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APPLYING VIM TO CONFORMABLE PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we used new conformable variational iteration method, by the conformable derivative, for solving fractional heat-like and wave-like equations. This method is simple and very effective in the solution procedures of the fractional partial differential equations that have complicated solutions with classical fractional derivative definitions like Caputo, Riemann-Liouville and etc. The results show that conformable variational iteration method is usable and convenient for the solution of fractional partial differential equations. Obtained results are compared to the exact solutions and their graphics are plotted to demonstrate efficiency and accuracy of the method.

Keywords: Variational iteration method, Conformable derivative, fractional partial differential equations, Factional derivative.

AMS Subject Classification: 34A08, 35R11, 30C70.

1. Introduction and Basic definitions

A fractional differentiation and integration operator has different kinds of definitions that we can mention, the Riemann-Liouville definition [29, 27], the Caputo definition [28] and so on. Lately, Khalil et al [23, 24] introduced a new simple definition of the fractional derivative named the conformable derivative; which can redress shortcomings of the other definitions [1, 12, 14, 6, 30]. Variational iteration method [15, 19, 20, 21, 25, 7, 8, 4, 5] based on the use of restricted variations, correction functional and Lagrange multiplier technique developed by Inokuti et al. (1978). This method does not require the presence of small parameters in the differential equation and provides the solution (or an approximation to it) as a sequence of iterates. The method does not require that the nonlinearities be differentiable with respect to the dependent variable and its derivatives.

This technique is, in fact, a modifying of the general Lagrange multiplier method into an iteration shown to solve effectively, easily, and accurately a large class of nonlinear

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problems, generally one or two iterations lead to highly accurate solutions.

The aim of this paper is to extend the variational iteration method to conformable variational iteration method by conformable derivative [2, 3] and was used to solve various kinds of fractional heat-like and wave-like equations. Some examples are given to show the reliability and the efficiency of the conformable variational iteration method.
In this section, we briefly present the basic definitions. Conformable variational iteration method C-VIM is introduced in Section 2. The proposed method is illustrated by solving two examples of conformable heat-like equations in Section 3. In Section 4 we present the mentioned method for solving conformable wave-like equations and proved by two examples. In Section 5 the conclusion is illustrated.

Definition 1.1. *Given a function $f : [0, \infty) \to \mathbb{R}$. Then the " conformable derivative" of f of order α is defined by*

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}$$

for all $t > 0$, $\alpha \in (0, 1)$. If f is α-differentiable in some $(0, a)$, $a > 0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t) \tag{1}$$

We will, sometimes, write $f^{(\alpha)}$ for $T_\alpha(f)(t)$, to denote the conformable derivatives of f of order α.
One can easily show that T_α satisfies all the properties in the following theorem [23].

Theorem 1.1. *Let $\alpha \in (0, 1]$ and f, g be α-differentiable at a point $t > 0$. Then*

(1) $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, *for all $a, b \in \mathbb{R}$.*
(2) $T_\alpha(t^p) = pt^{p-\alpha}$ *for all $p \in \mathbb{R}$*
(3) $T_\alpha(\lambda) = 0$, *for all constant functions $f(t) = \lambda$*
(4) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$
(5) $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$
(6) *In addition, if f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha \frac{df}{dt}(t)}$.*

2. Conformable variational iteration method C-VIM

We will introduce C-VIM for FPDEs in this section. We write the non-linear FPDEs in the standard operator form

$$T_\alpha u(x, t) + Lu(x, t) + Nu(x, t) = G(x, t) \tag{2}$$

where T_α is a linear operator with conformable derivative of order α with $n < \alpha \leq n + 1$, L is a linear operator, N is a non-linear operator and G is a nonhomogeneous term. If the linear operator in Eq (2) is applied to Theorem 1.1, the following equation is obtained :

$$t^{[\alpha]-\alpha} \frac{\partial^{[\alpha]}}{\partial t^{[\alpha]}} u(x, t) + Lu(x, t) + Nu(x, t) = G(x, t) \tag{3}$$

as in classical VIM, the correction function for equation (3) can be constructed as

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda(x, s) \left[s^{[\alpha]-\alpha} \frac{\partial^{[\alpha]}}{\partial t^{[\alpha]}} u_n(x, s) + Lu(x, s) + Nu(x, s) - G(x, s) \right] ds \tag{4}$$

where λ is a general Lagrangian multiplier and it can be optimally determined by the aid of variational theory [18, 19, 20, 21].
It is obvious that the successive approximations $u_n; n \geq 0$ can be established by determining λ, a general Lagrange multiplier, which can be identified optimally via the variational theory. The function $\tilde{u}_n$ is a restricted variation which means $\delta\widetilde{u}_n = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations $u_{n+1}(x,t) \geq 0$ of the solution $u(x,t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. The initial values are usually used for the selected zeroth approximation u_0. With λ determined, then several approximations can be determined. Consequently, the exact solution is given by

$$u(x,t) = \lim_{n\to\infty} u_n(x,t)$$

3. Conformable heat-like equations

Example 3.1. *Considering the following one-dimensional conformable heat-like equation*

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = \frac{1}{2}x^2\frac{\partial^2}{\partial x^2}u(x,t), \ 0 < \alpha \leq 1, \ 0 < x \leq 1 \ \textit{and} \ \ t > 0 \tag{5}$$

subject to the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = e^{\frac{t^\alpha}{\alpha}}, \tag{6}$$

and the initial condition

$$u(x,0) = x^2. \tag{7}$$

The exact solution of this problem (5) *is*

$$u(x,t) = x^2 e^{\frac{t^\alpha}{\alpha}} \tag{8}$$

For solving by C-VIM we obtain the recurrence relation

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(x,s)\left[s^{1-\alpha}\frac{\partial}{\partial s}u_n(x,s) - \frac{1}{2}x^2\frac{\partial^2}{\partial x^2}\widetilde{u}_n(x,s)\right]ds \tag{9}$$

where λ is a Lagrange multiplier, $\widetilde{u}_n$ is a restricted variation, i.e $\delta\widetilde{u}_n = 0$. Making Eq (9) *stationary with respect to u_n, we have*

$$\frac{\partial}{\partial s}\lambda(x,s) = 0, \qquad \textit{and} \qquad 1 + \lambda(x,s)|_{s=t} = 0. \tag{10}$$

The Lagrange multiplier can be identified as $\lambda = -1$. As a result we obtain the following iteration formula.

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[s^{1-\alpha}\frac{\partial}{\partial s}u_n(x,s) - \frac{1}{2}x^2\frac{\partial^2}{\partial x^2}\widetilde{u}_n(x,s)\right]ds, \tag{11}$$

Beginning with an initial approximation $u_0(x,t) = u(x,0) = x^2$, we can obtain the following successive approximations

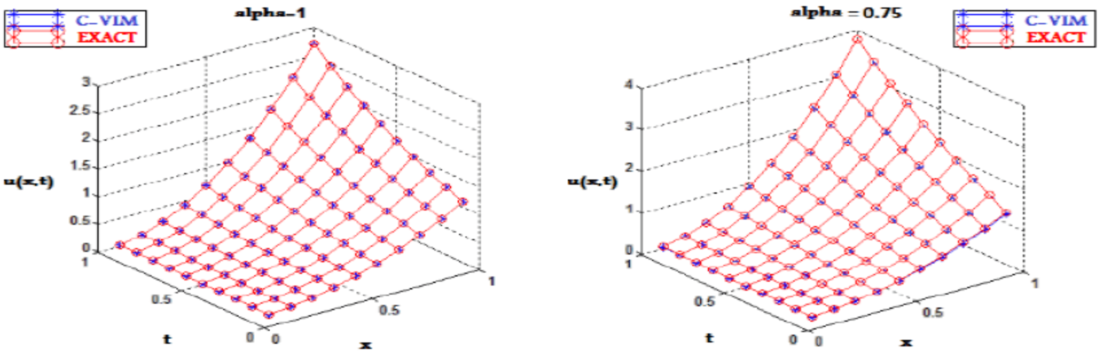

FIGURE 1. Comparison of four iteration C-VIM solutions with the exact solutions for Eq (5)

$$
\begin{aligned}
u_1(x,t) &= x^2[1+t] \\
u_2(x,t) &= x^2\left[1+2t-\frac{t^{2-\alpha}}{2-\alpha}+\frac{t^2}{2}\right] \\
u_3(x,t) &= x^2\left[1+3t-3\frac{t^{2-\alpha}}{2-\alpha}+\frac{t^{3-2\alpha}}{3-2\alpha}+3\frac{t^2}{2}-\frac{t^{3-\alpha}}{3-\alpha}-\frac{t^{3-\alpha}}{(2-\alpha)(3-\alpha)}+\frac{t^3}{3!}\right] \\
u_4(x,t) &= x^2\left[1+4t-6\frac{t^{2-\alpha}}{2-\alpha}+4\frac{t^{3-2\alpha}}{3-2\alpha}-\frac{t^{4-3\alpha}}{4-3\alpha}+2\frac{t^{4-2\alpha}}{4-2\alpha}-4\frac{t^{3-\alpha}}{3-\alpha}\right. \\
&- 4\frac{t^{3-\alpha}}{(2-\alpha)(3-\alpha)}+\frac{t^{4-2\alpha}}{(3-2\alpha)(4-2\alpha)}+6\frac{t^2}{2!}+4\frac{t^3}{3!}-\frac{t^{4-\alpha}}{(3-\alpha)(4-\alpha)} \\
&- \left.\frac{t^{4-\alpha}}{(2-\alpha)(3-\alpha)(4-\alpha)}-\frac{t^{4-\alpha}}{(2)(4-\alpha)}+\frac{t^4}{4!}\right] \\
&\vdots
\end{aligned}
$$

when we consider $\alpha=1$, the solution by C-VIM is obtained as

$$u_n(x,t)=x^2\left[1+t+\frac{t^2}{2!}+\cdots+\frac{t^n}{n!}\right] \tag{12}$$

From (12), the C-VIM solution $u(x,t)=\lim_{n\to\infty}u_n(x,t)=x^2e^t$ This is also exact solution. Now, we analyse the C-VIM and exact solutions graphically for some α values see fig 1.

Example 3.2. *We consider the two-dimensional conformable heat-like model*

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,y,t)=\frac{1}{2}\left(y^2\frac{\partial^2}{\partial x^2}u(x,y,t)+x^2\frac{\partial^2}{\partial y^2}u(x,y,t)\right),\ 0<\alpha\leq 1,\ 0<x,y\leq 1 \text{ and } t>0 \tag{13}$$

subject to the boundary conditions

$$\frac{\partial}{\partial x}u(0,y,t)=0,\quad \tfrac{\partial}{\partial x}u(1,y,t)=2\sinh\left(\tfrac{t^\alpha}{\alpha}\right) \tag{14}$$

$$\frac{\partial}{\partial y}u(x,0,t)=0,\quad \tfrac{\partial}{\partial y}u(x,1,t)=2\cosh\left(\tfrac{t^\alpha}{\alpha}\right) \tag{15}$$

and the initial condition

$$u(x,y,0)=y^2 \tag{16}$$

Exact solution of this problem (13) *is*

$$u(x,y,t)=x^2\sinh\left(\frac{t^\alpha}{\alpha}\right)+y^2\cosh\left(\frac{t^\alpha}{\alpha}\right) \tag{17}$$

Similarly we can establish an iteration formula in the form

$$\begin{aligned} u_{n+1}(x,y,t) &= u_n(x,y,t)-\int_0^t\Big[s^{1-\alpha}\frac{\partial}{\partial s}u_n(x,y,s)\\ &- \frac{1}{2}\left(y^2\frac{\partial^2}{\partial x^2}u_n(x,y,s)+x^2\frac{\partial^2}{\partial y^2}u_n(x,y,s)\right)\Big]ds \end{aligned} \tag{18}$$

Begininig with (16), *by the iteration formula* (18) , *we obtain the following successive approximations*

$$\begin{aligned} u_1(x,y,t) &= x^2t+y^2\\ u_2(x,y,t) &= x^2\left[2t-\frac{t^{2-\alpha}}{2-\alpha}\right]+y^2\left[1+\frac{t^2}{2!}\right]\\ u_3(x,y,t) &= x^2\left[3t-3\frac{t^{2-\alpha}}{2-\alpha}+\frac{t^{3-2\alpha}}{3-2\alpha}+\frac{t^3}{3!}\right]+y^2\left[1+3\frac{t^2}{2!}-\frac{t^{3-\alpha}}{3-\alpha}-\frac{t^{3-\alpha}}{(2-\alpha)(3-\alpha)}\right]\\ u_4(x,y,t) &= x^2\Big[4t-6\frac{t^{2-\alpha}}{2-\alpha}+4\frac{t^{3-2\alpha}}{3-2\alpha}-\frac{t^{4-3\alpha}}{4-3\alpha}\\ &+ 4\frac{t^3}{3!}-\frac{t^{4-\alpha}}{(3-\alpha)(4-\alpha)}-\frac{t^{4-\alpha}}{(2-\alpha)(3-\alpha)(4-\alpha)}-\frac{t^{4-\alpha}}{2(4-\alpha)}\Big]\\ &+ y^2\Big[1+6\frac{t^2}{2}-4\frac{t^{3-\alpha}}{(3-\alpha)}-4\frac{t^{3-\alpha}}{(2-\alpha)(3-\alpha)}+2\frac{t^{4-2\alpha}}{4-2\alpha}\\ &+ \frac{t^{4-2\alpha}}{(3-2\alpha)(4-2\alpha)}+\frac{t^4}{4!}\Big]\\ &\vdots \end{aligned}$$

when we consider $\alpha=1$, *the solution by C- VIM is obtained as*

$$u_n(x,t)=x^2\left[t+\frac{t^3}{3!}+\cdots+\frac{1-(-1)^n}{2}\frac{t^n}{n!}\right]+y^2\left[1+\frac{t^2}{2!}+\cdots+\frac{1+(-1)^n}{2}\frac{t^n}{n!}\right] \tag{19}$$

From (19), *the C-VIM solution* $u(x,t)=\lim_{n\to\infty}u_n(x,t)=x^2\sinh(t)+y^2\cosh(t)$ *This is also exact solution. Now, we analyse the C-VIM and exact solutions graphically for some* α *values fig2.*

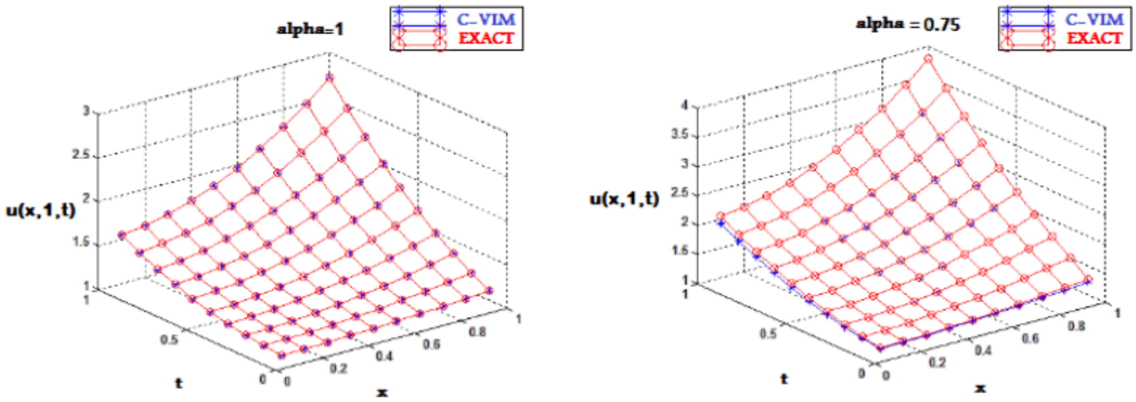

FIGURE 2. Comparison of four iteration C-VIM solutions with the exact solutions for Eq (13)

4. Conformable wave-like equations

Example 4.1. *We first consider the one-dimensional conformable wave-like equation*

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = \frac{1}{2}x^2\frac{\partial^2}{\partial x^2}u(x,t), 1 < \alpha \leq 2, 0 < x \leq 1 \text{ and } t > 0 \tag{20}$$

subject to the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 1 + \sinh\left(\frac{t^\alpha}{\alpha}\right) \tag{21}$$

and the initial conditions

$$u(x,0) = x, \quad \frac{\partial^\alpha}{\partial t^\alpha}u(x,0) = x^2 \text{ for } 0 < \alpha \leq 1 \tag{22}$$

Exact solution of this problem (20) *is*

$$u(x,t) = x + x^2\sinh\left(\frac{t^\alpha}{\alpha}\right) \tag{23}$$

According to the C-VIM, the correct functional for (20) *reads*

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(x,s)\left[s^{2-\alpha}\frac{\partial^2}{\partial s^2}u_n(x,s) - \frac{1}{2}x^2\frac{\partial^2}{\partial x^2}\widetilde{u}_n(x,s)\right]ds \tag{24}$$

The Lagrange can be identified by making Eq (24) *stationary with respect to* u_n *:*

$$\frac{\partial^2}{\partial s^2}\lambda(x,s) = 0, \quad 1 - \frac{\partial}{\partial s}\lambda(x,s)\bigg|_{t=s} = 0, \quad \lambda(x,s)|_{t=s} = 0$$

The Lagrange multiplier can be identified as $\lambda = s - t$. *We obtain the following iteration formula.*

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (s-t)\left[s^{2-\alpha}\frac{\partial^2}{\partial s^2}u_n(x,s) - \frac{1}{2}x^2\frac{\partial^2}{\partial x^2}\widetilde{u}_n(x,s)\right]ds \tag{25}$$

Beginning with an initial approximation

$$u_0(x,t) = u(x,0) + \frac{\partial^\alpha}{\partial t^\alpha}u(x,0) = x + x^2 t \text{ for } 0 < \alpha \leq 1$$

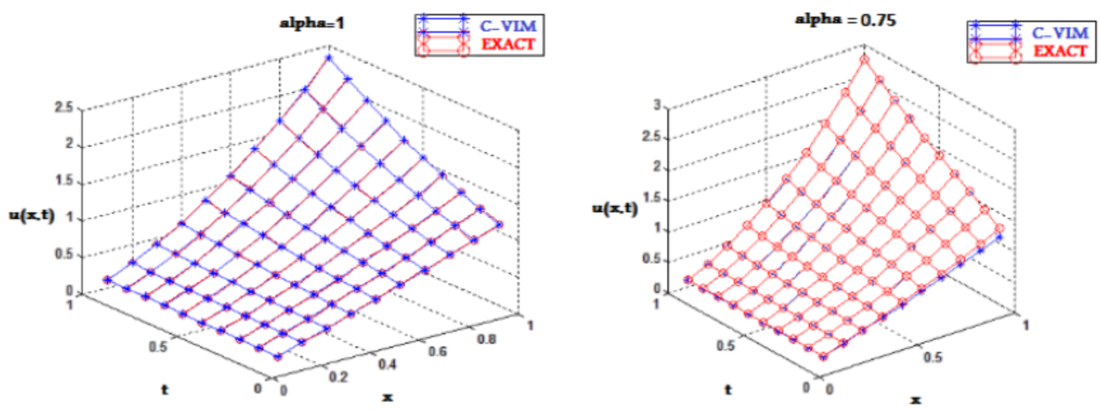

FIGURE 3. Comparison of three iteration C-VIM solutions with the exact solutions for Eq (20)

we obtain the following successive approximations

$$\begin{aligned} u_1(x,t) &= x + x^2\left[t + \frac{t^3}{3!}\right] \\ u_2(x,t) &= x + x^2\left[t + 2\frac{t^3}{3!} - \frac{t^{5-\alpha}}{(4-\alpha)(5-\alpha)} + \frac{t^5}{5!}\right] \\ u_3(x,t) &= x + x^2\left[t + 3\frac{t^3}{3!} - 2\frac{t^{5-\alpha}}{(4-\alpha)(5-\alpha)} + 3\frac{t^5}{5!}\right. \\ &- \left.\frac{t^{7-\alpha}}{3!(6-\alpha)(7-\alpha)} - \frac{t^{7-\alpha}}{(4-\alpha)(5-\alpha)(6-\alpha)(7-\alpha)} + \frac{t^7}{7!}\right] \end{aligned}$$

when we consider $\alpha = 1$, the solution by C-VIM is obtained as

$$u_n(x,t) = x + x^2\left[t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots + \frac{t^{2n+1}}{(2n+1)!}\right]$$

The approximate solution reads $u = \lim_{n\to\infty} u_n(x,t) = x + x^2\sinh(t)$, this is also exact solution.
Now, we compare the C-VIM solution with the exact solution on the graphs for some α These comparisons can be seen in fig 3

Example 4.2. *Consider the two-dimensional coformable wave-like equation*

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,y,t) = \frac{1}{12}\left(y^2\frac{\partial^2}{\partial x^2}u(x,y,t) + x^2\frac{\partial^2}{\partial y^2}u(x,y,t)\right), 1 < \alpha \leq 2, 0 < x, y \leq 1 \text{and } t > 0 \tag{26}$$

subject to the boundary conditions

$$\frac{\partial}{\partial x}u(0,y,t) = 0, \quad \frac{\partial}{\partial x}u(1,y,t) = 2\cosh\left(\frac{t^\alpha}{\alpha}\right) \tag{27}$$

$$\frac{\partial}{\partial y}u(x,0,t) = 0, \quad \frac{\partial}{\partial y}u(x,1,t) = 2\sinh\left(\frac{t^\alpha}{\alpha}\right) \tag{28}$$

and the initial condition

$$u(x,y,0) = x^4, \quad \frac{\partial^\alpha}{\partial t^\alpha}u(x,y,0) = y^4 \quad \text{for} \quad 0 < \alpha \leq 1 \tag{29}$$

Exact solution of this problem (26) *is*

$$u(x,y,t) = x^4 \cosh\left(\frac{t^\alpha}{\alpha}\right) + y^4 \sinh\left(\frac{t^\alpha}{\alpha}\right) \tag{30}$$

Similarly we can establish an iteration formula in the form

$$\begin{aligned} u_{n+1}(x,y,t) &= u_n(x,y,t) + \int_0^t (s-t)\Bigg[s^{2-\alpha}\frac{\partial^2}{\partial s^2}u_n(x,y,s) \\ &- \frac{1}{12}\Big(x^2\frac{\partial^2}{\partial x^2}u_n(x,y,s) + y^2\frac{\partial^2}{\partial y^2}u_n(x,y,s)\Big)\Bigg]ds \end{aligned} \tag{31}$$

We select an initial approximation

$$u_0(x,y,t) = u(x,y,0) + \frac{\partial^\alpha}{\partial t^\alpha}u(x,y,0) = x^4 + y^4 t \quad \textit{for} \quad 0 < \alpha \leq 1$$

by the following successive approximations can be obtained :

$$\begin{aligned} u_1(x,y,t) &= x^4\left[1 + \frac{t^2}{2!}\right] + y^4\left[t + \frac{t^3}{3!}\right] \\ u_2(x,y,t) &= x^4\left[1 + 2\frac{t^2}{2!} - \frac{t^{4-\alpha}}{(3-\alpha)(4-\alpha)} + \frac{t^4}{4!}\right] + y^4\Big[t + 2\frac{t^3}{3!} \\ &- \frac{t^{5-\alpha}}{(4-\alpha)(5-\alpha)} + \frac{t^5}{5!}\Big] \\ u_3(x,y,t) &= x^4\Bigg[1 + 3\frac{t^2}{2!} - 3\frac{t^{4-\alpha}}{(3-\alpha)(4-\alpha)} + \frac{t^{6-2\alpha}}{(5-2\alpha)(6-2\alpha)} \\ &- \frac{t^{6-\alpha}}{(4-\alpha)(5-\alpha)(6-\alpha)} - \frac{t^{6-\alpha}}{(3-\alpha)(4-\alpha)(5-\alpha)(6-\alpha)} \\ &+ 3\frac{t^4}{4!} + \frac{t^6}{6!}\Bigg] \\ &+ y^4\Bigg[t + 3\frac{t^3}{3!} - 3\frac{t^{5-\alpha}}{(4-\alpha)(5-\alpha)} + \frac{t^{7-2\alpha}}{(6-2\alpha)(7-2\alpha)} \\ &+ 3\frac{t^5}{5!} - \frac{t^{7-\alpha}}{3!(6-\alpha)(7-\alpha)} - \frac{t^{7-\alpha}}{(4-\alpha)(5-\alpha)(6-\alpha)(7-\alpha)} \\ &+ \frac{t^7}{7!}\Bigg] \end{aligned}$$

when we consider $\alpha = 1$, *the solution by C-VIM is obtained as*

$$\begin{aligned} u_n(x,y,t) &= x^4\Big[1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots + \frac{t^{2n}}{(2n)!}\Big] + y^4\Big[t + \frac{t^3}{3!} + \frac{t^5}{5!} \\ &\quad + \cdots + \frac{t^{2n+1}}{(2n+1)!}\Big] \end{aligned}$$

We, therefore, obtain the approximate solution reads

$$u = \lim_{n\to\infty} u_n(x,y,t) = x^4\cosh(t) + y^4\sinh(t)$$

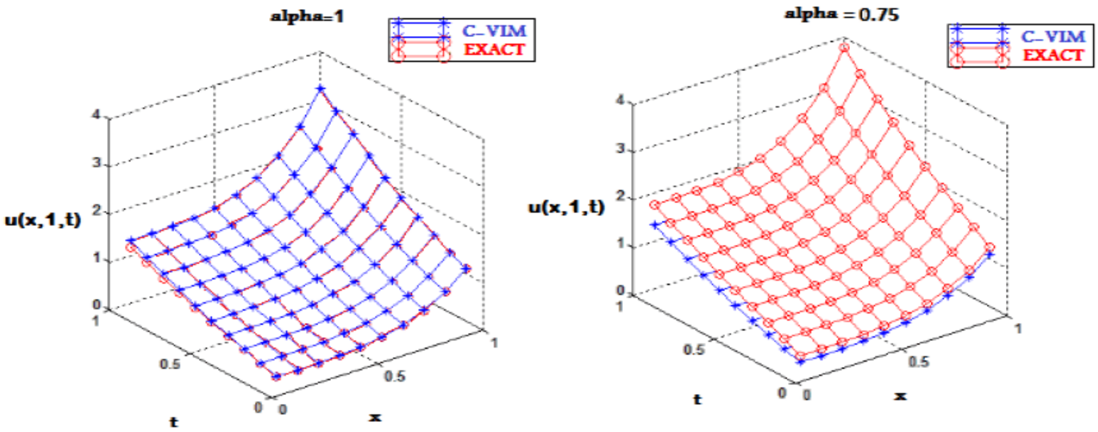

Figure 4. Comparison of three iteration C-VIM solutions with the exact solutions for Eq (26)

which is the exact solution.
Now, we compare the C-VIM solution with the exact solution on the graphs for some α*. These comparisons can be seen in fig 4.*

5. Conclusions

The main goal of this work was to propose a reliable method for solving conformable heat-like and wave-like equations with variable coefficients. The main advantage of this method is the flexibility to give approximate solutions to conformable problems because the proposed equations may not be solved by the method of separation of variables. The conformable variational iteration method has worked effectively and simply to handle these models see figures, and this gives it wider applicability. The proposed scheme was applied directly. The approach was tested by employing the method to obtain exact solutions for four numerical examples, two from each type (one-dimensional and two-dimensional for Conformable Heat-like and Conformable wave-like equations). The results obtained in all cases demonstrate the reliability and the efficiency of this method.
For further research, we propose the study of fractional differential dynamical systems [8, 26]. In addition, we propose to extend the results of the present paper and combine them with the results in the epidemic model [16, 17].

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