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ON RARELY FUZZY e-CONTINUOUS FUNCTIONS IN THE SENSE OF Š OSTAK'S

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Abstract. In this paper, we introduce the concepts of rarely fuzzy e-continuous functions in the sense of Šostak's is introduced. Some interesting properties and characterizations of rarely fuzzy e-continuous and weakly fuzzy e-continuous are investigated. Also, fuzzy $eT_{1/2}$-space, rarely fuzzy eT_2-spaces and some applications to fuzzy compact spaces are established.

Keywords: Rarely fuzzy e-continuous, fuzzy e-compact space, rarely fuzzy e-almost compact space, fuzzy $eT_{1/2}$-space, and rarely fe-T_2-spaces.

AMS Subject Classification: 54A40.

1. Introduction

Kubiak [12] and Šostak [17] introduced the fundamental concept of a fuzzy topological structure, as an extension of both crisp topology and fuzzy topology [2], in the sense that not only the objects are fuzzified, but also the axiomatics. In [18, 19], Šostak gave some rules and showed how such an extension can be realized. Chattopadhyay et al., [4] have redefined the same concept under the name gradation of openness.

In 2008, the initiations of e-open and e-closed sets in topological spaces was introduced by Ekici [7]. Thereafter Ekici [5, 6, 8, 9] has introduced new classes of sets called e^*-open sets and a-open sets to establish some new decompositions of continuous functions. By using new notions of e-continuous functions, e^*-continuous functions and a-continuous functions via e-open sets, e^*-open sets and a-open sets, respectively. Popa [15] introduced the notion of rarely continuity as a generalization of weak continuity [13] which has been further investigated by Long and Herrington [14] and Jafari [10] and [11].

Recently Sobana et al. [20] introduced the concept of fuzzy e-open and fuzzy e-closed sets in fuzzy topological spaces in the sense of Šostak's. In this paper, we introduce the concepts of rarely fuzzy e-continuous functions in the sense of Sostak's. Some interesting

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properties and characterizations of them are investigated. Also, some applications to fuzzy compact spaces are established.

2. Preliminaries

Throughout this paper, let X be a nonempty set, $I = [0,\ 1]$ and $I_0 = (0,\ 1]$. For $\lambda \in I^X$, $\overline{\lambda}(x) = \lambda$ for all $x \in X$. For $x \in X$ and $t \in I_0$, a fuzzy point x_t is defined by $x_t(y) = \begin{cases} t & \text{if } y = x \\ 0 & \text{if } y \neq x. \end{cases}$ Let $Pt(X)$ be the family of all fuzzy points in X. A fuzzy point $x_t \in \lambda$ iff $t < \lambda(x)$. All other notations and definitions are standard, for all in the fuzzy set theory.

Definition 2.1. [17] *A function $\tau : I^X \to I$ is called a fuzzy topology on X if it satisfies the following conditions:*

(O1) $\tau(\overline{0}) = \tau(\overline{1}) = 1$,
(O2) $\tau(\bigvee_{i\in\Gamma} \mu_i) \geq \bigwedge_{i\in\Gamma} \tau(\mu_i)$, *for any* $\{\mu_i\}_{i\in\Gamma} \subset I^X$,
(O3) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, *for any* $\mu_1,\ \mu_2 \in I^X$.

The pair $(X,\ \tau)$ is called a fuzzy topological space (for short, fts). A fuzzy set λ is called an r-fuzzy open (r-fo, for short) if $\tau(\lambda) \geq r$. A fuzzy set λ is called an r-fuzzy closed (r-fc, for short) set iff $\overline{1} - \lambda$ is an r-fo set.

Theorem 2.1. [3] *Let (X, τ) be a fts. Then for each $\lambda \in I^X$ and $r \in I_0$, we define an operator $C_\tau : I^X \times I_0 \to I^X$ as follows: $C_\tau(\lambda,\ r) = \bigwedge\{\mu \in I^X : \lambda \leq \mu,\ \tau(\overline{1} - \mu) \geq r\}$. For $\lambda,\ \mu \in I^X$ and $r, s \in I_0$, the operator C_τ satisfies the following statements:*

(C1) $C_\tau(\overline{0}, r) = \overline{0}$,
(C2) $\lambda \leq C_\tau(\lambda,\ r)$,
(C3) $C_\tau(\lambda,\ r) \vee C_\tau(\mu,\ r) = C_\tau(\lambda \vee \mu,\ r)$,
(C4) $C_\tau(\lambda,\ r) \leq C_\tau(\lambda,\ s)$ *if* $r \leq s$,
(C5) $C_\tau(C_\tau(\lambda,\ r),\ r) = C_\tau(\lambda,\ r)$.

Theorem 2.2. [3] *Let (X, τ) be a fts. Then for each $\lambda \in I^X$ and $r \in I_0$, we define an operator $I_\tau : I^X \times I_0 \to I^X$ as follows: $I_\tau(\lambda,\ r) = \bigvee\{\mu \in I^X : \mu \leq \lambda,\ \tau(\mu) \geq r\}$. For $\lambda,\ \mu \in I^X$ and $r, s \in I_0$, the operator I_τ satisfies the following statements:*

(I1) $I_\tau(\overline{1}, r) = \overline{1}$,
(I2) $I_\tau(\lambda,\ r) \leq \lambda$,
(I3) $I_\tau(\lambda,\ r) \wedge I_\tau(\mu,\ r) = I_\tau(\lambda \wedge \mu,\ r)$,
(I4) $I_\tau(\lambda,\ r) \leq I_\tau(\lambda,\ s)$ *if* $s \leq r$,
(I5) $I_\tau(I_\tau(\lambda,\ r),\ r) = I_\tau(\lambda,\ r)$.
(I6) $I_\tau(\overline{1} - \lambda, r) = \overline{1} - C_\tau(\lambda, r)$ *and* $C_\tau(\overline{1} - \lambda, r) = \overline{1} - I_\tau(\lambda,\ r)$

Definition 2.2. [16] *Let $(X,\ \tau)$ be a fts, $\lambda \in I^X$ and $r \in I_0$. Then*

(1) *a fuzzy set λ is called r-fuzzy regular open (for short, r-fro) if $\lambda = I_\tau(C_\tau(\lambda, r), r)$.*
(2) *a fuzzy set λ is called r-fuzzy regular closed (for short, r-frc) if $\lambda = C_\tau(I_\tau(\lambda, r), r)$.*

Definition 2.3. [20] *Let $(X,\ \tau)$ be a fts. For $\lambda, \mu \in I^X$ and $r \in I_0$.*

(1) *The r-fuzzy δ-closure of λ, denoted by δ-$C_\tau(\lambda, r)$, and is defined by δ-$C_\tau(\lambda, r) = \bigwedge\{\mu \in I^X | \mu \geq \lambda, \mu$ is r-frc $\}$.*
(2) *The r-fuzzy δ-interior of λ, denoted by δ-$I_\tau(\lambda, r)$, and is defined by δ-$I_\tau(\lambda, r) = \bigvee\{\mu \in I^X | \mu \leq \lambda, \mu$ is r-fro $\}$.*

Definition 2.4. [20] *Let $(X,\ \tau)$ be a fts and $\lambda \in I^X$, $r \in I_0$. Then*

(1) λ *is called* r*-fuzzy* δ*-semiopen (resp.* r*-fuzzy* δ*-semiclosed) if* $\lambda \leq C_\tau(\delta\text{-}I_\tau(\lambda, r), r)$ *(resp.* $\lambda \geq I_\tau(\delta\text{-}C_\tau(\lambda, r), r)$*).*
(2) λ *is called* r*-fuzzy* δ*-preopen (resp.* r*-fuzzy* δ*-preclosed) if* $\lambda \leq I_\tau(\delta\text{-}C_\tau(\lambda, r), r)$ *(resp.* $\lambda \geq C_\tau(\delta\text{-}I_\tau(\lambda, r), r)$*).*
(3) λ *is called* r*-fuzzy e-open (for short,* r*-feo) if* $\lambda \leq I_\tau(\delta\text{-}C_\tau(\lambda, r), r) \vee C_\tau(\delta\text{-}I_\tau(\lambda, r), r)$*.*
(4) λ *is called* r*-fuzzy e-closed (for short,* r*-fec) if* $\lambda \geq I_\tau(\delta\text{-}C_\tau(\lambda, r), r) \wedge C_\tau(\delta\text{-}I_\tau(\lambda, r), r)$*.*

Definition 2.5. [20] *Let* $(X,\ \tau)$ *be a fts. For* $\lambda, \mu \in I^X$ *and* $r \in I_0$*.*

(1) *The* r*-fuzzy e-closure of* λ*, denoted by* $eC_\tau(\lambda, r)$*, and is defined by* $eC_\tau(\lambda, r) = \bigwedge\{\mu \in I^X | \mu \geq \lambda, \mu \text{ is } r\text{-fec}\}$*.*
(2) *The* r*-fuzzy e-interior of* λ*, denoted by* $eI_\tau(\lambda, r)$*, and is defined by* $eI_\tau(\lambda, r) = \bigvee\{\mu \in I^X | \mu \leq \lambda, \mu \text{ is } r\text{-feo}\}$*.*

Definition 2.6. [20] *Let* $(X,\ \tau)$ *and* $(Y,\ \eta)$ *be a fts's. Let* $f : (X,\ \tau) \to (Y,\ \eta)$ *be a function. Then* f *is called*

(1) *fuzzy e-continuous (for short,* fe*-continuous) iff* $f^{-1}(\mu)$ *is* r*-feo for each* $\mu \in I^Y,\ r \in I_0$ *with* $\eta(\mu) \geq r$*.*
(2) *fuzzy e-open (for short,* fe*-open) iff* $f(\lambda)$ *is* r*-feo for each* $\lambda \in I^X,\ r \in I_0$ *with* $\tau(\lambda) \geq r$*.*
(3) *fuzzy e-closed (for short,* fe*-closed) iff* $f(\lambda)$ *is* r*-fec for each* $\lambda \in I^X,\ r \in I_0$ *with* $\tau(\overline{1} - \lambda) \geq r$*.*
(4) *fuzzy e-irresolute (for short,* fe*-irresolute) iff* $f^{-1}(\mu)$ *is* r*-fec for each* r*-fec set* $\mu \in I^Y$*.*

Definition 2.7. [1] *Let* (X, τ) *be a fts and* $r \in I_0$*. For* $\lambda \in I^X$*,* λ *is called an* r*-fuzzy rare set if* $I_\tau(\lambda, r) = \overline{0}$*.*

Definition 2.8. [1] *Let* $(X,\ \tau)$ *and* $(Y,\ \eta)$ *be a fts's. Let* $f : (X,\ \tau) \to (Y,\ \eta)$ *be a function. Then* f *is called*

(1) *weakly continuous if for each* $\mu \in I^Y$*, where* $\sigma(\mu) \geq r,\ r \in I_0,\ f^{-1}(\mu) \leq I_\tau(f^{-1}(C_\sigma(\mu, r)), r)$*.*
(2) *rarely continuous if for each* $\mu \in I^Y$*, where* $\sigma(\mu) \geq r,\ r \in I_0$*, there exists an* r*-fuzzy rare set* $\lambda \in I^Y$ *with* $\mu + C_\sigma(\lambda, r) \geq 1$ *and* $\rho \in I^X$*, where* $\tau(\rho) \geq r$ *such that* $f(\rho) \leq \mu \vee \lambda$*.*

Proposition 2.1. [1] *Let* $(X,\ \tau)$ *and* $(Y,\ \sigma)$ *be any two fts's,* $r \in I_0$ *and* $f : (X,\ \tau) \to (Y,\ \sigma)$ *is fuzzy open and one-to-one, then* f *preserves* r*-fuzzy rare sets.*

3. Rarely fuzzy e-continuous functions

Definition 3.1. *Let* $(X,\ \tau)$ *and* $(Y,\ \sigma)$ *be a fts's, and* $f : (X,\ \tau) \to (Y,\ \sigma)$ *be a function. Then* f *is called*

(1) *rarely fuzzy* δ*-semicontinuous (for short, rarely* $f\delta s$*-continuous) if for each* $\mu \in I^Y$*, where* $\sigma(\mu) \geq r,\ r \in I_0$*, there exists an* r*-fuzzy rare set* $\lambda \in I^Y$ *with* $\mu + C_\sigma(\lambda, r) \geq 1$ *and a* r*-fuzzy* δ*-semiopen set* $\rho \in I^X$ *such that* $f(\rho) \leq \mu \vee \lambda$*.*
(2) *rarely fuzzy* δ*-precontinuous (for short, rarely* $f\delta p$*-continuous) if for each* $\mu \in I^Y$*, where* $\sigma(\mu) \geq r,\ r \in I_0$*, there exists an* r*-fuzzy rare set* $\lambda \in I^Y$ *with* $\mu + C_\sigma(\lambda, r) \geq 1$ *and a* r*-fuzzy* δ*-preopen set* $\rho \in I^X$ *such that* $f(\rho) \leq \mu \vee \lambda$*.*
(3) *rarely fuzzy e-continuous (for short, rarely* fe*-continuous) if for each* $\mu \in I^Y$*, where* $\sigma(\mu) \geq r,\ r \in I_0$*, there exists an* r*-fuzzy rare set* $\lambda \in I^Y$ *with* $\mu + C_\sigma(\lambda, r) \geq 1$ *and a* r*-feo set* $\rho \in I^X$ *such that* $f(\rho) \leq \mu \vee \lambda$*.*

Remark 3.1. (1) *Every weakly continuous (resp. fuzzy continuous) function is rarely continuous* [1] *(resp. fuzzy e-continuous* [20]*) but converse is not true.*

(2) *Every rarely continuous function is rarely $f\delta s$-continuous (resp. rarely $f\delta p$-continuous) function but converse is not true.*
(3) *Every rarely $f\delta s$-continuous (resp. rarely $f\delta p$-continuous) function is rarely fe-continuous but converse is not true.*

From the above definition and remarks it is not difficult to conclude that the following diagram of implications is true.

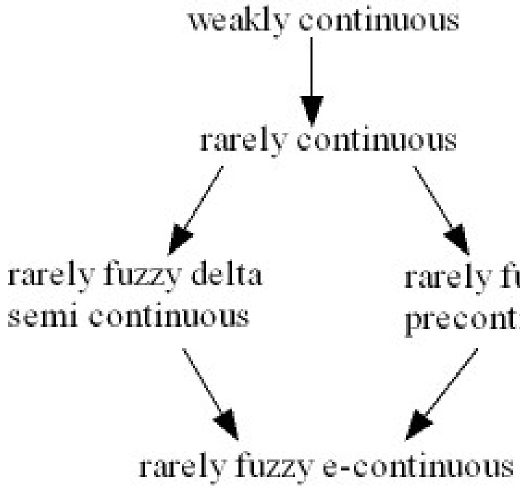

Diagram -I

Example 3.1. *Let $X = \{a, b, c\} = Y$. Define $\lambda_1, \lambda_2 \in I^X$, $\lambda_3 \in I^Y$ as follows:* $\lambda_1(a) = 0.4$, $\lambda_1(b) = 0.6$, $\lambda_1(c) = 0.5$, $\lambda_2(a) = 0.6$, $\lambda_2(b) = 0.4$, $\lambda_2(c) = 0.4$, $\lambda_3(a) = 0.6$, $\lambda_3(b) = 0.4$, $\lambda_3(c) = 0.5$. *Define the fuzzy topologies τ, $\sigma : I^X \to I$ as follows:*

$$\tau(\lambda) = \begin{cases} 1 & \textit{if } \lambda = \overline{0} \textit{ or } \overline{1}, \\ \frac{1}{10} & \textit{if } \lambda = \lambda_1, \\ \frac{1}{10} & \textit{if } \lambda = \lambda_2, \\ \frac{1}{10} & \textit{if } \lambda = \lambda_1 \vee \lambda_2, \\ \frac{1}{10} & \textit{if } \lambda = \lambda_1 \wedge \lambda_2, \\ 0 & \textit{otherwise}, \end{cases} \quad \sigma(\lambda) = \begin{cases} 1 & \textit{if } \lambda = \overline{0} \textit{ or } \overline{1}, \\ \frac{1}{10} & \textit{if } \lambda = \lambda_3, \\ 0 & \textit{otherwise}. \end{cases}$$

Let $r = 1/10$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = a$, $f(b) = b$, $f(c) = c$ and $\gamma \in I^Y$ be a $1/10$-fuzzy rare set defined by $\gamma(a) = 0.4$, $\gamma(b) = 0.5$, $\gamma(c) = 0.5$ and a r-feo set $\lambda_4 \in I^X$ is defined by $\lambda_4(a) = 0.6$, $\lambda_4(b) = 0.4$, $\lambda_4(b) = 0.5$, $f(\lambda_4) = (0.6,\ 0.4,\ 0.5) \leq \lambda_3 \vee \gamma = (0.6,\ 0.4,\ 0.5)$. Then f is rarely fe-continuous but not rarely $f\delta p$-continuous, because $\lambda_4 \in I^X$ is not r-fuzzy δ-preopen set.

Example 3.2. *In Example 3.1, Let $Y = \{a, b, c\}$. Define $\lambda_3 \in I^Y$ as follows:* $\lambda_3(a) = 0.4$, $\lambda_3(b) = 0.5$, $\lambda_3(c) = 0.5$. *Define the fuzzy topologies $\sigma : I^X \to I$ as follows:*

$$\sigma(\lambda) = \begin{cases} 1 & \textit{if } \lambda = \overline{0} \textit{ or } \overline{1}, \\ \frac{1}{10} & \textit{if } \lambda = \lambda_3, \\ 0 & \textit{otherwise}. \end{cases}$$

Let $r = 1/10$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = a$, $f(b) = b$, $f(c) = c$ and $\gamma \in I^Y$ be a $1/10$-fuzzy rare set defined by $\gamma(a) = 0.4$, $\gamma(b) = 0.4$, $\gamma(c) = 0.4$ and a r-feo *set* $\lambda_4 \in I^X$ *is defined by* $\lambda_4(a) = 0.4,\ \lambda_4(b) = 0.5,\ \lambda_4(b) = 0.5,\ f(\lambda_4) = (0.4,\ 0.5,\ 0.5) \leq \lambda_3 \vee \gamma = (0.4,\ 0.5,\ 0.5)$. *Then* f *is rarely* fe*-continuous but not rarely* $f\delta s$*-continuous, because* $\lambda_4 \in I^X$ *is not* r*-fuzzy* δ*-semiopen set.*

Example 3.3. *Let* $X = \{a, b, c\} = Y$. *Define* $\lambda_1, \lambda_2 \in I^X,\ \lambda_3 \in I^Y$ *as follows:* $\lambda_1(a) = 0.2,\ \lambda_1(b) = 0.3,\ \lambda_1(c) = 0.4,\ \lambda_2(a) = 0.3,\ \lambda_2(b) = 0.4,\ \lambda_2(c) = 0.5,\ \lambda_3(a) = 0.3,\ \lambda_3(b) = 0.4,\ \lambda_3(c) = 0.5$. *Define the fuzzy topologies* $\tau,\ \sigma : I^X \to I$ *as follows:*

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{10} & \text{if } \lambda = \lambda_1, \\ \frac{1}{10} & \text{if } \lambda = \lambda_2, \\ 0 & \text{otherwise}, \end{cases} \qquad \sigma(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{10} & \text{if } \lambda = \lambda_3, \\ 0 & \text{otherwise.} \end{cases}$$

Let $r = 1/10$. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be defined by* $f(a) = a,\ f(b) = b,\ f(c) = c$ *and* $\gamma \in I^Y$ *be a* $1/10$*-fuzzy rare set defined by* $\gamma(a) = 0.4,\ \gamma(b) = 0.3,\ \gamma(c) = 0.4$ *and a* r*-f*δ*so set* $\lambda_4 \in I^X$ *is defined by* $\lambda_4(a) = 0.4,\ \lambda_4(b) = 0.4,\ \lambda_4(b) = 0.5,\ f(\lambda_4) = (0.4,\ 0.4,\ 0.5) \leq \lambda_3 \vee \gamma = (0.4,\ 0.4,\ 0.5)$. *Then* f *is rarely* $f\delta s$*-continuous but not rarely continuous, because* $\lambda_4 \in I^X$ *is not* r*-fo set.*

Example 3.4. *Let* $X = \{a, b, c\} = Y$. *Define* $\lambda_1, \lambda_2 \in I^X,\ \lambda_3 \in I^Y$ *as follows:* $\lambda_1(a) = 0.6,\ \lambda_1(b) = 0.5,\ \lambda_1(c) = 0.7,\ \lambda_2(a) = 0.6,\ \lambda_2(b) = 0.5,\ \lambda_2(c) = 0.5,\ \lambda_3(a) = 0.6,\ \lambda_3(b) = 0.5,\ \lambda_3(c) = 0.6$. *Define the fuzzy topologies* $\tau,\ \sigma : I^X \to I$ *as follows:*

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{10} & \text{if } \lambda = \lambda_1, \\ \frac{1}{10} & \text{if } \lambda = \lambda_2, \\ 0 & \text{otherwise}, \end{cases} \qquad \sigma(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{10} & \text{if } \lambda = \lambda_3, \\ 0 & \text{otherwise.} \end{cases}$$

Let $r = 1/10$. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be defined by* $f(a) = a,\ f(b) = b,\ f(c) = c$ *and* $\gamma \in I^Y$ *be a* $1/10$*-fuzzy rare set defined by* $\gamma(a) = 0.4,\ \gamma(b) = 0.5,\ \gamma(c) = 0.5$ *and a* r*-f*δ*po set* $\lambda_4 \in I^X$ *is defined by* $\lambda_4(a) = 0.6,\ \lambda_4(b) = 0.5,\ \lambda_4(b) = 0.6,\ f(\lambda_4) = (0.6,\ 0.5,\ 0.6) \leq \lambda_3 \vee \gamma = (0.6,\ 0.5,\ 0.6)$. *Then* f *is rarely* $f\delta p$*-continuous but not rarely continuous, because* $\lambda_4 \in I^X$ *is not* r*-fo set.*

Definition 3.2. *Let* $(X,\ \tau)$ *and* $(Y,\ \sigma)$ *be a fts's, and* $f : (X,\ \tau) \to (Y,\ \eta)$ *be a function. Then* f *is called weakly fuzzy e-continuous (for short, weakly* fe*-continuous) if for each* r*-feo set* $\mu \in I^Y$, $r \in I_0$, $f^{-1}(\mu) \leq I_\tau(f^{-1}(C_\sigma(\mu, r)), r)$.

Definition 3.3. *A fts* (X, τ) *is said to be* fe*-*$T_{1/2}$*-space if every* r*-feo set* $\lambda \in I^X,\ r \in I_0$ *is* r*-fo set.*

Theorem 3.1. *Let* $(X,\ \tau)$ *and* $(Y,\ \sigma)$ *be any two fuzzy topological spaces. If* $f : (X,\ \tau) \to (Y,\ \sigma)$ *is both* fe*-open,* fe*-irresolute and* $(X,\ \tau)$ *is* fe*-*$T_{1/2}$ *space, then it is weakly* fe*-continuous.*

Proof. Let $\lambda \in I^X,\ r \in I_0$ with $\tau(\lambda) \geq r$. Since f is fe-open $f(\lambda) \in I^Y$ is r-feo. Also, since f is fe-irresolute, $f^{-1}(f(\lambda)) \in I^X$ is r-feo set. Since $(X,\ \tau)$ is fe-$T_{1/2}$ space, every r-feo set is r-fo set, now, $\tau(f^{-1}(f(\lambda))) \geq r$. Consider $f^{-1}(f(\lambda)) \leq f^{-1}(C_\sigma(f(\lambda), r))$ from which $I_\tau(f^{-1}(f(\lambda)), r) \leq I_\tau(f^{-1}(C_\sigma(f(\lambda), r)), r)$. Since $\tau(f^{-1}(f(\lambda))) \geq r$, $f^{-1}(f(\lambda)) \leq I_\tau(f^{-1}(C_\sigma(f(\lambda), r)), r)$, thus f is weakly fe-continuous. □

Definition 3.4. *Let* (X, τ) *be a fts. A* r*-fuzzy e-open cover of* (X, τ) *is the collection* $\{\lambda_i \in I^X,\ \lambda_i \text{ is } r\text{-feo},\ i \in J\}$ *such that* $\bigvee_{i \in J} \lambda_i = \overline{1}$.

Definition 3.5. *A fts (X, τ) is said to be r-fuzzy e-compact space if every r-fuzzy e-open cover of (X, τ) has a finite sub cover.*

Definition 3.6. *A fts (X, τ) is said to be rarely fuzzy e-almost compact if every r-fuzzy e-open cover $\{\lambda_i \in I^X,\ \lambda_i$ is r-feo, $i \in J\}$ of (X, τ), there exists a finite subset J_0 of J such that $\bigvee_{i \in J} \lambda_i \vee \rho_i = \overline{1}$ where $\rho_i \in I^X$ are r-fuzzy rare sets.*

Theorem 3.2. *Let $(X,\ \tau)$ and $(Y,\ \sigma)$ be any two fuzzy topological spaces, and $f : (X,\ \tau) \to (Y,\ \sigma)$ be rarely fe-continuous. If $(X,\ \tau)$ is r-fuzzy e-compact then $(Y,\ \sigma)$ is rarely fuzzy e-almost compact.*

Proof. Let $\{\lambda_i \in I^Y,\ i \in J\}$ be r-fuzzy e-open cover of (Y, σ). Then $\overline{1} = \bigvee_{i \in J} \lambda_i$. Since f is rarely fe-continuous, there exists an r-fuzzy rare sets $\rho_i \in I^Y$ such that $\lambda_i + C_\sigma(\rho_i, r) \geq \overline{1}$ and an r-feo set $\mu_i \in I^X$ such that $f(\mu_i) \leq \lambda_i \vee \rho_i$. Since (X, τ) is r-fuzzy e-compact, every fuzzy e-open cover of (X, τ) has a finite sub cover. Thus $\overline{1} \leq \bigvee_{i \in J_0} \mu_i$. Hence $\overline{1} = f(\overline{1}) = f(\bigvee_{i \in J_0} \mu_i) = \vee_{i \in J_0} f(\mu_i) \leq \bigvee_{i \in J_0} \lambda_i \vee \rho_i$. Therefore (Y, σ) is rarely fuzzy e-almost compact. □

Theorem 3.3. *Let $(X,\ \tau)$ and $(Y,\ \sigma)$ be any two fts's, and $f : (X,\ \tau) \to (Y,\ \sigma)$ be rarely $f\delta p$-continuous. If $(X,\ \tau)$ is r-fuzzy e-compact then $(Y,\ \sigma)$ is rarely fuzzy e-almost compact.*

Proof. Since every rarely $f\delta p$-continuous function is rarely fe-continuous, then proof follows immediately from the Theorem 3.2. □

Theorem 3.4. *Let $(X,\ \tau)$, $(Y,\ \sigma)$ and $(Z,\ \eta)$ be any fts's. If $f : (X,\ \tau) \to (Y,\ \sigma)$ be rarely fe-continuous, fe-open and $g : (Y,\ \sigma) \to (Z,\ \eta)$ is fuzzy open and one-to-one, then $g \circ f : (X,\ \tau) \to (Z,\ \eta)$ is rarely fe-continuous.*

Proof. Let $\lambda \in I^X$ with $\tau(\lambda) \geq r$. Since f is fe-open $f(\lambda) \in I^Y$ with $\sigma(f(\lambda)) \geq r$. Since f is rarely fe-continuous, there exists a r-fuzzy rare set $\rho \in I^Y$ with $f(\lambda) + C_\sigma(\rho, r) \geq \overline{1}$ and an r-feo set $\mu \in I^X$ such that $f(\mu) \leq f(\lambda) \vee \rho$. By the proposition 2.1, $g(\rho) \in I^Z$ is also a r-fuzzy rare set. Since $\rho \in I^Y$ is such that $\rho < \gamma$ for all $\gamma \in I^Y$ with $\sigma(\gamma) \geq r$, and g is injective, it follows that $(g \circ f)(\lambda) + C_\eta(g(\rho), r) \geq \overline{1}$. Then $(g \circ f)(\mu) = g(f(\mu)) \leq g(f(\lambda) \vee \rho) \leq g(f(\lambda)) \vee g(\rho) \leq (g \circ f)(\lambda) \vee g(\rho)$. Hence the result. □

Theorem 3.5. *Let $(X,\ \tau)$, $(Y,\ \sigma)$ and $(Z,\ \eta)$ be any fts's. If $f : (X,\ \tau) \to (Y,\ \sigma)$ be fe-open, onto and $g : (Y,\ \sigma) \to (Z,\ \eta)$ be a function such that $g \circ f : (X,\ \tau) \to (Z,\ \eta)$ is rarely fe-continuous, then g is rarely fe-continuous.*

Proof. Let $\lambda \in I^X$ and $\mu \in I^Y$ be such that $f(\lambda) = \mu$. Let $(g \circ f)(\lambda) = \gamma \in I^Z$ with $\eta(\gamma) \geq r$. Since $(g \circ f)$ is rarely fe-continuous, there exists a r-fuzzy rare set $\rho \in I^Z$ with $\gamma + C_\eta(\rho, r) \geq \overline{1}$ and a r-feo set $\delta \in I^X$ such that $(g \circ f)(\delta) \leq \gamma \vee \rho$. Since f is fe-open, $f(\delta) \in I^Y$ is a r-feo set. Thus there exists a r-fuzzy rare set $\rho \in I^Z$ with $\gamma + C_\eta(\rho, r) \geq \overline{1}$ and a r-feo set $f(\delta) \in I^Y$ such that $g(f(\delta)) \leq \gamma \vee \rho$. Hence g is rarely fe-continuous. □

Theorem 3.6. *Let $(X,\ \tau)$ and $(Y,\ \sigma)$ be any two fts's. If $f : (X,\ \tau) \to (Y,\ \sigma)$ is rarely fe-continuous and $(X,\ \tau)$ is fe-$T_{1/2}$-space, then f is rarely continuous.*

Proof. The proof is trivial. □

Definition 3.7. *A fts (X,τ) is said to be rarely fe-T_2-space if for each pair $\lambda,\ \mu \in I^X$ with $\lambda \neq \mu$ there exist r-feo sets $\rho_1,\ \rho_2 \in I^X$ with $\rho_1 \neq \rho_2$ and a r-fuzzy rare set $\gamma \in I^X$ with $\rho_1 + C_\tau(\gamma, r) \geq \overline{1}$ and $\rho_2 + C_\tau(\gamma, r) \geq \overline{1}$ such that $\lambda \leq \rho_1 \vee \gamma$ and $\mu \leq \rho_2 \vee \gamma$.*

Theorem 3.7. *Let $(X,\ \tau)$ and $(Y,\ \sigma)$ be any two fuzzy topological spaces. If $f : (X,\ \tau) \to (Y,\ \sigma)$ is fe-open and injective and $(X,\ \tau)$ is rarely fe-T_2 space, then $(Y,\ \sigma)$ is also a rarely fe-T_2 space.*

Proof. $\lambda,\ \mu \in I^X$ with $\lambda \neq \mu$. Since f is injective, $f(\lambda) \neq f(\mu)$. Since $(X,\ \tau)$ is rarely fe-T_2-space, there exist r-feo sets $\rho_1,\ \rho_2 \in I^X$ with $\rho_1 \neq \rho_2$ and a r-fuzzy rare set $\gamma \in I^X$ with $\rho_1 + C_\tau(\gamma, r) \geq \overline{1}$ and $\rho_2 + C_\tau(\gamma, r) \geq \overline{1}$ such that $\lambda \leq \rho_1 \vee \gamma$ and $\mu \leq \rho_2 \vee \gamma$. Since f is fe-open, $f(\rho_1),\ f(\rho_2) \in I^Y$ are r-feo sets with $f(\rho_1) \neq f(\rho_2)$. Since f is fe-open and one-to-one, $f(\gamma)$ is also a r-fuzzy rare set with $f(\rho_1) + C_\sigma(\gamma, r) \geq \overline{1}$ and $f(\rho_2) + C_\sigma(\gamma, r) \geq \overline{1}$ such that $f(\lambda) \leq f(\rho_1 \vee \gamma)$ and $f(\mu) \leq f(\rho_1 \vee \gamma)$. Thus $(Y,\ \sigma)$ is a rarely fe-T_2-space. □

4. Conclusions

Šostak's fuzzy topology has been recently of major interest among fuzzy topologies. In this paper, we have introduced rarely fuzzy e-continuous functions in fuzzy topological spaces of Šostak's. We have also introduced fuzzy e-compact space, rarely fuzzy e-almost compact space, rarely fe-T_2-spaces and some properties and characterizations of them are investigated.

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