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RESULTS ON MÖBIUS INDEX FOR STANDARD GRAPHS

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Abstract. This paper is concerned with calculating the Möbius index values and arrived results for a few standard graphs. Some standard graphs which we considered are Path graph, Cycle graph, Complete Graph, Star Graph, Shell graph, Wheel Graph, Gear graph, Helm Graph, the Web graph, Flower Graph.

Keywords: Möbius index, Standard Graphs, G_n, H_n, F_n, Web Graph.

AMS Subject Classification: 05C10.

1. Introduction

The department of chemical sciences that deals with the chemical formation with the assistance of scientific instruments are known as computational chemistry. In a chemical graph nodes represent atoms or molecules and edges represent the chemical bonding between the atoms or molecules. The number of nodes of G, adjoining to a given node v, is the valency of that node. The concept of the degree in graph theory is closely related to the concept of valence in chemistry. Various researchers have conducted studies on topological indices for different graph families; these records have vital chemical significance in the fields of chemical graph theory, molecular topology, and numerical chemistry. Vasumathi [5] carried an unused class of graph specifically Möbius graph through allotting the Möbius function on the nearness of 2 vertices in a graph in the year 1994. In 2019, Aravinth and R. Vignesh [1] derived the concept of Möbius function graphs and provided results regarding the basic properties of $M_n(G)$ and induced proper coloring were given. All graphs utilized in this article are basic and undirected for the reason that there would not be any numerous edges or loops. In this article, we compute some newly defined Möbius index values of some known standard graphs. For notations and graph theory terminologies which we defined here are followed as in [6].

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2. Preliminaries

Definition 2.1. [4] *A gear graph G_n is constructed from W_n by attaching a vertex between every pair of adjacent vertices of rim of the wheel W_n.*

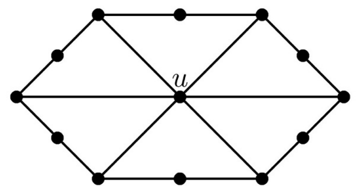

Figure 1. Gear Graph on 6 vertices

Definition 2.2. [4] *A helm graph H_n, $n > 2$ is the graph obtained from the wheel W_n by adding a pendant edge at each vertex on the rim of the wheel W_n.*

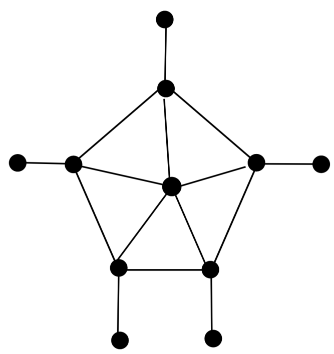

Figure 2. Helm Graph on 5 vertices

Definition 2.3. [2] *A flower graph F_n is the graph obtained from a H_n by joining each pendant vertex to the central vertex of the helm H_n.*

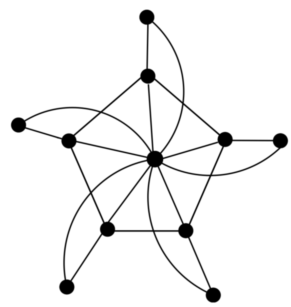

Figure 3. Flower Graph on 5 vertices

Definition 2.4. [3] *A generalized web graph $W(n, m)$, $n > 2$, $m > 1$, is a graph obtained by joining all vertices of the generalized prism P_n^m to a central vertex c.*

Definition 2.5. [1] *The Möbius relation is characterized through $\mu(1) = 1$ and in case $n > 1$, at that point compose $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$.*

$$\mu(n) = \begin{cases} (-1)^r, & \text{if } a_1 = a_2 = \dots = a_r = 1; \\ 0, & \text{otherwise.} \end{cases}$$

3. Results on Möbius index of a graph

Definition 3.1. Let G be a simple graph. Now we define a Möbius index of G as $M(G) = \sum\limits_{u \in V} \left[d_u\right]^{\mu(d_u)}$, where d_u is the degree of the vertex u.

Example 3.1. *Let $G = K_4$. Each vertices in K_4 will be of degree 3, which has been denoted at the vertices in the figure 4.*

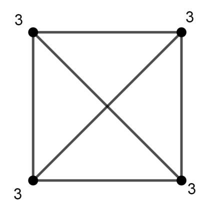

Figure 4. Complete Graph on 4 vertices

$$\begin{aligned} M(G) &= \sum_{u \in V} [3]^{\mu(3)} \\ &= 4(3^{-1}) = \frac{4}{3}. \end{aligned}$$

Theorem 3.1. $M(P_n) = \frac{n+2}{2}$.

Proof. For any path P_n, there will be exactly 2 vertices of degree 1 which are also known as pendant vertices and $\mu(1) = 1$. Thus, these 2 vertices contribute 2 to the sum and the remaining $n-2$ nodes have the degree 2 then $\mu(2) = -1$. Hence we have

$$\begin{aligned} M(P_n) &= 2(1)^{\mu(1)} + (n-2)(2)^{\mu(2)} \\ &= 2 + (n-2)2^{-1} \\ &= 2 + \frac{n-2}{2} \\ &= \frac{4+n-2}{2} \\ &= \frac{n+2}{2}. \end{aligned}$$

□

Result 3.1. $M(C_n) = \frac{n}{2}$.

Proof. It is clear that all vertices in C_n receives the degree 2 and we know that $\mu(2) = -1$ (from the definition of Möbius Function), we have

$$\begin{aligned} M(C_n) &= \sum_{u \in V} [d_u]^{\mu(d_u)} \\ &= \sum_{u \in V} [2]^{\mu(2)} = \sum_{u \in V} \left[2^{-1}\right] \\ &= n\left[[2^{-1}\right] \\ &= \frac{n}{2}. \end{aligned}$$

□

Note 1. *For $n \geq 4$, $M(C_n) = \frac{n}{2} = M(P_{n-2})$.*

Theorem 3.2. $M(K_n) = \begin{cases} \frac{n}{n-1} & ;\ \mu(n-1) = -1, \\ n^2 - n & ;\ \mu(n-1) = 1, \\ n & ;\ \mu(n-1) = 0. \end{cases}$

Proof. Since every complete graph is $n-1$ regular, so the valency of all n nodes is $n-1$.

Case (i.) Supposing that $n-1$ has odd number of non - composite factors, we have $\mu(n-1) = -1$. So,

$$\begin{aligned} M(K_n) &= n(n-1)^{\mu(n-1)} \\ &= n(n-1)^{-1} \\ &= \frac{n}{n-1}. \end{aligned}$$

Case (ii.) In the event that $n-1$ has even number of non - composite factors, then $\mu(n-1) = 1$. So,

$$\begin{aligned} M(K_n) &= n(n-1)^{\mu(n-1)} \\ &= n(n-1)^{1} \\ &= n^2 - n. \end{aligned}$$

Case (iii.) In the event that $n-1$ contains a squared prime factor, then $\mu(n-1) = 0$. So,

$$\begin{aligned} M(K_n) &= n(n-1)^{\mu(n-1)} \\ &= n(n-1)^{0} \\ &= n. \end{aligned}$$

□

Theorem 3.3. $M(K_{1,n}) = \begin{cases} \frac{n^2+1}{n} & ;\ \mu(n) = -1, \\ 2n & ;\ \mu(n) = 1, \\ n+1 & ;\ \mu(n) = 0. \end{cases}$

Proof. $K_{1,n}$ is a bipartite graph also known as star graph with 2 partitions consisting of 1 in a set of degree $n-1$ and the remaining $n-1$ independent vertices in the another set of degree 1.

Case (i.) If n has odd number of prime factors, then $\mu(n) = -1$. So,

$$\begin{aligned} M(K_{1,n}) &= n^{\mu(n)} + n(1^1) \\ &= n^{-1} + n \\ &= \frac{1}{n} + n \\ &= \frac{n^2+1}{n}. \end{aligned}$$

Case (ii.) If n contains even number of non - composite factors, then $\mu(n) = 1$. So,

$$\begin{aligned} M(K_{1,n}) &= n^{\mu(n)} + n(1^1) \\ &= n^1 + n \\ &= 2n. \end{aligned}$$

Case (iii.) Suppose n contains a squared non - composite factor, then $\mu(n) = 0$. So,

$$\begin{aligned} M(K_{1,n}) &= n^{\mu(n)} + n \\ &= n^0 + n \\ &= n + 1. \end{aligned}$$

□

Theorem 3.4. $M(S_n) = \begin{cases} \frac{n^2-n+3}{3(n-1)} & ;\ \mu(n-1) = -1, \\ \frac{4n-3}{3} & ;\ \mu(n-1) = 1, \\ \frac{n+3}{3} & ;\ \mu(n-1) = 0. \end{cases}$

Proof. S_n is an apex in which a vertex sharing $(n-3)$ chords has the degree $n-1$. Further, it is adjacent to all the vertices of $n-$cycle and there will be exactly 2 vertices of degree 2 and $(n-3)$ vertices of degree 3.

Case (i.) Supposing that $n-1$ has odd number of non - composite factors, at that point $\mu(n-1) = -1$. Now we have

$$\begin{aligned} M(S_n) &= (n-1)^{\mu(n-1)} + (n-3)(3^{-1}) + 2(2^{-1}) \\ &= \frac{1}{n-1} + \frac{n-3}{3} + 1 \\ &= \frac{n^2-n+3}{3(n-1)}. \end{aligned}$$

Case (ii.) In the event that $n-1$ contains an even number of non - composite factors, then $\mu(n-1) = 1$. So,

$$\begin{aligned} M(S_n) &= (n-1)^{\mu(n-1)} + (n-3)(3^{-1}) + 2(2^{-1}) \\ &= (n-1)^1 + \frac{n-3}{3} + 1 \\ &= n + \frac{n-3}{3} \\ &= 4n - 3 \end{aligned}$$

Case (iii.) Suppose $n-1$ contains a squared non - composite factor, then $\mu(n-1) = 0$. So,

$$\begin{aligned} M(S_n) &= (n-1)^{\mu(n-1)} + \frac{n-3}{3} + 1 \\ &= (n-1)^0 + \frac{n-3}{3} + 1 \\ &= 2 + \frac{n-3}{3} \\ &= \frac{n+3}{3}. \end{aligned}$$

□

Theorem 3.5. $M(W_n) = \begin{cases} \frac{n^2+3}{3n} & ; \mu(n) = -1, \\ \frac{4n}{3} & ; \mu(n) = 1, \\ \frac{n+3}{3} & ; \mu(n) = 0. \end{cases}$

Proof. Since W_n has a cycle C_n with K_1 at the centre, so we have all n vertices of degree 3 and C_n and the centre vertex is of degree exactly n.

Case (i.) Supposing that n contains odd number of non - composite factors, then $\mu(n) = -1$.

$$\begin{aligned} M(W_n) &= n(3)^{-1} + n^{\mu(n)} \\ &= \frac{1}{n} + \frac{n}{3} \\ &= \frac{n^2+3}{3n}. \end{aligned}$$

Case (ii.) Supposing that n contains even number of non - composite factors, then $\mu(n) = 1$. So,

$$\begin{aligned} M(W_n) &= n(3)^{-1} + n^{\mu(n)} \\ &= \frac{n}{3} + n^1 \\ &= n + \frac{n}{3} \\ &= \frac{4n}{3}. \end{aligned}$$

Case (iii.) Suppose n contains a squared non - composite factor, then $\mu(n) = 0$. So,

$$\begin{aligned} M(W_n) &= n(3)^{-1} + n^{\mu(n)} \\ &= \frac{n}{3} + 1 \\ &= \frac{n+3}{3}. \end{aligned}$$

□

Theorem 3.6. $M(G_n) = \begin{cases} \frac{5n^2+6}{6n} & ; \mu(n) = -1, \\ \frac{11n}{6} & ; \mu(n) = 1, \\ \frac{5n+6}{6} & ; \mu(n) = 0. \end{cases}$

Proof. Since G_n is obtained from W_n and so there will be n more vertices of degree 2.

Case (i.) Supposing that n features an odd number of non - composite factors, then $\mu(n) = -1$. It is enough to add $\frac{n}{2}$ to the value of W_n.

$$\begin{aligned} M(G_n) &= \frac{n^2+3}{3n} + \frac{n}{2} \\ &= \frac{2n^2+3n^2+6}{6n} \\ &= \frac{5n^2+6}{6n}. \end{aligned}$$

Case (ii.) Supposing that n features an even number of non - composite factors,, then $\mu(n) = 1$. Now we have to add same $\frac{n}{2}$ to the value of W_n.

$$\begin{aligned} M(G_n) &= \frac{4n}{3} + \frac{n}{2} \\ &= \frac{8n+3n}{3} \\ &= \frac{11n}{6}. \end{aligned}$$

Case (iii.) Suppose n features a squared non - composite factor, then $\mu(n) = 0$. Now we have to add same $\frac{n}{2}$ to the value of W_n.

$$\begin{aligned} M(G_n) &= \frac{n+3}{3} + \frac{n}{2} \\ &= \frac{2n+6+3n}{6} \\ &= \frac{5n+6}{6}. \end{aligned}$$

□

Theorem 3.7. $M(H_n) = \begin{cases} \frac{2n^2+1}{n} & ;\ \mu(n) = -1, \\ 3n & ;\ \mu(n) = -1, \\ 2n+1 & ;\ \mu(n) = 0. \end{cases}$

Proof. Let H_n be a Helm graph on n vertices.

Case (i.) In case n includes an odd number of non - composite factors, then $\mu(n) = -1$. Since H_n is obtained from W_n through joining a terminal edge at every node of C_n, then the degree of each n vertices on the $n-$cycle is 4. We know that $\mu(4) = 0$. Thus, it contributes $n(4^0) = n$ to the sum. Already the central node features a degree n and all those pendant edges contributes the degree 1 to all the n vertices outside the $n-$cycle. We have

$$\begin{aligned} M(H_n) &= n(4^0) + n^{\mu(n)} + n(1^{\mu(1)}) \\ &= n + n^{-1} + n(1^1) \\ &= 2n + \frac{1}{n} \\ &= \frac{2n^2+1}{n}. \end{aligned}$$

Case (ii.) If n includes an even number of non - composite factors, then $\mu(n) = 1$ and hence we have

$$\begin{aligned} M(H_n) &= n(4^0) + n^{\mu(n)} + n \\ &= 3n. \end{aligned}$$

Case (iii.) In case n includes a squared non - composite factor, then $\mu(n) = 0$.

$$\begin{aligned} M(H_n) &= n(4^0) + n^{\mu(n)} + n \\ &= 2n + 1. \end{aligned}$$

□

Theorem 3.8. $M(Web) = \begin{cases} \frac{3n^2+1}{n} & ;\ \mu(n) \ = \ -1, \\ 4n & ;\ \mu(n) \ = \ 1, \\ 3n+1 & if\ \mu(n) \ = \ 0. \end{cases}$

Proof. In Web graph of n vertices central vertex has degree n. There are two $n-$cycles containing $2n$ vertices and the degree of each vertex is 4, so we have $\mu(4) = 0$. There are n vertices of degree 1 lies outside of the outer $n-$cycle. So we have $\mu(1) = 1$.

Case (i.) If n has odd number of prime factors, then $\mu(n) = -1$. We have

$$\begin{aligned} M(Web) &= (n)^{\mu(n)} + (2n)(4^0) + n(1)^1 \\ &= \frac{1}{n} + 3n \\ &= \frac{3n^2+1}{n}. \end{aligned}$$

Case (ii.) If n contains an even number of non - composite factors, then $\mu(n) = 1$ and hence we have

$$\begin{aligned} M(Web) &= 2n(4^0) + n^{\mu(n)} + n \\ &= 4n. \end{aligned}$$

Case (iii.) In case n features a squared non - composite factor, then $\mu(n) = 0$.

$$\begin{aligned} M(Web) &= n^{\mu(n)} + 3n \\ &= 3n + 1. \end{aligned}$$

□

Theorem 3.9. $M(F_n) = \begin{cases} \frac{7n}{2} & ;\ n = odd\ with\ \ \mu(n) \ = \ -1, \\ \frac{3n^2+1}{2n} & ;\ n = odd\ with\ \ \mu(n) \ = \ 1, \\ \frac{3n+2}{2} & for\ all\ other\ values\ of\ n. \end{cases}$

Proof. Let F_n be a Flower graph on n vertices.

Case (i.) In the event that n has odd number of non - composite factors, so it means there must not be 2 as its factor. Therefore, $\mu(n) = -1$. Now the central vertex is of degree $2n$. Now $2n$ features an even number of non - composite factors and thus $\mu(2n) = 1$. Already there are n vertices of degree each 4 on $n-$cycle and n vertices outside the $n-$cycle have degree each 2. We have

$$\begin{aligned}M(F_n) &= (2n)^{\mu(2n)} + n(4^0) + n(2^{-1})\\ &= (2n)^1 + n(1) + \frac{n}{2}\\ &= 3n + \frac{n}{2}\\ &= \frac{7n}{2}.\end{aligned}$$

Case (ii.) If n is odd with even number of prime factors, then $\mu(n) = 1$. Now $2n$ must have odd number of prime factors and also n has no even prime factor which means $\mu(2n) = -1$. Hence we have

$$\begin{aligned}M(F_n) &= (2n)^{\mu(2n)} + n(4^0) + n(2^{-1})\\ &= (2n)^{-1} + n + \frac{n}{2}\\ &= \frac{1}{2n} + \frac{3n}{2}\\ &= \frac{1}{2}\left[\frac{1}{n} + 3n\right]\\ &= \frac{1}{2}\left[\frac{3n^2+1}{n}\right]\\ &= \frac{3n^2+1}{2n}\end{aligned}$$

Case (iii.)

(1) In case $n =$ even with even number of non - composite factors, then we have $\mu(n) = 1$ and the number n has 2 as its one of the prime factor. Hence the number $2n$ incorporates a squared non - composite factor, so $\mu(2n) = 0$.
(2) In case $n =$ even with odd number of prime factors, then we have $\mu(n) = -1$, since n is even it has already 2 as its factor and hence the number $2n$ incorporates a squared non - composite factor, so $\mu(2n) = 0$.
(3) In case n has a squared prime factor, hence $2n$ also. Therefore $\mu(2n) = 0$.

In the above all sub-cases, we have

$$\begin{aligned}M(F_n) &= (2n)^{\mu(2n)} + n(4^0) + n(2^{-1})\\ &= (2n)^0 + n + \frac{n}{2}\\ &= 1 + \frac{3n}{2}\\ &= \frac{3n+2}{2}.\end{aligned}$$

□

4. Numerical Examples

In this section, we give few numerical examples consisting of Helm Graph, Flower Graph and Web Graph.

On considering the Helm graph H_n, we have

i.) $n = 4, \mu(4) = 0$, we have $M(H_4) = 2(4) + 1 = 9$.
ii.) $n = 5, \mu(5) = -1$, we have $M(H_5) = \frac{2(5)^2+1}{5} = 10.2$.
iii.) $n = 6, \mu(4) = 1$, we have $M(H_6) = 3(6) = 18$.

On considering the Web graph Web_n, we have
i.) $n = 4, \mu(4) = 0$, we have $M(Web_4) = 3(4) + 1 = 13$.
ii.) $n = 5, \mu(5) = -1$, we have $M(Web_5) = \frac{3(5)^2+1}{5} = 15.2$.
iii.) $n = 6, \mu(4) = 1$, we have $M(Web_6) = 4(6) = 24$.

On considering the Flower graph F_n, we have
i.) $n = 4, \mu(4) = 0$, we have $M(F_4) = \frac{3(4)+2}{2} = 7$.
ii.) $n = 5, \mu(5) = -1$, we have $M(F_5) = \frac{7(5)}{2} = 17.5$.
iii.) $n = 6, \mu(4) = 1$, we have $M(F_6) = \frac{3(6)^2+1}{12} = 9.08$.

Here, we gave the numerical calculations of the Möbius Index of Helm, Web, and Flower graphs.

Graph Type	M($n = 4$)	M($n = 5$)	M($n = 6$)
H_n	9	10.2	18
Web_n	13	15.2	24
F_n	7	17.5	9.08

Table 1. Numerical Examples for Helm, Web and Flower graphs for n = 4, 5 and 6.

5. Conclusions

Whereas this paper gives an introduction to the think about of Möbius index values. Results regarding the study of Möbius index values for distinctive standard graphs. As future work, we envisage applying these definitions to the line graph, subdivision graphs, and, total graphs of the Standard Graphs. This Möbius index values may be mainly used for the formulation of medicines, industry, and electronics which may indicate their threshold values based on their nodes.

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R. Vignesh for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.10, N.3.