$ with $uv = 1$. If $u = v$ then v is a self unit and so there is no edge in $V(UG2(R))$ with end point v. So

degree of v is zero. If $u \neq v$ then by definition of $UG2(R)$ there exists an edge between v and u. Since there are no other edges with end point v We get that the degree of v is 1. Hence we conclude that for every v in $UG2(R)$ the degree is either 0 or 1.
(i) $v \in V(UG2(R))$ is an isolated vertex.
$\Leftrightarrow$ degree of v is equal to zero and v is a unit.
$\Leftrightarrow v = v^{-1}$. $\square$

Theorem 4.2. *(i)* $|E(UG2(R))| = \frac{1}{2}$ $\{|U(R)|$*-number of self units*$\}$.
(ii) Sum of the degrees of the vertices is equal to ($|U(R)|$*-number of self units).*

Proof. (i) Let R be a ring and $U(R)$ be the set of all self units in R.
write $NSU(R)$={x$\in U(R)$/x$\neq$x^{-1}} and $SU(R)$={x$\in U(R)$/x=x^{-1}}. Now $U(R)$=$NSU(R)$ $\cup$ $SU(R)$, a disjoint union. If x$\neq$x^{-1} then there is an edge between x and x^{-1}. The number of such {x, x^{-1}} pairs that can be formed is $\frac{|NSU(R)|}{2}$
Thus $|E(UG2(R))| = \frac{1}{2}$ $|NSU(R)| = \frac{1}{2}$ $|U(R) - SU(R)|$.
(ii)Sum of the degrees of the vertices in $UG2(R)$
$= 2\times$ number of edges (by Theorem 13.1, p337 [24])
$= |U(R)| - SU(R)$ (by (i)). $\square$

5. Conclusions

A new concept 'Unit graph of type-2 of a finite commutative ring R (denoted by UG2(R))' was introduced. Two important results proved are
(i). R is a division ring if and only if $|E(UG2(R))| = \frac{|R|-1-(\text{number of self units in } R)}{2}$;
(ii). The sum of the degrees of the vertices in the graph G is equal to ($|U(R)|-$ number of self units). These results provide some new fundamental important relations between 'Graph Theory', and 'Ring Theory'.

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References

[1] Lambeck J., (1986), Lectures on Rings & Modules, 3rd Edition. Chelsea publishing company, New York.
[2] Satyanarayana Bh., Godloza L., Nagaraju D., (2011), Some results on Principal Ideal graph of a ring, African Journal of Mathematics and Computer Science Research 4, (6), pp. 235-241.
[3] Satyanarayana Bh., Mohiddin S., Mallikarjun Bh., Pradeep K. T. V., (2010), On a Graph related to the Ring of Integers Modulo n, Proceedings of the International Conference on Challenges and Applications of Mathematics in Science and Technology, (CAMIST), January 11-13 (Publisher: Macmillan Research Series), pp. 688-697. (India).
[4] Satyanarayana Bh., Mohiddin S. and Venkata V. K. A., (2010). Prime Graph of an Integral Domain, Proceedings of the National Seminar on Present Trends in Mathematics and its Applications, November 11-12, pp. 124-134.
[5] Satyanarayana Bh., Nagaraju D., (2011), Dimension and Graph Theoretic Aspects of Rings, VDM verlag Dr Muller, Germany.
[6] Satyanarayana Bh., Pradeep K. T.V, Mohiddin S.,(2016), Mathematical Foundations of Computer Science, BS publications, Hyderabad, A. P., India.
[7] Satyanarayana Bh., Srinivasulu D., (2015), Cartesian Product of Graphs Vs. Prime Graphs of Rings, Global Journal of Pure and Applied Mathematics (GJPAM), 11, (2), pp. 199-205.

[8] Satyanarayana Bh., Srinivasulu D., Mallikarjun Bh., Abul B.,(2019), Power Chains in a Divisor Graph, International Journal of Research in Humanities, Arts and Literature. 7, (4), pp.383-390.

[9] Satyanarayana Bh., Srinivasulu D., Hymavathi T., Mallikarjun Bh.., (2017), One Sided Zero Divisor Graphs of Rings with a Total order, Asian Journal of Mathematics and Computer Research, 15, (1), pp. 9-17.

[10] Satyanarayana Bh., Srinivasulu D., Mallikarjun Bh., (2016), A Theorem on the Prime Graph of 2×2 - matrix ring of Z2, International Journal on Recent and Innovation Trends in Computing and Communication, 4, (5), pp. 571-573.

[11] Satyanarayana Bh., Srinivasulu D., Mallikarjun Bh., (2016), A Theorem on the Zero Divisor Graph of the ring of 2×2- matrices over Z2, International Educational Scientific and Research Journal 2, (6), pp. 45-46.

[12] Satyanarayana Bh., Srinivasulu D., Mallikarjun Bh., (2016). Left Zero Divisor Graphs of Totally Ordered Rings, International Journal of Advanced Engineering, Management and Science, (IJAEMS) 2, (6), pp. 877-880.

[13] Satyanarayana Bh., Srinivasulu D., Mallikarjun Bh., (2016), Star Number of a Graph, The International Journal of Research Publication's Research Journal of Science & IT Management, 5, (11), pp. 18-22.

[14] Satyanarayana Bh. , Srinivasulu D., Mallikarjun Bh., (2016), Prime Graph vs. Zero Divisor Graph, IOSR Journal of Mathematics, 12, (5), ver. VI, pp.75-78.

[15] Satyanarayana Bh., Srinivasulu D., Mallikarjun Bh., (2017), Cycle Number of a Graph., Global journal of Pure and Applied Mathematics, 13, (4), pp.07-10.

[16] Satyanarayana Bh., Srinivasulu D., Mallikarjun Bh., (2019). A Note on Unit Graph of a Ring, International Journal of Research and Analytical Reviews, (IJRAR), 6, (1), pp. 84-87.

[17] Satyanarayana Bh., Srinivasulu D., Mallikarjun Bh., Eswaraiah S. S., (2017), 1-quasitotal Graphs Vs. Degree of Vertices with respect to a Vertex Set, International Journal of Information Research and Review, 4, (2), pp. 3780 -3797.

[18] Satyanarayana Bh., Srinivasulu D., Syam P. K., (2012), Some Results on Degree of Vertices in Semitotal Block Graph and Total – Block Graph, International Journal of computer Applications, 50, (9), pp.19-22.

[19] Satyanarayana Bh., Srinivasulu D., Syam P. K., (2014), Line Graphs & Quasi- total Graphs, International Journal of Computer Applications 105, (3), pp. 12-16.

[20] Satyanarayana Bh., Srinivasulu D., Syam P. K., (2017), A Note on Line Graphs, TWMS Journal of Applied and Engineering Mathematics, 7, (1), pp. 173-179.

[21] Satyanarayana Bh., Syam P. K., (2003), An Isomorphism Theorem on Directed Hypercubes of Dimension, Indian J. Pure & Appl. Math., 34, (10), pp. 1453-1457.

[22] Satyanarayana Bh., Syam P. K., (2011), Dimension of N-groups and Fuzzy ideals in Gamma Near–rings, VDM verlag Dr Muller, Germany.

[23] Satyanarayana Bh., Syam P. K., (2013), Nearrings, Fuzzy Ideals and Graph Theory, CRC Press (Taylor & Francis Group, London, New York).

[24] Satyanarayana Bh., Syam P. K., (2014), Discrete Mathematics and Graph Theory, Prentice Hall India Pvt. Ltd., New Delhi, 2014, (2nd Edition).

[25] Satyanarayana Bh., Syam P. K., Nagaraju D., (2010). Prime Graph of a Ring, J. Combinatorics, Informations & System Sciences, 35, pp. 27-42.

[26] Thomas K., (2009), Elementary Number Theory with Applications, (2nd Edition), Academic press, (An imprint of Elsevier).

[27] Vasantha K. W. B., Florentin S., (2009), Groups as Graphs, Editura Cu Art.

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