TWMS J. App. Eng. Math. V.13, N.2, 2023, pp. 440-447

ON FRACTIONAL INTEGRAL OPERATOR OVER NON-NEWTONIAN CALCULUS

M. MOMENZADEH[1], S. NOROZPOUR[2], A. E. BASHIROV[3*], §

Abstract. The definition of a non-Newtonian calculus is based on the homeomorphism which customary denoted by $y = \alpha(x)$. In the mean of this function, elementary algebraic operations can be modified and we reach to the world of new calculus that is called a Non-Newtonian calculus. Nowadays, fractional operators role an important topic in mathematics because of their applications in many area of interest. In this paper we use an old technique of Cauchy iterated integrals to define biα-fractional integral operator. The allocated method makes the new class of fractional integral operators which are successfully compatible with the non-Newtonian calculi and supported with several examples. Since the non-Newtonian calculi were introduced, the bigeometric calculus has been considered as a brilliant example of these kind of calculi. The definition of fractional integral operator in this calculus leads to Hadamard type fractional integral operator which answers many questions about the behavior of this operator. Classic property of fractional integral operator, semigroup property is stablished and this operator is studied. Moreover, Jensen's inequality provide boundness theorem for general biα-fractional integral operator.

Keywords: Fractional differential operators, bigeometric calculus, Cauchy iterated integrals, integral operator.

AMS Subject Classification: 30E20.

1. Introduction

The recent studies on fractional differential equations indicate that a variety of interesting and important results have been obtained, and the surge for investigating more and more results are underway. This concerns the existence and uniqueness of solutions, the stability properties of solutions, the analytic and numerical methods of solutions etc.

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[1] Near East University, Lefkosa, Mersin 10, Turkey.
e-mail: mohammad.momenzadeh@neu.edu.tr; ORCID: https://orcid.org//0000-0002-7830-9820.
[2] Faculty of Economics, Administrative and Social Sciences, Nişantaşı University, Istanbul, Turkey.
e-mail: snorozpour@gelisim.edu.tr; ORCID: https://orcid.org//0000-0003-3542-1932.
[3] Eastern Mediterranean University, Gazimagusa, Mersin 10, Turkey.
e-mail: agamirza.bashirov@emu.edu.tr; ORCID: https://orcid.org//0000-0002-3083-6314.
[*] Corresponding author.
§ Manuscript received: March 10, 2021; accepted: June 07, 2021.
TWMS Journal of Applied and Engineering Mathematics, Vol.13, No.2

for these equations. The tools of fractional calculus have played a significant role in improving the modeling techniques for several real-world problems. However, it has been noticed that most of the works in the area are based on Riemann-Liouville and Caputo fractional differential equations. Another kind of fractional derivatives that appears side by side to Riemann-Liouville and Caputo derivatives in the literature is the fractional derivative due to Hadamard, which contains logarithmic function of arbitrary exponent in the kernel of the integral. In 1892, Hadamard began the publication of series of articles under the common title [1]. Third section of this article gave an underlying idea for creating different form of fractional integral operators. In this section, Hadamard investigated the relation between coefficients of series with unite radius of convergent and singularity of function. He applied similar method of Riemann [2] and extend Reimann fractional integral operator to the form that is known as the Hadamard fractional integral operator. In part (33) of section 3 of this paper, he considered that if $x = e^y$, where the function is with respect to y, then the Reimann formulation is changed to the Hadamard form. Hadamard-type integrals arise in the formulation of many problems in mechanics such as in fracture analysis.

2. Overview of Bigeometric Calculus

We start our discussion by rewriting some concepts and definitions of bigeometric calculus. In the 60th decade Michael Grossman and Robert Katz [3] gave an underlying idea for creating different presentations of Newtonian calculus. Briefly, if α is a homeomorphism from the system $\mathbb{R}$ of real numbers to the interval $\mathbb{I}$, then the algebraic operations in $\mathbb{R}$ can be isometrically transferred to $\mathbb{I}$ by letting

(i) $a \oplus_\alpha b = \alpha(\alpha^{-1}(a) + \alpha^{-1}(b))$,
(ii) $a \otimes_\alpha b = \alpha(\alpha^{-1}(a) \times \alpha^{-1}(b))$,
(iii) $a \ominus_\alpha b = \alpha(\alpha^{-1}(a) - \alpha^{-1}(b))$,
(iv) $a \oslash_\alpha b = \alpha(\alpha^{-1}(a)/\alpha^{-1}(b))$.

These α-operations form a field on $\mathbb{I}$. Based on α-operations, α-calculus with the α-derivative and α-integral can be created. All these calculi are isometric and present different views to the same phenomena so that an easily developing issue in one of them may cause complications in the other ones. The case when α is the identity function, α-calculus is simply a familiar Newtonian calculus. Otherwise, it presents a non-Newtonian calculus.

In the existing literature, the most investigated non-Newtonian calculus is the case of the exponential function $\alpha(x) = e^x$. The α-operations in the case of the exponential function become

(i) $a \oplus_{\exp} b = e^{\ln a+\ln b} = ab$,
(ii) $a \otimes_{\exp} b = e^{\ln a \ln b} = a^{\ln b} = b^{\ln a}$,
(iii) $a \ominus_{\exp} b = e^{\ln a-\ln b} = a/b$,
(iv) $a \oslash_{\exp} b = e^{\ln a/\ln b} = a^{1/\ln b}$.

These operations define a field in the range $\mathbb{I} = (0, \infty)$ of the exponential function in which the neutral elements of exp-addition and exp-multiplication are 1 and e, respectively.

On the basis of α-operations, it is possible to set two calculi. One of them is called α-calculus, the other one biα-calculus. The derivative of biα-calculus, called biα-derivative and denoted by $f^{\hat{\alpha}}$, is defined as the limit

$$f^{\hat{\alpha}}(x) = \lim_{y\to x}[(f(y) \ominus_\alpha f(x)) \oslash_\alpha (y \ominus_\alpha x)].$$

Similarly, the integral of biα-calculus, called biα-integralis defined as the limit of the integral sums

$$\int_a^b f(x)\, d^{\hat{\alpha}}(x) = \lim \bigoplus_{i=1}^{n} {}_{\alpha}[f(c_i) \otimes_\alpha (x_{i+1} \ominus_\alpha x_i)].$$

They are related to the ordinary derivative and integral of Newtonian calculus as

$$f^{\hat{\alpha}}(x) = \alpha\bigg(\frac{\alpha^{-1}(f(x))'}{\alpha^{-1}(x)'}\bigg)$$

and

$$\int_a^b f(x)\, d^{\hat{\alpha}}x = \alpha\bigg(\int_a^b \alpha^{-1}(f(x))\alpha^{-1}(x)'\, dx\bigg).$$

It is not difficult to see that the removal of $\alpha^{-1}(x)'$ from these formulae modifies them to a new form as

$$f^{\alpha}(x) = \alpha(\alpha^{-1}(f(x))')$$

and

$$\int_a^b f(x)\, d^{\alpha}x = \alpha\bigg(\int_a^b \alpha^{-1}(f(x))\, dx\bigg).$$

These are called α-derivative and α-integral and they define α-calculus.

In the case of the exponential function $\alpha(x) = e^x$, these calculi are called geometric (frequently, it is called multiplicative as well) and bigeometric. Bigeometric derivative and integral are also called π-derivative and π-integral and defined by

$$f^{\pi}(x) = e^{x(\ln f(x))'} \quad \text{and} \quad \int_a^b f(x)\, d^{\pi}(x) = e^{\int_a^b \frac{\ln f(x)}{x}\, dx}.$$

Thus, $\hat{\alpha}$ is replaced by π. Similarly, geometric derivative and integral are defined by

$$f^{*}(x) = e^{(\ln f(x))'} \quad \text{and} \quad \int_a^b f(x)\, d^{*}(x) = e^{\int_a^b \ln f(x)\, dx},$$

where α is replaced by $*$.

Bigeometric calculus was prompted in [5]. Recently, it was successfully extended to complex bigeometric calculus in [6]. Multiplicative calculus was pointed out in [7]. Since then it was investigated andapplied to different areas [8-17]. In the sequel, we will use the symbols $Df(x)$ and $I_a^b f$ for the biα-derivative and integral, that is,

$$Df(x) = \alpha\bigg(\frac{\alpha^{-1}(f(x))'}{\alpha^{-1}(x)'}\bigg) \quad \text{and} \quad I_a^b f = \alpha\bigg(\int_a^b \alpha^{-1}(f(x))\alpha^{-1}(x)'\bigg)dx.$$

3. Fractional Integral Operator over biα-Calculus

Now we try to extend the meaning of biα-derivative and biα-integral to reach fractional biα-integral. Iteration of two biα-integrals of $f(x)$ leads to

$$\begin{aligned}\int_a^w \int_a^v f(s)\, d^{\hat{\alpha}}s d^{\hat{\alpha}}v &= \alpha\bigg(\int_a^w \int_a^v \alpha^{-1}(f(s))\alpha^{-1}(s)'^{-1}(v)'\, dsdv\bigg)\\ &= \alpha\bigg(\int_a^w \alpha^{-1}(f(s))\alpha^{-1}(s)'^{-1}(w) - \alpha^{-1}(s)]\, ds\bigg).\end{aligned}$$

For $\alpha(x) = e^x$,

$$\int_a^w \int_a^v f(s)\, d^\pi s d^\pi v = e^{\int_a^w \ln f(s)(\ln s)'[\ln w - \ln s]\, ds}$$
$$= e^{\int_a^w \ln\left(f(s)^{\ln \frac{w}{s}}\right)(\ln s)'\, ds}$$
$$= \int_a^w f(s)^{\ln \frac{w}{s}}\, d^\pi s. \tag{1}$$

Because of the form of the integrand, this formula is well suited to the fractional calculus. The argument that leads to a definition of the fractional integral can be studied by considering n-fold integral which is called Cauchy iterated integration:

$$\int_a^x dx_1 \int_a^{x_1} dx_2 \cdots \int_a^{x_{n-1}} f(x_n) dx_n = \frac{1}{(n-1)!} \int_a^x (x-s)^{n-1} f(s) ds$$

If we consider $\frac{dx_i}{x_i}$ instead of dx_i in the chain ofintegration, then we have Hadamard approaches as

$$\int_a^x \frac{dx_1}{x_1} \int_a^{x_1} \frac{dx_2}{x_2} \cdots \int_a^{x_{n-1}} f(x_n) \frac{dx_n}{x_n} = \frac{1}{(n-1)!} \int_a^x \left(\ln \frac{x}{s}\right)^{n-1} f(s) \frac{ds}{s}.$$

There are several approaches to generalize fractional integral operator which are inspired by the form of Cauchy iterated integral [18, 19]. For instance, assume the following n-fold biα-integral as an extension of Cauchy iterated integral

$$I_\alpha^n(f)(x) = \int_a^x \int_a^{x_1} \cdots \int_a^{x_{n-1}} f(x_n) d^{\hat{\alpha}} x_n d^{\hat{\alpha}} x_{n-1} \cdots d^{\hat{\alpha}} x_1. \tag{2}$$

Applying (1) repeatedly and changing the order of integration leads to

$$I_\alpha^n(f)(x) = \alpha\left(\frac{1}{(n-1)!} \int_a^x \left(\alpha^{-1}(x) - \alpha^{-1}(s)\right)^{n-1} \alpha^{-1}(f(s)) \alpha^{-1}(s)'\, ds\right). \tag{3}$$

This identity can be proved easily by induction. So we can define fractional integral operator as follow:

Definition 3.1. *Let $\alpha(x)$ be a homeomorphism from the system $\mathbb{R}$ of real numbers to the interval $\mathbb{I}$ and $\beta > 0$, then biα-fractional integral operator is defined by*

$$I_\alpha^\beta(f)(x) = \alpha\left(\frac{1}{\Gamma(\beta)} \int_a^x \left(\alpha^{-1}(x) - \alpha^{-1}(s)\right)^{\beta-1} \alpha^{-1}(f(s)) \alpha^{-1}(s)'\, ds\right). \tag{4}$$

Let us consider this definition deeply. In the case that $\alpha(x) = x$, wehave Reimann fractional integral operator. For instance, let $\alpha(x) = e^x$, then we can express bigeometric fractional integral operator as

$$I_{\exp}^\beta(f)(x) = \exp\left(\frac{1}{\Gamma(\beta)} \int_a^x \left(\ln\left(\frac{x}{s}\right)\right)^{\beta-1} \ln f(s)\, \frac{ds}{s}\right) = e^{J^\beta(\ln(f(x))}.$$

Here, J^β denotes Hadamard fractional integral operator. When $\beta \to 1$, given operator tends to π-derivative, i.e., $I_{\exp}^\beta(f)(x) \to f^\pi(x)$. We should mention that defined operator is not only the simple transformation of fractional integral operator. In fact, nature of function determine suitable biα-calculus and related biα-derivative, then related biα-fractional integral operator can be expressed by last definition. Recently, unification of Riemann and Hadamard operator was studied [18]. In fact, the author used x_i^p terms in chain of iterated integration. The parameter p ranges between 0 and 1. Therefore, different values of p create different types of fractional integral operator. Let us consider $\alpha(x) = x^{\frac{1}{p}}$ with $0 < p < 1$. Then biα-fractional integral operator can be written as

$$I^{\beta}_{\sqrt[p]{\cdot}}(f)(x) = \left(\frac{p}{\Gamma(\beta)} \int_a^x (x^p - s^p)^{\beta-1} s^{p-1} (f^p(s)) ds \right)^{\frac{1}{p}} = \left(p^{\alpha} J^{\beta}_p (f^p(x)) \right)^{\frac{1}{p}} \tag{5}$$

Here J^{β}_p is Katugampola fractional integral operator. We should remark that $\alpha(x) = x^{\frac{1}{p}} = \exp\left(\frac{\ln x}{p}\right)$ is not satisfying the mentioned conditions for makeing the calculus generally. In chapter 7 of Grossman's book [3], quadratic calculus for $p = 2$ was introduced and similar discussion can be used to make a sure about this kind of calculi.

Proposition 3.1. *Let α be a homeomorphism from the system $\mathbb{R}$ of real numbers to the interval $\mathbb{I}$. Then the semigroup property*

$$I^{\beta}_{\alpha} I^{\nu}_{\alpha}(f)(x) = I^{\beta+\nu}_{\alpha}(f)(x)$$

holds for $\beta > 0$ and $\nu > 0$ provided that the respective integrals exist.

Proof. Procedure of prove is similar to the prove of semigroup property for fractional integral operator. If we expand left side of expression, we have

$$\begin{aligned} I^{\beta}_{\alpha}\left(I^{\nu}_{\alpha}(f)(x)\right) = \alpha \Bigg(\frac{1}{\Gamma(\nu)\Gamma(\beta)} \int_a^x \Bigg(\int_w^x & \left(\alpha^{-1}(x) - \alpha^{-1}(s)\right)^{\beta-1} \\ & \times \left(\alpha^{-1}(s) - \alpha^{-1}(w)\right)^{\nu-1} \alpha^{-1}(s)' ds \Bigg) \alpha^{-1}(f(w)) \alpha^{-1}(w)' \, dw \Bigg). \end{aligned}$$

Applying the substitution $Y = \left(\alpha^{-1}(s) - \alpha^{-1}(w)\right) / \left(\alpha^{-1}(x) - \alpha^{-1}(w)\right)$ and using the definition of the beta function, we obtain

$$\begin{aligned} I^{\beta}_{\alpha}\left(I^{\nu}_{\alpha}(f)(x)\right) = \alpha \Bigg(\frac{1}{\Gamma(\nu+\beta)} \int_a^x \Bigg(& \left(\alpha^{-1}(x) - \alpha^{-1}(w)\right)^{\nu+\beta-1} \\ & \times \int_0^1 (1-Y)^{\beta-1} (Y)^{\nu-1} dY \Bigg) \alpha^{-1}(f(w)) \alpha^{-1}(w)' \, dw \Bigg). \end{aligned}$$

This proves the semi-group property. □

4. Properties of biα-Fractional Integral Operator

In this section, we apply the Jensen's inequality to the biα-fractional integral operator. Let us start our discussion by one probability concept, mean. We consider the following definition of mean in biα-calculus

$$\overline{X}_{\alpha} = \left(\bigoplus_{i=1}^{n} {}_{\alpha} x_i \right) \oslash_{\alpha} n.$$

If we put $\alpha(x) = x$, we obtain the definition of arithmetic mean. On the other hand, letting $\alpha(x) = e^x$ and changing n with the corresponding value e^n in bigeometric calculus lead to geometric mean. The relation between arithmetic and geometric means can be answered by convexity of e^x which generally reaches to Jensen's inequality [20].

Theorem 4.1. *Let μ be a positive measure on a σ-algebra $\mathcal{M}$ in a set Ω so that $\mu(\Omega) = 1$. If f is a real μ-integrable function, if $-\infty \leq a \leq f(x) \leq b \leq \infty$ for all $x \in \Omega$ and if $\alpha(x)$ is convex on (a, b), then*

$$\alpha\left(\int_{\Omega} f d\mu \right) \leq \int_{\Omega} \alpha(f) d\mu \tag{6}$$

If $\alpha(x)$ is concave, then (6) holds in the reversed direction.

In the case when $\Omega = \mathbb{R}^+$, $\mu = \frac{1}{n}\sum_{k=1}^{n}\delta_k$, $\alpha(x) = \ln x$, where δ_k is the unity mass at $t = k$, then Jensen's inequality coincides with the inequality of arithmetic and geometric means. Since boundedness of introduced operator is closely related to behavior of $\alpha(x)$, we consider the restriction on this function to be concave. Definitely, concavity of $\alpha(x)$ leads to similar inequality with reversed direction. One of brilliant examples of α is the case that $\alpha(x) = e^x$ which is convex function. For instance, it is easy to see that

$$e^{\frac{1}{b-a}\int_a^b \frac{\ln f(x)}{x}dx} \leq \frac{1}{b-a}\int_a^b f(x)^{\frac{1}{x}}dx.$$

Proposition 4.1. *Let $\alpha(x)$ be a convex homeomorphism and let (4) be defined both for f and $\alpha \circ f$. In addition, let $0 \leq K \leq 1$ for*

$$K = \frac{1}{\Gamma(\beta+1)}\left(\alpha^{-1}(x) - \alpha^{-1}(a)\right)^{\beta}. \tag{7}$$

Then

$$I_\alpha^\beta(f)(x) \leq I_\alpha^\beta(\alpha \circ f)(x) + (1-K)\,\alpha(0) \tag{8}$$

Proof. Convexity of $\alpha(x)$ and condition of $0 \leq K \leq 1$, implies that

$$I_\alpha^\beta(f)(x) \leq K\alpha\Bigg(\frac{1}{\int_a^x \left(\alpha^{-1}(x) - \alpha^{-1}(r)\right)^{\beta-1}\left(\alpha^{-1}\right)'(r)dr}$$
$$\times \int_a^x f(s)\left(\alpha^{-1}(x) - \alpha^{-1}(s)\right)^{\beta-1}\left(\alpha^{-1}\right)'(s)ds\Bigg) + (1-K)\,\alpha(0).$$

Now, apply Jensen's inequality and complete the proof. □

Moreover, we can consider following inequality for $\alpha^{-1}\left(I_\alpha^\beta(f)(x)\right)$. This inequality makes relation clear:

Theorem 4.2. *Let $\alpha(x)$ be a convex homeomorphism and let (4) be defined both for f and $\alpha^{-1} \circ f$. Assume that $\alpha^{-1} \circ f \in L^p(a,b)$ where $1 \leq p \leq \infty$, $a \geq 0$ and $\beta > 0$. Then*

$$\left\|\alpha^{-1}\left(I_\alpha^\beta(f)(x)\right)\right\|_p \leq K\left\|\alpha^{-1} \circ f\right\|_p$$

where K is defined by (7).

Proof. In the aid of substitution $u = \alpha^{-1}(x) - \alpha^{-1}(a)$ and using generalized Minkowski inequality, we have

$$\left\|\alpha^{-1}\left(I_\alpha^\beta(f)(x)\right)\right\|_p$$
$$= \left(\int_a^b \left|\frac{1}{\Gamma(\beta)}\int_0^{\alpha^{-1}(x)-\alpha^{-1}(a)} \left(\alpha^{-1} \circ f\right)\left(\alpha\left(\alpha^{-1}(x) - u\right)\right)u^{\beta-1}du\right|^p dx\right)^{\frac{1}{p}}$$
$$\leq \frac{1}{\Gamma(\beta)}\int_0^{\alpha^{-1}(x)-\alpha^{-1}(a)} u^{\beta-1}\left(\int_a^b \left|\left(\alpha^{-1} \circ f\right)\left(\alpha\left(\alpha^{-1}(x) - u\right)\right)\right|^p dx\right)^{\frac{1}{p}} du.$$

This completes the proof. □

By putting $\alpha(x) = x^{\frac{1}{p}}$, bi$\alpha$-fractional integraloperator is related to Katugampola's fractional operator [18] by equation (5). Actually, Katugampola's operator is a unification of Reimann and Hadamard's operators and this has done by the parameter $0 < p < 1$

which makes α as a concave function. In the aid of Theorem 2 and relation (5), we can reach to the following inequality

$$\left\| \left(p^{\alpha} J_p^{\beta}(f^p(x)) \right) \right\|_q \leq K \left\| f^p(x) \right\|_q .$$

This inequality can be compared by Theorem 3.1 from [18]. In fact, the author of that paper has used weighted L^p space which is denoted by $X_c^p(a,b)$ and found boundness of operator similar to above equation. Discussion about boundness of Hadamard's fractional integral operator in the general form is studied at [21].

5. Conclusion

In this article, we introduced fractional integral operator in biα-calculus. This definition is based on iterated Cauchy integral. Some related theorems are investigated. We introduced vast number of fractional integral operators and bijection in this space easily convert fractional integral operators. There are a lot of number of papers that discussed about the application of different forms of fractional integral operator and new approach in this article can be useful there.

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Mohammad Momenzadeh was born in 1982 in Tehran, Iran. He completed his undergraduate degree at the Shiraz State University, Department of Mathematics (2000-2004) Graduate degree at Urmiyeh State University (2006) PhD at Eastern Mediterranean University, Department of Mathematics In Jan. 2016. He has worked as an assistant professor at Near East University since 2016 with interests of special functions, q-calculus and fractional calculus.

Sajedeh Norozpour was born in 1990, in the Rasht, Iran. She completed her undergraduate and master degrees at Guilan University, Faculty of Art and Science, Department of Mathematics (2006-2013). She followed her PhD (2018) at Eastern Mediterranean University, Department of Mathematics. She presented her thesis in title of "On an Alternative View to Complex Calculus", and was awarded the title of Doctor of Mathematics. Currently she is working at Istanbul Nişantaşı University.

Agamirza E. Bashirov for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.1, N.1.