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CHARACTERIZATION OF UNIFORM AND HYBRID CELLULAR AUTOMATA WITH NULL BOUNDARY

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ABSTRACT. In this article, we dispute about the characterization of Cellular automata with restricted vertical neighborhood and Von neumann neighborhood of null boundary conditions over the field $\mathbb{Z}_2$ in uniform cellular automata and hybrid cellular automata. Transition rule matrix for uniform and hybrid cellular automara with null boundary condition is obtained and the reversibility of the uniform cellular automata studied.

Keywords: 2$\mathbb{D}$ $\mathcal{CA}$,VNN, restricted vertical neighborhood, ternary field, null boundary, matrix algebra, transition rule matrix, reversible cellular automata.

AMS Subject Classification: 68Q80, 03D05, 15A04.

1. Introduction

Cellular Automata shortly ($\mathcal{CA}$). The embody of ($\mathcal{CA}$) was present in $1950's$ by John Von neumann [12] visible that a ($\mathcal{CA}$) can be global. The read of ($\mathcal{CA}$) has obtain significant assiduity in the ultimate few years [1, 5, 14], as the ($\mathcal{CA}$) has been widely interrogate in many field (For Example Mathematics, physics, computer science, chemistry, etc.) with different scope(For Example simulation of image processing, natural phenomena, pseudo-random number generation, analysis of universal model of computations, cryptography). The behaviour of 2$\mathbb{D}$ adjecent neighbors linear ($\mathcal{CA}$) with some basic and accurate mathematical model Null Boundary (NB) or Periodic Boundary (PB) condition using matrix algebraic domain in the domain with two elements $\mathbb{Z}_2$ [4, 5, 14].

A ($\mathcal{CA}$) is system of a set of "cell" objects with the following properties.

* The cell live is at one point.

* Every cell has a state. The number of state possibilities is generally limited. The easy model has the two possibilities of 1 and 0 (otherwise indicate to as "ON", "OFF" (or) "alive", "dead").

* Every cell has an ambient. This can be distinct in number of path, but it is usually a list of proximate cells.

$m \times n$ cells regulate the 2$\mathbb{D}$ $\mathcal{CA}$. Each cell lease of the one values in $\mathbb{Z}_2$. The akin plight

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of the cells are said neighbor of the midst cell. The state of these neighbors are cash to count the new state of the midst cell.

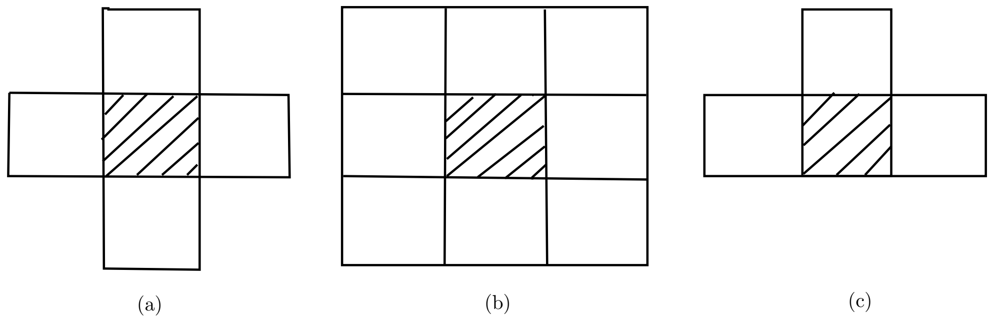

FIGURE 1. (a) Von Neumann Neighborhood (b) Moore Neighborhood (c)Restricted Vertical Neighborhood

Type of neighborhood cells in Figure 1. (a) Von neumann Neighborhood (VNN) (top, right, bottom and left) (b) Moore Neighborhood (All the nearest surrounding the center cell)(c) Restricted Vertical Neighborhood (RVN) (The class of ($\mathcal{CA}$) both cells in horizontal and either top (or) bottom not both in vertically)

The article is provide as pursue. In 2^{nd} section, the notion cash in the article are formally apparent. In 3^{rd} section, the algebraic structure of RVN and VnN of $2\mathbb{D}$ $\mathcal{CA}$ is obtained. In 4^{th} section, the rule of the uniform cellular automata ($\mathcal{UCA}$) with (RVN) is studied. In 5^{th} section, using both VNN and RVN the transition rule matrix of the hybrid cellular automata ($\mathcal{HYDCA}$) is obtained. In 6^{th} section, we count the rank of rule matrices relevant to $2\mathbb{D}$ $\mathcal{UCA}$ and reversibility of the $\mathcal{UCA}$ with RVN is studied.

2. PRELIMINARIES

Definition 2.1. [8]*A PB - $\mathcal{CA}$ is the left and right boundary cells connected to proximate cells.*

Definition 2.2. [8]*A NB - $\mathcal{CA}$ is both left and right boundary cells connected to 0 logic.*

Definition 2.3. [9]*Uniform Cellular Automata ($\mathcal{UCA}$): $\mathcal{UCA}$ is said to be all the cells applied to same rule.*

Definition 2.4. [9]*Linear Cellular Automata ($\mathcal{LCA}$): $\mathcal{LCA}$ is said to be the rule of $\mathcal{CA}$ involving only XOR logic then it is called linear CA.*

Definition 2.5. [2]*Hybrid Cellular Automata ($\mathcal{HYDCA}$): $\mathcal{HYDCA}$ is said to be the different rule applied for different cells.*

Definition 2.6. [7]*Reversibility of Cellular Automata: It is said that a $\mathcal{CA}$ can always be replaced by a $\mathcal{CA}$ that will return to its original state.*

Definition 2.7. [10]*Restricted Vertical Neighborhood* (RVN)*: The Restricted Vertical Neighborhood* RVN *rule is the class of $\mathcal{CA}$ horizontal both cells and vertical either top (or) bottom not both.*

Definition 2.8. [3]*Cellular Automata($\mathcal{CA}$): $\mathcal{CA}$ is defined as a quadruplets $\mathcal{A} = \{\mathbb{D}, \mathbb{Q}, \mathbb{N}, f\}$*
** $\mathbb{D} \in \mathbb{Z}_+$ is the dimension of the $\mathcal{CA}$.*
** $\mathbb{Q} = \{1, 2, ..., p\}$ is a countable set of states.*
** $\mathbb{N} = (\vec{n_1}, \vec{n_2}, ..., \vec{n_m})$ is the neighbor vector*

$*$ $f:\mathbb{Q}^m \to \mathbb{Q}$ is the local rule. f given the new states of a cell from the old neighbors states of the cells.

A configuration is a mapping $\mathbb{C}:\mathbb{Z}^{\mathbb{D}} \to \mathbb{Q}$. $\mathbb{C}^t$ is denote the time t, the cell move to next state at time t+1.

$\mathbb{C}^{t+1}(\vec{n}) = f(\mathbb{C}^t(\vec{n_1}), \mathbb{C}^t(\vec{n_2}), ..., \mathbb{C}^t(\vec{n_m}))$

now we consider the local rule f is a linear function

$\mathbb{C}^{t+1}(\vec{n}) = \lambda_1\mathbb{C}^t(\vec{n_1}) + \lambda_2\mathbb{C}^t(\vec{n_2}) + ... + \lambda_m\mathbb{C}^t(\vec{n_m}))$

λ_i is the co-efficient for neighborhood.

In this article, we discuss with $\mathcal{CA}$ defined by VNN and RVN under (NB) of $\mathcal{UCA}$ and $\mathcal{HYDCA}$. The state of the cell (r,s) at time t is represented by $\alpha^{(t)}_{(r,s)}$. The state of the cell (r,s) at time (t+1) is represented by $\alpha^{(t+1)}_{(r,s)} = \beta^{(t)}_{(r,s)}$.
In [6], show the configuration matrix. We associate RVN presentations with row vectors by transforming them from
$\mathbb{C}^{(t)}$ to $([\alpha]_{1\times mn})$=$(\alpha^{(t)}_{11}, \alpha^{(t)}_{12}, \cdots, \alpha^{(t)}_{1n}, \cdots, \alpha^{(t)}_{m1}, \cdots, \alpha^{(t)}_{mn})$.
where $\alpha^{(t)}_{(i,j)} \in \mathbb{Z}_2$.

Hence, the transition matrix $\mathcal{T_R}$ that changes set of states of cellular automata from (t) to (t+1) such that
$[(\alpha)_{(1\times mn)}.(\mathcal{T_R})_{(mn\times mn)}] = [(\beta)_{(mn\times 1)}]$,
where,
$[(\beta)_{mn\times 1}] = [(\beta^{(t)}_{11}, \beta^{(t)}_{12}, \cdots, \beta^{(t)}_{1n}, \cdots, \beta^{(t)}_{m1}, \cdots, \beta^{(t)}_{mn})]$.

3. 2$\mathbb{D}$ $\mathcal{CA}$ over the field $\mathbb{Z}_2$

In this section, we discuss with some types of neighbors. The cell arranged in m rows and n column in $m \times n$ grid. The cell move to next state at the time is denoted t+1.

3.1. **Restricted Vertical Neighborhood** (RVN)**-162 rule.** In 2$\mathbb{D}$ $\mathcal{CA}$ theory, there are a few classic sample of neighbors but in this article we only constrain ourselves to the specific neighbors which is called (RVN).

FIGURE 2. (a) Element of restricted vertical neighborhood surrounding the mid cell (b) Numbering of rules with respect to neighbors

In figure 2, we show the (RVN) which comprises 3 cells which encircle the midst cell $\alpha_{(r,s)}$. The next state of the cell at time t+1 is denoted by the local rule function.
$f:\mathbb{Z}_2^3 \to \mathbb{Z}_2$ as follows,

$$\alpha^{(t+1)}_{(r,s)} = f(\alpha_{(r,s+1)}, \alpha_{(r,s-1)}, \alpha_{(r-1,s)}) = \alpha_{(r,s+1)} + \alpha_{(r,s-1)} + \alpha_{(r-1,s)} \quad ... (1)$$

where, $\alpha_{(r,s)} \in \mathbb{Z}_2, r = 1,2,...,m$ and $s = 1,2,...,n$.

3.2. **Von Neumann Neighborhood** (VNN)**-170 rule.**

	$\alpha_{(p-1,q)}$	
$\alpha_{(p,q-1)}$	$\alpha_{(p,q)}$	$\alpha_{(p,q+1)}$
	$\alpha_{(p+1,q)}$	

(a)

	$2^7 = 128$	
$2^5 = 32$	$2^0 = 1$	$2^1 = 2$
	$2^3 = 8$	

(b)

FIGURE 3. (a) Element of Von Neumann neighborhood surrounding the mid cell (b) Numbering of rules with respect to neighbors

In figure 3, we show the VNN which comprises 4 square cell surrounding the midst cell $\alpha_{(r,s)}$. The state $\alpha^{(t+1)}_{(r,s)}$ function $f : \mathbb{Z}_2^4 \to \mathbb{Z}_2$ as follows.

$$\begin{aligned}\alpha^{(t+1)}_{(r,s)} &= f(\alpha_{(r,s+1)}, \alpha_{(r,s-1)}, \alpha_{(r-1,s)}, \alpha_{(r+1,s)}) \\ &= \alpha_{(r,s+1)} + \alpha_{(r,s-1)} + \alpha_{(r-1,s)} + \alpha_{(r+1,s)} \quad \ldots (2)\end{aligned}$$

where, $\alpha_{(r,s)} \in \mathbb{Z}_2, r = 1, 2, ..., m$ and $s = 1, 2, ..., n$.

4. Uniform Cellular Automata ($\mathcal{UCA}$):

4.1. **Rule Matrix of the Restricted Vertical Neighborhood.** We get the rule matrix of $2\mathbb{D}$ $\mathcal{CA}$ with (RVN) rule over the field $\mathbb{Z}_2$ underneath the (NB) condition. The rule matrix which takes the t^{th} finite configuration matrix $\mathbb{C}^t$ of order $m \times n$ to the $t+1^{th}$ state $\mathbb{C}^{t+1}$.

Theorem 4.1. *Let $m > 2$ and $n > 2$. Then the rule matrix $\mathcal{T}_{R-162}$ from $\mathbb{Z}_2^{mn} \to \mathbb{Z}_2^{mn}$ corresponding to the $2\mathbb{D}$ $\mathcal{UCA}$ that takes from configuration the state $\mathbb{C}^t$ of order $m \times n$ to the $t+1^{th}$ state $\mathbb{C}^{t+1}$ is given by,*

$$\mathcal{T}_{\mathcal{R}} = \begin{pmatrix} \mathcal{P} & \mathcal{I} & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & \mathcal{P} & \mathcal{I} & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \mathcal{P} & \mathcal{I} & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \mathcal{P} & \mathcal{I} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & \mathcal{P} & \mathcal{I} \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \mathcal{P} \end{pmatrix}_{mn \times mn}$$

where each partitioned matrix is of order $n \times n$.

$$\mathcal{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \end{pmatrix}_{n \times n}$$

where, $\mathcal{I}$ is an identity matrix and 0 is zero matrix is of order $n \times n$

Proof. Let us consider $\mathcal{T}_{\mathcal{R}}.\alpha_{(r,s)} = \beta_{(r,s)}$. $\beta_{(r,s)} = \alpha^{(t+1)}_{(r,s)}$ is a analogous to the linear clubbing of the neighbors in the following equation (1). The co-efficient of $\alpha_{(r,s)} = 0$ if

$r \leq 0$ *or* $s \leq 0$. By use the local rule of the $\mathcal{CA}$ we have congest the following,
$\beta_{(1,1)} = \alpha_{(1,2)}$
$\beta_{(1,2)} = \alpha_{(1,3)} + \alpha_{(1,1)}$, $2 \leq s \leq n-1$
$\beta_{(1,s)} = \alpha_{(1,s+1)} + \alpha_{(1,s-1)}$
$\beta_{(1,n)} = \alpha_{(1,n-1)}$
For $2 \leq r \leq m-1$ we have,
$\beta_{(r,1)} = \alpha_{(r,2)} + \alpha_{(r-1,1)}$
$\beta_{(r,s)} = \alpha_{(r,s+1)} + \alpha_{(r,s-1)} + \alpha_{(r-1,s)}$, $2 \leq s \leq n-1$
$\beta_{(r,n)} = \alpha_{(r,n-1)} + \alpha_{(r-1,n)}$
Finally, we have
$\beta_{(m,1)} = \alpha_{(m,2)} + \alpha_{(m-1,1)}$
$\beta_{(m,s)} = \alpha_{(m,s+1)} + \alpha_{(m,s-1)} + \alpha_{(m-1,s)}$, $2 \leq s \leq n-1$
$\beta_{(m,n)} = \alpha_{(m,n-1)} + \alpha_{(m-1,n)}$
Finally we get the rule matrix $\mathcal{T}_{R-162}$. □

Example 4.1. *Let* $m = 4$ *and* $n = 4$ *then, we obtain the rule matrix* $\mathcal{T}_{R-162}$ *of* $2\mathbb{D}$ $\mathcal{CA}$ *with* RVN *rule over the field* $\mathbb{Z}_2$ *be as follows,*

$$\mathcal{T}_{R-162} = \begin{pmatrix} \mathcal{P} & \mathcal{I} & 0 & 0 \\ 0 & \mathcal{P} & \mathcal{I} & 0 \\ 0 & 0 & \mathcal{P} & \mathcal{I} \\ 0 & 0 & 0 & \mathcal{P} \end{pmatrix}$$

where, $$\mathcal{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\mathcal{P}, \mathcal{I}$ *are the sub matrices of order* (4×4)*, 0 is the zero matrices and I is an identity matrix.*

4.2. **Rule Matrix of the Von Neumann Neighborhood.** We get the rule matrix of $2\mathbb{D}$ $\mathcal{CA}$ with (VNN) rule over the field $\mathbb{Z}_2$ underneath the (NB) condition. The rule matrix which takes the t^{th} finite configuration matrix $\mathbb{C}^t$ of order $m \times n$ to the $t+1^{th}$ state $\mathbb{C}^{t+1}$.

Theorem 4.2. *Let* $m > 2$ *and* $n > 2$*. Then the rule matrix* $\mathcal{T}_{R-170}$ *from* $\mathbb{Z}_2^{mn} \to \mathbb{Z}_2^{mn}$ *corresponding to the* $2\mathbb{D}$ $\mathcal{UCA}$ *that takes from configuration the state* $\mathbb{C}^t$ *of order* $m \times n$ *to the* $t+1^{th}$ *state* $\mathbb{C}^{t+1}$ *is given by,*

$$\mathcal{T}_{\mathcal{R}} = \begin{pmatrix} \mathcal{P} & \mathcal{I} & 0 & 0 & 0 & \dots & 0 & 0 \\ \mathcal{I} & \mathcal{P} & \mathcal{I} & 0 & 0 & \dots & 0 & 0 \\ 0 & \mathcal{I} & \mathcal{P} & \mathcal{I} & 0 & \dots & 0 & 0 \\ 0 & 0 & \mathcal{I} & \mathcal{P} & \mathcal{I} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & \mathcal{P} & \mathcal{I} \\ 0 & 0 & 0 & 0 & 0 & \dots & \mathcal{I} & \mathcal{P} \end{pmatrix}_{mn \times mn}$$

where each partitioned matrix is of order $n \times n$*.*

$$\mathcal{P}=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0\\ 1 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0\\ 0 & 1 & 0 & 1 & 0 & \dots & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 1 & \dots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}_{n\times n}$$

where, $\mathcal{I}$ is an identity matrix and 0 is zero matrix is of order $n \times n$

Proof. The proof of the Theorem following alike steps as the proof of Theorem 4.1 □

5. Hybrid Cellular Automata ($\mathcal{HDYCA}$)

In the aspect study, the work with specific $2\mathbb{D}$ $\mathcal{CA}$ defined by ($\mathcal{HDYCA}$) rule over the field $\mathbb{Z}_2$ under (NB) condition. We will resolve the rule matrix $\mathbb{C}^t$ of matrix of order $m \times n$.

Case(i). m is even, the rule matrix $\mathcal{T}_\mathcal{R}$ is given in the following theorem

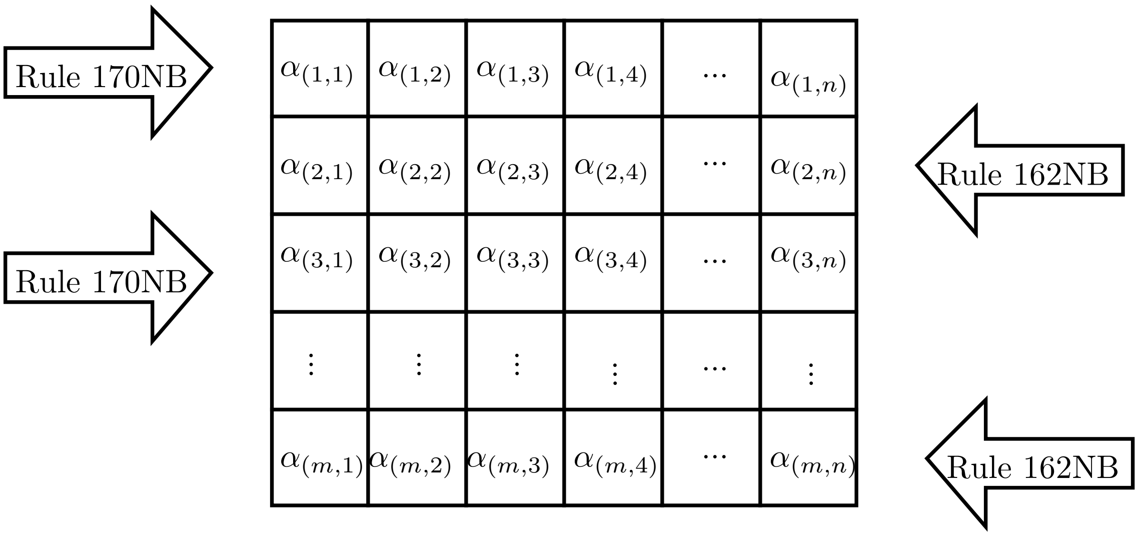

FIGURE 4. Hybrid rule means that it is applied 170 NB and 162 NB respectively for each rows when m is even on $(m \times n)$ CA

Theorem 5.1. *Let us consider $m > 2$ and $n > 2$. Then the rule matrix $\mathcal{T}_\mathcal{R} - (HYDCA)$ from $\mathbb{Z}_2^{mn} \to \mathbb{Z}_2^{mn}$ which takes t^{th} state to $(t+1)^{th}$ state is given by*

$$\mathcal{T}_\mathcal{R} - (HYDCA)=\begin{pmatrix} \mathcal{P} & \mathcal{I} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0\\ \mathcal{I} & \mathcal{P} & \mathcal{I} & 0 & 0 & 0 & \dots & 0 & 0 & 0\\ 0 & 0 & \mathcal{P} & \mathcal{I} & 0 & 0 & \dots & 0 & 0 & 0\\ 0 & 0 & \mathcal{I} & \mathcal{P} & \mathcal{I} & 0 & \dots & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \mathcal{P} & \mathcal{I} & \dots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \mathcal{P} & \mathcal{I}\\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \mathcal{I} & \mathcal{P} \end{pmatrix}_{mn\times mn}$$

where each partitioned matrix is of order $n \times n$.

$$\mathcal{P}=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0\\ 1 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0\\ 0 & 1 & 0 & 1 & 0 & \dots & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 1 & \dots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}_{n\times n}$$

where, $\mathcal{I}$ is an identity matrix and 0 is zero matrix is of order $n \times n$

Proof. Let us consider $\mathcal{T}_{\mathcal{R}}.\alpha_{(r,s)} = \beta_{(r,s)}$. $\beta_{(r,s)} = \alpha_{(r,s)}^{(t+1)}$ is a analogous to the linear clubbing of the neighbors in the following equation (1) and (2). The co-efficient of $\alpha_{(r,s)} = 0$ if $r \leq 0$ *or* $s \leq 0$. By use the local rule of the $\mathcal{CA}$ we have congest the following,
$\beta_{(1,1)} = \alpha_{(1,2)} + \alpha_{(2,1)}$
$\beta_{(1,s)} = \alpha_{(1,s+1)} + \alpha_{(1,s-1)} + \alpha_{(2,s)},\ 2 \leq s \leq n-1$
$\beta_{(1,n)} = \alpha_{(1,n-1)} + \alpha_{(2,n)}$
when m is odd
$\beta_{(r,s)} = \alpha_{(r,s+1)} + \alpha_{(r,s-1)} + \alpha_{(r-1,s)} + \alpha_{(r+1,s)},\ 2 \leq r \leq m-1$
$\beta_{(2,1)} = \alpha_{(2,2)} + \alpha_{(1,1)}$
$\beta_{(2,s)} = \alpha_{(2,s+1)} + \alpha_{(2,s-1)} + \alpha_{(1,s)},\ 2 \leq s \leq n-1$
$\beta_{(2,n)} = \alpha_{(2,n-1)} + \alpha_{(1,n)}$
when m is even
$\beta_{(r,s)} = \alpha_{(r,s+1)} + \alpha_{(r,s-1)} + \alpha_{(r-1,s)},\ 2 \leq r \leq m-1$
Finally we get the rule matrix $\mathcal{T}_{R-HYDCA}$. □

Case(ii). m is odd, the rule matrix $\mathcal{T}_{\mathcal{R}}$ is given in the following theorem

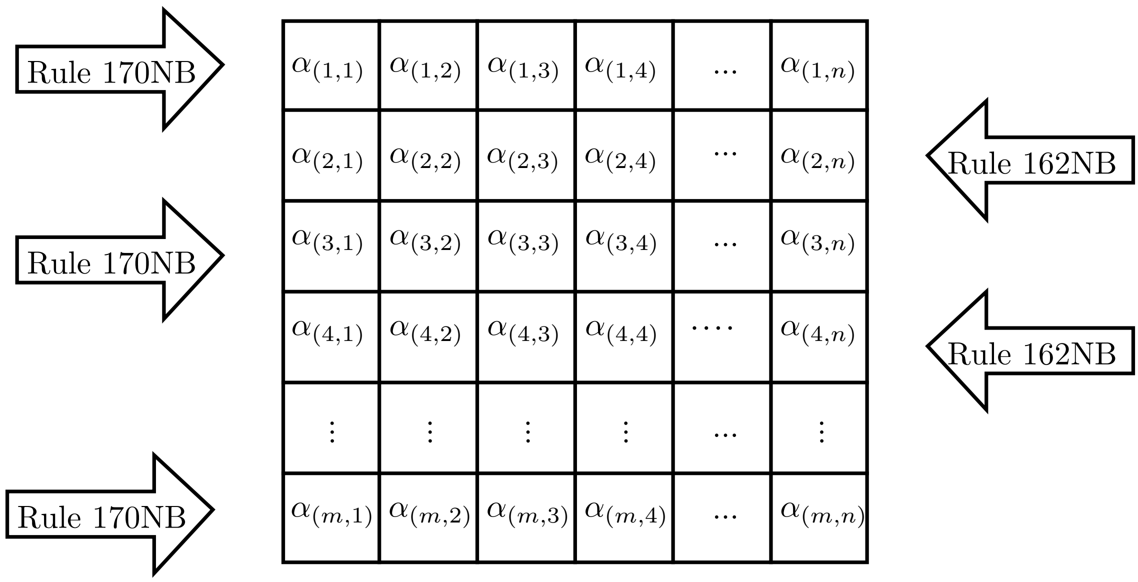

FIGURE 5. Hybrid rule means that it is applied 170 NB and 162 NB respectively for each rows when m is odd on $(m \times n)$ CA

Theorem 5.2. *Let us consider $m > 2$ and $n > 2$. Then the rule matrix $\mathcal{T}_{\mathcal{R}} - (HYDCA)$ from $\mathbb{Z}_2^{mn} \to \mathbb{Z}_2^{mn}$ which takes t^{th} state to $(t+1)^{th}$ state is given by*

$$\mathcal{T_R} - HYDCA = \begin{pmatrix} \mathcal{P} & \mathcal{I} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \mathcal{I} & \mathcal{P} & \mathcal{I} & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \mathcal{P} & \mathcal{I} & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \mathcal{I} & \mathcal{P} & \mathcal{I} & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{P} & \mathcal{I} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \mathcal{P} & \mathcal{I} \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \mathcal{P} \end{pmatrix}_{mn \times mn}$$

where each partitioned matrix is of order $n \times n$.

$$\mathcal{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}_{n \times n}$$

where, $\mathcal{I}$ is an identity matrix and 0 is zero matrix is of order $n \times n$

Proof. The proof of the Theorem following alike steps as the proof of Theorem 5.1 □

6. Reversible of $\mathcal{UCA}$ with RVN

This is section we discuss with reversible of $\mathcal{CA}$. If the rule matrix has full rank then the matrix is invertible, then $2\mathbb{D}$ $\mathcal{CA}$ is reversible otherwise it is irreversible.

Lemma 6.1. *For $n > 2$ is odd, hence rank ($\mathcal{P}$)= n-1.*

Proof. We put induction on n.
For n=3, then

$$\mathcal{P} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$C_1 \Leftrightarrow C_2$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} R_3 \Rightarrow R_1 - R_3$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

the rank of 3×3 matrix is equals to 2.
for n = 5, then

$$\mathcal{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$C_1 \Leftrightarrow C_2$

$$\mathcal{P}=\begin{pmatrix}1&0&0&0&0\\0&1&1&0&0\\1&0&0&1&0\\0&0&1&0&1\\0&0&0&1&0\end{pmatrix}\ R_3 \Rightarrow R_1 - R_3$$

$$\mathcal{P}=\begin{pmatrix}1&0&0&0&0\\0&1&1&0&0\\0&0&0&1&0\\0&0&1&0&1\\0&0&0&1&0\end{pmatrix}\ R_5 \Rightarrow R_3 + R_5$$

$C_4 \Leftrightarrow C_3$

$$\begin{pmatrix}1&0&0&0&0\\0&1&1&0&0\\0&0&1&0&1\\0&0&0&1&0\\0&0&0&0&0\end{pmatrix}$$

the rank of 5×5 matrix is equals to 4.
Inductively, by use correspondent row operation, we get the result.
Hence, when n is odd the rank of matrix $\mathcal{P}$ is (n-1). □

Lemma 6.2. *For all even $n > 3$, rank $(\mathcal{P}) = n$. Then $\mathcal{P}$ is invertible.*

Proof. We put induction on n.
For n = 4

$$\mathcal{P} = \begin{pmatrix}0&1&0&0\\1&0&1&0\\0&1&0&1\\0&0&1&0\end{pmatrix}$$

$$det(\mathcal{P}_{(4\times4)}) = (-1)\begin{pmatrix}1&1&0\\0&0&1\\0&1&0\end{pmatrix}$$

$$=(-1)^2$$

$det(\mathcal{P}_{(4\times4)}) = (-1)^2$
For n = 6

$$\mathcal{P} = \begin{pmatrix}0&1&0&0&0&0\\1&0&1&0&0&0\\0&1&0&1&0&0\\0&0&1&0&1&0\\0&0&0&1&0&1\\0&0&0&0&1&0\end{pmatrix}$$

$$= -1[1(det(\mathcal{P}_{(4\times4)})) - (0)]$$

$$=(-1)^3$$

$det(\mathcal{P}_{(6\times6)}) = (-1)^3$
for n = 2k+2

$$\mathcal{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}_{(2k+2)\times(2k+2)}$$

$= -1[1(det(P_{(2k\times 2k)})) - 0]$

$=(-1)^{k+1}$

If the result is true for $n = 2k + 2$, we prove that it is also true for $n = 2k + 4$.

$$\mathcal{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots\vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}_{(2k+4)\times(2k+4)}$$

$= -1[1(det(P_{(2k+2\times 2k+2)})) - 0]$

$=(-1)^{k+2}$

then $\mathcal{P}$ is invertible.

□

Theorem 6.1. *For every rule matrix of n is odd, then T_{R-162} is not reversible in uniform cellular automata.*

Theorem 6.2. *For every rule matrix of n is even, then T_{R-162} is reversible in uniform cellular automata.*

7. Conclusions (mandatory)

In this article, $2\mathbb{D}$ $\mathcal{CA}$ of RVN and VNN are discussed. The rule matrix $\mathcal{T}_{\mathcal{R}}$ of $2\mathbb{D}$ $\mathcal{CA}$ is computed. We show that (i) Every rule matrix of n is odd, hence $\mathcal{T}_{\mathcal{R}} - 162$ is not reversible. (ii) Every rule matrix of n is even, hence $\mathcal{T}_{\mathcal{R}} - 162$ is reversible in Uniform Cellular Automata. In future i will try to apply the concept of this article to the new cellular automata over the field $\mathbb{Z}_3$.

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