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FIXED POINTS RESULTS ON A PARTIALLY ORDERED MULTIPLICATIVE METRIC SPACES

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Abstract. In the present paper, we show the existence of a fixed point for a monotone mapping in partially ordered complete multiplicative metric space using a partial order induced by an appropriate function ϕ. Moreover, common fixed points for two and three weakly comparable mappings are also proved in the same space.

Keywords: Multiplicative Metric space, Partially Ordered space, Weakly comparable mappings, Bounded above functions.

AMS Subject Classification: 47H10.

1. Introduction and Preliminaries

Bashirov et al. [1] introduced the concept of multiplicative metric on a nonempty set X. Ozavsar et al. [2] gave the topological properties of multiplicative metric spaces. They noticed that R_+ is a complete multiplicative metric space with respect to the multiplicative metric. They also introduced the concept of multiplicative contraction mapping and prove some fixed point theorems of multiplicative contraction mappings on multiplicative metric spaces. He et al. [3] discussed the common fixed points of two pairs of weak commutative mappings on a complete multiplicative metric space. Some recent results on mulplicative metric spaces and its variants can be found in [8]-[11].

A common fixed point theorem for different mappings was obtained on a 2-metric space in [4, 5]. Agarwal [6] proved some fixed point results for monotone operators in a metric space endowed with a partial order using a weak generalized contraction-type mapping.

Our technique of proof is essentially different. In this paper, we show the existence of a fixed point for a non-decreasing mapping in partially ordered complete multiplicative metric space using a partial order induced by an appropriate function ϕ without using any multiplicative contractive condition. Some examples are also given in order to illustrate the effectiveness of our result. Moreover common fixed point for two and three mappings satisfying the weakly comparable condition are also proved in the same space.

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Recall that if $(X, \preceq)$ is a partially ordered set and $T : X \to X$ such that for $x, y \in X$, $x \preceq y$ implies $Tx \preceq Ty$ then a mapping F is said to be non-decreasing.

Definition 1.1. [1] *Let X be a nonempty set. Multiplicative metric is a mapping $d : X \times X \to R$ satisfying the following conditions:*
(1) $d(x, y) \geq 1$ *for all* $x, y \in X$ *and* $d(x, y) = 1$ *if and only if* $x = y$,
(2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$,
(3) $d(x, z) \leq d(x, y).d(y, z)$ *for all* $x, y, z \in X$ *(multiplicative triangle inequality).*

Definition 1.2. [2] *Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. If for every multiplicative open ball $B_\epsilon(x)$, there exists a natural number N such that $n \geq N \implies x_n \in B_\epsilon(x)$, then the sequence $\{x_n\}$ is said to be multiplicative convergent to x, denoted by $x_n \to_* x (n \to \infty)$.*

Lemma 1.1. [2] *Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $x_n \to_* x (n \to \infty)$ if and only if $(x_n, x) \to_* 1 (n \to \infty)$.*

Theorem 1.1. [2] *Let (X, d_X) and (Y, d_Y) be two multiplicative metric spaces, $f : X \to Y$ be a mapping and $\{x_n\}$ be any sequence in X. Then f is multiplicative continuous at the point $x \in X$ if and only if $f(x_n) \to_* f(x)$ for every sequence $\{x_n\}$ with $x_n \to_* x (n \to \infty)$.*

Definition 1.3. [2] *Let (X, d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a multiplicative Cauchy sequence if and only if $d(x_m, x_n) \to_* 1 (m, n \to \infty)$.*

Definition 1.4. [2] *A monotone multiplicative bounded sequence in $(R_+, |.|^*)$ is multiplicative convergent.*

Definition 1.5. [2] *We call a multiplicative metric space complete if every multiplicative Cauchy sequence in it is multiplicative convergent to $x \in X$.*

Definition 1.6. [7] *Let $(X, \preceq)$ be an ordered space. Two mappings $f, g : X \to X$ are said to be weakly comparable if $fx \preceq gfx$ and $gx \preceq fgx$ for all $x \in X$.*

2. Main Results

We first prove the following lemma:

Lemma 2.1. *Let (X, d) be a multiplicative metric space and $\phi : X \to R$. Define the relation $\preceq$ on X as follows:*

$$x \preceq y \Leftrightarrow d(x, y) \leq \phi(y)/\phi(x) \tag{1}$$

Then $\preceq$ is partial order on X, called the partial order induced by ϕ.

Proof. For all $x \in X$, $d(x, x) = 1 = \phi(x)/\phi(x)$ then $x \preceq x$ that is $\preceq$ is reflexive.
Now for $x, y \in X$ s.t. $x \preceq y$ and $y \preceq x$ then

$$d(x, y) \leq \phi(y)/\phi(x) \tag{2}$$

And

$$d(y, x) \leq \phi(x)/\phi(y) \tag{3}$$

This shows that $d(x, y) = 1$ i.e.$x = y$. Thus $\preceq$ is antisymmetric.
Again for $x, y, z \in X$ s.t. $x \preceq y$ and $y \preceq z$ then

$$d(x, y) \leq \phi(y)/\phi(x) \tag{4}$$

And

$$d(y, z) \leq \phi(z)/\phi(y) \tag{5}$$

We have

$$d(x,z) \leq d(x,y).d(y,z) \leq \frac{\phi(y)}{\phi(x)}.\frac{\phi(z)}{\phi(y)} = \phi(z)/\phi(x) \tag{6}$$

Then $x \preceq z$. Thus $\preceq$ is transitive.
And so the relation $\preceq$ is a partial order on X. □

Example 2.1. *Let $X = [0, \infty)$ and $d(x,y) = |x/y|^*$ then (X, d) is a multiplicative metric space.*
Where

$$|a|^* = \begin{cases} a \; if \; a \geq 1 \\ \frac{1}{a} \; if \; a < 1 \end{cases} \tag{7}$$

Then it is obvious that all conditions of multiplicative metric are satisfied.
Let $\phi : X \to R$, $\phi(x) = 2x$. Then for $x, y \in X$

$$x \preceq y \Leftrightarrow (x,y) \leq \phi(y)/\phi(x) \tag{8}$$

And so

$$|x/y|^* \leq 2y/2x = y/x \tag{9}$$

It follows that $1 \preceq 2$, $1/2 \preceq 1$, $1 \preceq 1$, $3 \preceq 5$ but $3 \npreceq 1$ and $6 \npreceq 5$ etc.
Therefore X is a partially ordered space.

Now we prove our main theorems:

Theorem 2.1. *Let (X, d) be a complete multiplicative metric space and $\phi : X \to R$ a bounded from above function and $\preceq$ is a partial order induced by ϕ. Suppose $f : X \to X$ be a continuous non-decreasing function with $x_0 \preceq fx_0$ for some $x_0 \in X$ then f has a fixed point in X.*

Proof. Consider a point $x_0 \in X$ such that $x_0 \preceq fx_0$. We define a sequence x_n s.t. $x_n = fx_{n-1}$. Then there exists $x_1 = fx_0$ and continuing in this way we get a sequence x_n which satisfies

$$x_1 \preceq x_2 \preceq x_3 \ldots \preceq x_n \preceq x_{n+1} \ldots \tag{10}$$

That is, the sequence x_n is non- decreasing . By definition of $\preceq$, we have

$$\phi(x_0) \leq \phi(x_1) \leq \phi(x_2) \leq \ldots \tag{11}$$

for all $t > 0$. In other words, the sequence $\phi(x_n)$ is nondecreasing sequence of real numbers for all $t > 0$. Since ϕ is bounded from above, $\phi(x_n)$ is convergent and hence Cauchy. So, for all $\epsilon > 0$ there exists $n_0 \in N$ such that for all $m > n > n_0$, we have

$$d(\phi(x_m), \phi(x_n)) < \epsilon \tag{12}$$

or

$$\frac{\phi(x_m)}{\phi(x_n)} < \epsilon. \tag{13}$$

Since $x_n \preceq x_m$, therefore by the definition of $\preceq$ we have

$$d(x_n, x_m) \leq \frac{\phi(x_m)}{(\phi(x_n))} < \epsilon \tag{14}$$

This shows that the sequence x_n is Cauchy in X and since X is complete, it converges to a point $z \in X$. Consequently by the continuity of f , we have $fz = z$. □

Now we prove the theorem based on the concept of weakly comparable mappings.

Theorem 2.2. *Let (X, d) be a complete multiplicative metric space and $\phi : X \to R$ a bounded from above function and $\preceq$ is a partial order induced by ϕ. Suppose $f, g : X \to X$ are two continuous and weakly comparable mappings then f and g have a common fixed point in X.*

Proof. We define a sequence x_n in X s.t. $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$. Since f and g are weakly comparable, $fx_0 \preceq gfx_0$ i.e. $x_1 \preceq gx_1$ or $x_1 \preceq x_2$ for some $x_0 \in X$. Again, by weak comparability of f and g we have $gx_1 \preceq fgx_1$ i.e. $x_2 \preceq x_3$. Continuing in this way we get a sequence x_n such that

$$x_1 \preceq x_2 \preceq x_3 \ldots \preceq x_n \preceq x_{n+1} \ldots \tag{15}$$

That is, the sequence x_n is non- decreasing.By definition of $\preceq$, we have

$$\phi(x_0) \leq \phi(x_1) \leq \phi(x_2) \leq \ldots \tag{16}$$

for all $t > 0$. Since ϕ is bounded from above, $\phi(x_n)$ is convergent and hence Cauchy. So, for all $\epsilon > 0$ there exist $n_0 \in N$ such that for all $m > n > n_0$, we have

$$d(\phi(x_m), \phi(x_n)) < \epsilon \tag{17}$$

or

$$\frac{\phi(x_m)}{\phi(x_n)} < \epsilon. \tag{18}$$

Since $x_n \preceq x_m$, therefore by the definition of $\preceq$ we have

$$d(x_n, x_m) \leq \frac{\phi(x_m)}{(\phi(x_n))} < \epsilon \tag{19}$$

This shows that the sequence $\{x_n\}$ is Cauchy in X and since X is complete, it converges to a point $z \in X$. Since the sequences $\{x_{2n+2}\}$ and $\{x_{2n+1}\}$ are subsequences of $\{x_n\}$ therefore $x_{2n} \to z$ and $x_{2n+1} \to z$ with $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$. Now since f and g are continuous, we have $z = fz$ and $z = gz$ i.e. $z = fz = gz$. Hence z is common fixed point of f and g.

□

We give an example to furnish the above theorem:

Example 2.2. *Let $X = R_+$, $d(x, y) = |x/y|^*$ then (X, d) is a multiplicative metric space. Where*

$$|a|^* = \begin{cases} a \; if \; a \geq 1 \\ \frac{1}{a} \; if a < 1 \end{cases} \tag{20}$$

Then it is obvious that all conditions of multiplicative metric are satisfied.

And so (X, d) is an complete multiplicative metric space. Let $\phi : X \to R$ be defined as $\phi(x) = x/5$. Now define $f, g : X \to X$ as $fx = x^2$ and $gx = x + 2$.
It is clear that f and g are continuous and weakly comparable. And so, all the conditions of theorem 2.4 are satisfied. Therefore f and g have a common fixed point.

Theorem 2.3. *Let (X, d) be a complete multiplicative metric space and $\phi : X \to R$ a bounded from above function and $\preceq$ a partial order induced by ϕ. Suppose $f, g, h : X \to X$ are three continuous mappings s.t. the pairs $\{f, g\}$ and $\{h, g\}$ are weakly comparable then f,g and h have a common fixed point.*

Proof. We construct a sequence x_n in X such that $x_{3n} = fx_{3n-1}$,$x_{3n-1} = gx_{3n-2}$ and $x_{3n-2} = hx_{3n-3}$ for all $n = 1, 2, 3 \ldots$ We have $x_1 = hx_0$ and since the pair $\{h, g\}$ is weakly comparable, we have $hx_0 \preceq gx_1$ and so there exist $x_2 = gx_1$ such that $x_1 \preceq x_2$. Again since $x_2 = gx_1$ and the pair $\{f, g\}$ is weakly comparable, we have $gx_1 \preceq fx_2$ and so there exist $x_3 = Fx_2$ such that $x_2 \preceq x_3$. Continuing in this way, we get

$$x_1 \preceq x_2 \preceq x_3 \ldots \preceq x_n \preceq x_{n+1} \ldots \tag{21}$$

That is, the sequence x_n is non- decreasing.By definition of $\preceq$, we have

$$\phi(x_0) \leq \phi(x_1) \leq \phi(x_2) \leq \ldots \tag{22}$$

for all $t > 0$. there exist $n_0 \in N$ such that for all $m > n > n_0$, we have

$$d(\phi(x_m), \phi(x_n)) < \epsilon \tag{23}$$

or

$$\frac{\phi(x_m)}{\phi(x_n)} < \epsilon. \tag{24}$$

Since $x_n \preceq x_m$, therefore by the definition of $\preceq$ we have

$$d(x_n, x_m) \leq \frac{\phi(x_m)}{(\phi(x_n))} < \epsilon \tag{25}$$

This shows that the sequence $\{x_n\}$ is Cauchy in X and since X is complete, it converges to a point $z \in X$. Since the sequences $\{x_{3n}\}$,$\{x_{3n-1}\}$ and $\{x_{3n-2}\}$ are subsequences of $\{x_n\}$ therefore $x_{3n} \to z$,$x_{3n-1} \to z$ and $x_{3n-2} \to z$ with $x_{3n} = fx_{3n-1}$, $x_{3n-1} = gx_{3n-2}$ and $x_{3n-2} = hx_{3n-3}$. Now since f,g and h are continuous, we have $z = fz$,$z = gz$ and $z = hz$ i.e. $z = fz = gz = hz$. Hence z is a common fixed point of f,g and h. □

3. Conclusions

In the present paper we have defined a partial order on a multiplicative metric space. Moreover, we have proved some fixed point theorems for weakly comparable mappings in this partially ordered multiplicative metric space.

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