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GENERALIZED CONTRACTIVE MAPPINGS ON QUASIMETRIC SPACES

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Abstract. In this paper, we introduce a new class of generalized contractive mappings to establish a fixed point theorem for this class of mappings in complete quasimetric spaces. In fact, at first we present the notion of ordered cyclic weakly (ψ, φ, A, B)-contraction and then we establish a fixed point theorem for such mappings in complete ordered s-quasimetric spaces. This can be considered as an improvement of some old fixed point theorems in the literature. Finally, we provide an example to show that our result is genuine generalizations of some fixed point theorems.

Keywords: Complete s-quasimetric spaces, cyclic contraction, fixed point.

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1. Introduction

Let (X, d) be a metric space and $T : X \to X$ be a mappings. The mappings T is called a φ-weak contraction if for each $x, y \in X$, there exists a function $\varphi : [0, \infty) \to [0, \infty)$ such that φ is positive on $(0, \infty)$, $\varphi(0) = 0$ and

$$d(Tx, Ty) \le d(x, y) - \varphi(d(x, y)).$$

The concept of the weak contraction was defined by Alber and Guerre-Delabriere [1] in 1997. In 2001, Roades proved that under certain circumstances every φ-weak contraction on a complete metric space has a fixed point.

Theorem 1.1. [10] *Let (X, d) be a nonempty complete metric space and $T : X \to X$ be a φ-weak contraction on X. If φ is a continuous and nondecreasing function with $\varphi(t) > 0$ for all $t > 0$ and $\varphi(0) = 0$, then T has a unique fixed point.*

Dutta and Choudhury improved the above result as follow.

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Theorem 1.2. [6] *Let (X, d) be a nonempty complete metric space and let $T : X \to X$ be a self mapping satisfying the inequality*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y))$$

where ψ,φ:$[0,\infty) \to [0,\infty)$ are both continuous and monotone non decreasing functions with $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

In 2009, Doric obtained the following generalization of Theorem 1.2.

Theorem 1.3. [5] *Let (X, d) be a nonempty complete metric space and let $T : X \to X$ be a self mapping satisfying the inequality*

$$\psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y)),$$

for any $x, y \in X$, where M is given by

$$M(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\},$$

and

(a) $\psi : [0,\infty) \to [0,\infty)$ *is a continuous monotone non decreasing function with* $\psi(t) = 0$ *if and only if* $t = 0$,

(b) $\varphi : [0,\infty) \to [0,\infty)$ *is a lower semi-continuous function with* $\varphi(t) = 0$ *if and only if* $t = 0$.

Then T has a unique fixed point.

The aim of this paper is to obtain some improvements of the above results, when the underlying space is a b-metric or s-quasimetric space.

2. Preliminaries

Throughout this paper, $\mathbb{R}$ denotes the real line, and $\mathbb{N}$ is the set of all natural numbers. We recall some preliminaries that will be needed in the sequel.

Definition 2.1. [4] *Let X be a nonempty set and $s \geq 1$ a given real number. A function $d : X \times X \to \mathbb{R}^+$ is called a b-metric provided that, for all $x, y, z \in X$, the following conditions hold:*

i) $d(x, y) = 0$ *if and only if* $x = y$;

ii) $d(x, y) = d(y, x)$;

iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space with the parameter s.

Example 2.1. *Let (X, ρ) be a metric space and $d(x, y) = (\rho(x, y))^p$, where $p > 1$ is a real number. Using convexity of the function $f(x) = x^p$ for $x > 0$, we see that d is a b-metric with the parameter $s = 2^{p-1}$.*

Remark 2.1. *Every metric space is a b-metric space with $s = 1$, however, the converse is not true in general. For example if X is the real numbers with usual metric, according to the above example, $(X, |.|^2)$ is a b-metric space which is not metric, since $|3 - 1|^2 \not\leq |3 - 2|^2 + |2 - 1|^2$.*

Definition 2.2. [3] *Let (X, d) be a b-metric space.*

(i) A sequence $\{x_n\}$ in X is called convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\}$ in X is called Cauchy if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

(*iii*) *The b-metric space* (X, d) *is said to be complete if every Cauchy sequence in* X *is convergent.*

(*iv*) *A set* $B \subseteq X$ *is said to be closed if for any sequence* $\{x_n\}$ *in* B *which* $\{x_n\}$ *is convergent to* $z \in X$*, we have* $z \in B$.

Definition 2.3. [4] *A pair* (X, d) *consisting of a non-empty set* X *and a function* $d : X \times X \to [0, \infty)$ *is said to be a semimetric space if it satisfies the following conditions:*

i) $d(x, y) = 0$ *if and only if* $x = y$;

ii) $d(x, y) = d(y, x)$;

for all $x, y \in X$. *The function* d *is then called a semimetric.*

Definition 2.4. [2] *A semimetric space* (X, d) *is called a quasimetric space or, more specifically, a s-quasimetric space, where* $s \geq 1$ *is fixed, if it satisfies the following condition:*

(*iii*) $d(x, z) \leq s.max\{d(x, y), d(y, z)\}$

for all $x, y, z \in X$. *In this case, the function* d *called a s-quasimetric.*

A 1-quasimetric space is known broadly in the literature as an ultrametric space. Every 1-quasimetric space is therefore a metric space. On the other hand, every metric space is, in fact, a 2-quasimetric space. The reverse, however, does not hold even for $1 < s \leq 2$, i.e., there exist s-quasimetric spaces which are not metric, see e. g. the following example.

Example 2.2. *Consider a set* $X := \{a, b, c\}$ *consisting of three distinct elements. Define a function* $d : X \times X \to \mathbb{R}$ *as follows: put* 0 *for* $d(a, a)$, $d(b, b)$ *and* $d(c, c)$, *and* $d(a, b) = d(b, a) = 1$, $d(a, c) = d(c, a) = 4$, $d(c, b) = d(b, c) = 2$. *One may think of such a space as a non-existent triangle, where the lengths of its sides are equal to* 1, 2 *and* 4, *respectively.*

In [7] Frink provided an innovative method for constructing a metric equivalent to a 2-quasimetric.

Theorem 2.1. [7]*(Frink) Let* (X, d) *be a* 2*-quasimetric space. Then there exists a metric* ρ *on* X *such that*

$$\forall x, y \in X, \rho(x, y) \leq d(x, y) \leq 4\rho(x, y)$$

In this theorem the metric ρ is obtained by the so-called chain approach, namely, ρ is defined by

$$\rho(x, y) = \inf \sum_{i=1}^{n} d(x_{i-1}, x_i);$$

where the infimum is taken over all finite sequences of points $x_0, x_1, x_2, \ldots, x_n$, where $x_0 = x$ and $x_n = y$, guaranteeing that the inequality is satisfied.

Schroeder[12] improved Frink's theorem as follows.

Theorem 2.2. [12]*(Schroeder) Let* (X, d) *be a s-quasimetric space with* $s \leq 2$. *Then there exists a metric* ρ *on* X *such that*

$$\forall x, y \in X, \rho(x, y) \leq d(x, y) \leq 2s\rho(x, y).$$

Recently, the above result generalized as follows.

Theorem 2.3. [2] *If* (X, d) *is a s-quasimetric space with* $s \leq 2$, *then there exists a metric* ρ *on* X *for which the following inequalities hold:*

$$\forall x, y \in X, \rho(x, y) \leq d(x, y) \leq s^2\rho(x, y).$$

Definition 2.5. *A function $f : X \to \mathbb{R}$ is called lower semi-continuous, if for any $\{x_n\} \subset X$ and $x \in X$*

$$x_n \to x \Rightarrow f(x) \leq \liminf_{n\to\infty} f(x_n).$$

Definition 2.6. [9] *A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:*

(i) ψ is continuous and nondecreasing;
(ii) $\psi(0) = 0$ and $\psi(t) > 0$ for all $t > 0$.

We define Ψ to be the set of all function ψ.

Throughout the paper, let Φ be the set of all functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(i) φ is lower semi-continuous;
(ii) $\varphi(0) = 0$ and $\varphi(t) > 0$ for each $t > 0$.

In [11], the following common fixed point result for contractions in ordered b-metric spaces were proved.

Theorem 2.4. [11] *Let $(X, \leq, d)$ be a complete ordered b-metric space and let $T, S : X \to X$ be two weakly increasing mappings. Suppose that there exist two altering distance functions ψ,φ and a constant $L \geq 0$ such that the inequality*

$$\psi(s^4 d(Tx, Sy)) \leq \psi(M_s(x, y)) - \varphi((M_s(x, y))) + L\psi(N(x, y)),$$

holds for all comparable $x, y \in X$, where

$$M_s(x, y) = \max\{d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2s}[d(x, Sy) + d(y, Tx)]\}$$

and

$$N(x, y) = min\{d(y, Sy), d(x, Sy), d(y, Tx)\}$$

If either [T or S is continuous], or the space $(X, \leq, d)$ is regular, then T and S have a common fixed point.
Here, the ordered metric space $(X, \leq, d)$ is called regular if for any non-decreasing sequence $\{x_n\}$ in X such that $x_n \to x \in X$, as $n \to \infty$, one has $x_n \leq x$ for all $n \in \mathbb{N}$.

3. Main results

Definition 3.1. [8] *Let $(X, \leq, d)$ be an ordered b-metric space, let $T, S : X \to X$ be two mappings, and let A and B be nonempty closed subsets of X. The pair (T, S) is called an ordered cyclic weakly (ψ, φ, L, A, B)-contraction if*

(1) *$X = A \cup B$ is a cyclic representation of X w.r.t. the pair (T, S); that is, $TA \subseteq B$ and $SB \subseteq A$;*
(2) *there exist two altering distance functions ψ,φ and a constant $L \geq 0$, such that for arbitrary comparable element $x, y \in X$ with $x \in A$ and $y \in B$, we have*

$$\psi(s^2 d(Tx, Sy)) \leq \psi(M_s(x, y)) - \varphi((M_s(x, y))) + L\psi(N(x, y)),$$

where

$$M_s(x, y) = \max\{d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2s}[d(x, Sy) + d(y, Tx)]\}$$

and

$$N(x, y) = min\{d(y, Sy), d(x, Sy), d(y, Tx)\}$$

Definition 3.2. [13] *Let $(X, \leq)$ be a partially ordered set, and let A and B be closed subsets of X with $X = A \cup B$.*
Let $T, S : X \to X$ be two mappings. The pair (T, S) is said to be (A, B)-weakly increasing if $Tx \leq STx$ for all $x \in A$ and $Sy \leq TSy$ for all $y \in B$.

In [8] Hussain et al. proved the following:

Theorem 3.1. *Let $(X, \leq, d)$ be a complete ordered b-metric space, and let $A$and B be closed subsets of X. Let $T, S : X \to X$ be two (A, B)-weakly increasing mappings with respect to $\leq$. Suppose that*

(a) *the pair (T, S) is an orderer cyclic weakly (ψ, φ, L, A, B)-contraction*
(b) *T or S continuous.*

Then T and S have a common fixed point $u \in A \cap B$.

Definition 3.3. *Let $(X, \leq, d)$ be an ordered s-quasimetric space with $1 < s \leq 2$, let $T, S : X \to X$ be two mappings, and let A and B be nonempty closed subsets of X. The pair (T, S) is called an ordered cyclic weakly (ψ, φ, A, B)-contraction if*

(1) *$X = A \cup B$ is a cyclic representation of X w.r.t. the pair (T, S); that is, $TA \subseteq B$ and $SB \subseteq A$;*
(2) *there exist two altering distance functions ψ, φ such that for arbitrary comparable element $x, y \in X$ with $x \in A$ and $y \in B$, we have*

$$\psi(2sd(Tx, Sy)) \leq \psi(M(x, y)) - \varphi((M(x, y))), \tag{1}$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Sy), \frac{1}{s}\max[d(x, Sy), d(y, Tx)]\right\}$$

Theorem 3.2. *Let $(X, \leq, d)$ be a complete ordered s-quasimetric space with $1 < s \leq 2$, let A and B be closed subsets of X. Let $T, S : X \to X$ be two (A, B)-weakly increasing mappings with respect to $\leq$. Suppose that*

(a) *the pair (T, S) is an orderer cyclic weakly (ψ, φ, A, B)-contraction*
(b) *T or S is continuous.*

Then T and S have a common fixed point $u \in A \cap B$.

Proof. Let us divide the proof into five steps.
Step 1: We prove that $u \in A \cap B$ is a fixed point of T if and only if u is a fixed point of S. Suppose that u is a fixed point of T. By (1), we have

$$\begin{aligned}
\psi(d(u, Su)) &\leq \psi(2sd(Tu, Su)) \\
&\leq \psi(max\{d(u, Tu), d(u, Su), \frac{1}{s}\max(d(u, Su), d(u, Tu))\}) \\
&- \varphi(max\{d(u, Tu), d(u, Su), \frac{1}{s}\max(d(u, Su), d(u, Tu))\}) \\
&= \psi(d(u, Su)) - \varphi(d(u, Su)).
\end{aligned}$$

It follows that $\varphi(d(u, Su)) = 0$. Therefore, $d(u, Su) = 0$, So $Su = u$. Similarly, we can show that if u is a fixed point of S, Then u is a fixed point of T.
Step 2: Let $x_0 \in A$, and let $x_1 = Tx_0$. Since $TA \subseteq B$, we have $x_1 \in B$. Also, let $x_2 = Sx_1$. Since $SB \subseteq A$, We have $x_2 \in A$. Continuing this process, we can construct a

sequence $\{x_n\}$ in X such that $x_{2n+1} = Tx_{2n}$, $x_{2n+2} = Sx_{2n+1}$, $x_{2n} \in A$, $x_{2n+1} \in B$. Since T and S are (A, B)-weakly increasing we have

$$x_1 = Tx_0 \leq STx_0 = x_2 = Sx_1 \leq TSx_1 = x_3 \leq \dots$$

$$x_{2n+1} = Tx_{2n} \leq STx_{2n} = x_{2n+2} \leq \dots$$

If $x_{2n} = x_{2n+1}$, for some $n \in \mathbb{N}$, then $x_{2n} = Tx_{2n}$. Thus, x_{2n} is a fixed point of T. By the first part of proof, we conclude that x_{2n} is also fixed point of S.
Similarly, if $x_{2n+1} = x_{2n+2}$, for some $n \in \mathbb{N}$, then $x_{2n+1} = Sx_{2n+1}$. Thus, x_{2n+1} is a fixed point of S. By the first part of proof, we conclude that x_{2n+1} is also fixed point of T.
Therefore we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Step 3: We will prove that

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0.$$

By (1), we have

$$\begin{aligned}
\psi(d(x_{2n+1}, x_{2n+2})) &\leq \psi(2sd(x_{2n+1}, x_{2n+2})) \\
&= \psi(2sd(Tx_{2n}, Sx_{2n+1})) \\
&\leq \psi(M(x_{2n}, x_{2n+1})) - \varphi(M(x_{2n}, x_{2n+1})).
\end{aligned}$$

where

$$\begin{aligned}
M(x_{2n}, x_{2n+1}) &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Tx_{2n}), d(x_{2n+1}, Sx_{2n+1}), \\
&\frac{1}{s}\max[d(x_{2n+1}, Tx_{2n}), d(x_{2n}, Sx_{2n+1})]\} \\
&= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), 0, \frac{d(x_{2n}, x_{2n+2})}{s}\} \\
&= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\psi(d(x_{2n+1}, x_{2n+2})) &\leq \psi(2sd(Tx_{2n}, Sx_{2n+1})) \\
&\leq \psi(\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}) \\
&- \varphi(\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} \qquad (2)
\end{aligned}$$

If $\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = d(x_{2n+1}, x_{2n+2})$, then (2) becomes

$$\begin{aligned}
\psi(d(x_{2n+1}, x_{2n+2})) &\leq \psi(2sd(Tx_{2n}, Sx_{2n+1})) \\
&\leq \psi(d(x_{2n+1}, x_{2n+2})) - \varphi(d(x_{2n+1}, x_{2n+2})) \\
&< \psi(d(x_{2n+1}, x_{2n+2})),
\end{aligned}$$

which is a contradiction. So that

$$\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = d(x_{2n}, x_{2n+1}).$$

Hence (2) becomes

$$\begin{aligned}
\psi(d(x_{2n+1}, x_{2n+2})) &\leq \psi(d(x_{2n}, x_{2n+1})) - \varphi(d(x_{2n}, x_{2n+1})) \\
&< \psi(d(x_{2n}, x_{2n+1})), \qquad (3)
\end{aligned}$$

is a non-increasing sequence of positive numbers.
Hence, there is $e \geq 0$ such that

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = e.$$

Letting $n \to \infty$ in (3), we get

$$\psi(e) \leq \psi(e) - \varphi(e),$$

which implies that $\varphi(e) = 0$, and hence $e = 0$. So, we have

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0. \tag{4}$$

Using (2.4) we have

$$\lim_{n\to+\infty} \rho(x_n, x_{n+1}) = 0. \tag{5}$$

Step 4: We will prove that $\{x_n\}$ is a Cauchy sequence.
Otherwise there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$ and

$$\rho(x_{m(k)}, x_{n(k)}) \geq \varepsilon. \tag{6}$$

This implies that $\rho(x_{m(k)}, x_{n(k)-1}) < \varepsilon$ for all $k \geq 1$ using the triangle inequality we have

$$\varepsilon \leq \rho(x_{m(k)}, x_{n(k)}) \leq \rho(x_{m(k)}, x_{n(k)-1}) + \rho(x_{n(k)-1}, x_{n(k)}).$$

Letting $k \to \infty$ and using (5) we obtain

$$\lim_{n\to\infty} \rho(x_{m(k)}, x_{n(k)}) = \varepsilon. \tag{7}$$

Again,

$$\rho(x_{m(k)}, x_{n(k)-1}) \leq \rho(x_{m(k)}, x_{n(k)}) + \rho(x_{n(k)}, x_{n(k)-1})$$

and

$$\rho(x_{m(k)}, x_{n(k)}) \leq \rho(x_{m(k)}, x_{n(k)-1}) + \rho(x_{n(k)-1}, x_{n(k)}).$$

Then we have

$$|\rho(x_{m(k)}, x_{n(k)-1}) - \rho(x_{m(k)}, x_{n(k)})| \leq \rho(x_{n(k)}, x_{n(k)-1})$$

Letting $k \to \infty$ and using (5) and (7) it follows that

$$\lim_{k\to\infty} \rho(x_{m(k)}, x_{n(k)-1}) = \varepsilon.$$

Similarly, we can prove that

$$\begin{aligned}
&\lim_{k\to\infty} \rho(x_{m(k)-1}, x_{n(k)}) = \lim_{k\to\infty} \rho(x_{m(k)-1}, x_{n(k)-1}) \\
&\lim_{k\to\infty} \rho(x_{m(k)+1}, x_{n(k)}) = \lim_{k\to\infty} \rho(x_{m(k)}, x_{n(k)+1}) = \varepsilon.
\end{aligned}$$

Therefore

$$\begin{aligned}
M(x_{m(k)}, x_{n(k)-1}) = \max\{&d(x_{m(k)}, x_{n(k)-1}), d(x_{m(k)}, x_{m(k)+1}), d(x_{n(k)-1}, x_{n(k)}), \\
&\frac{1}{s} \max\left\{(d(x_{m(k)}, x_{n(k)}), d(x_{n(k)-1}, x_{m(k)+1}))\right\} \\
&\leq \max\{s^2 \rho(x_{m(k)}, x_{n(k)-1}), s^2 \rho(x_{m(k)}, x_{m(k)+1}), s^2 \rho(x_{n(k)-1}, x_{n(k)}), \\
&\frac{1}{s}(s^2 \max\left\{\rho(x_{m(k)}, x_{n(k)}), \rho(x_{n(k)-1}, x_{m(k)+1})\right\})\}.
\end{aligned}$$

Letting $k \to \infty$ we have

$$\limsup_{k\to\infty} M(x_{m(k)}, x_{n(k)-1}) \leq 2s\varepsilon.$$

Let $x = x_{m(k)+1}$, $y = x_{n(k)}$, by (1) we have

$$\begin{aligned}\psi(2s\rho(x_{m(k)+1}, x_{n(k)})) &\le \psi(2sd(x_{m(k)+1}, x_{n(k)}))\\ &\le \psi(M(x_{m(k)}, x_{n(k)-1})) - \varphi(M(x_{m(k)}, x_{n(k)-1})).\end{aligned}$$

Letting $k \to \infty$

$$\begin{aligned}\psi(2s\varepsilon) = \psi(2s \lim_{k\to\infty} \rho(x_{m(k)+1}, x_{n(k)})) \le\\ \psi(\limsup_{k\to\infty} M(x_{m(k)}, x_{n(k)-1})) - \varphi(\liminf_{k\to\infty} M(x_{m(k)}, x_{n(k)-1})) \le\\ \psi(2s\varepsilon) - \varphi(\liminf_{k\to\infty} M(x_{m(k)}, x_{n(k)-1})),\end{aligned}$$

which implies that $\varphi(\liminf_{k\to\infty} M(x_{m(k)}, x_{n(k)-1})) = 0$. So that

$$\liminf_{k\to\infty} M(x_{m(k)}, x_{n(k)-1})) = 0.$$

It follows that

$$\liminf_{k\to\infty} d(x_{m(k)}, x_{n(k)})) = 0.$$

Therefore $\liminf_{k\to\infty}\rho(x_{m(k)}, x_{n(k)}) = 0$, which is a contradiction with (6). Therefore $\{x_n\}$ is a Cauchy sequence.

Step 5: By the completeness of X, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$, and $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = z$. Now, without loss of generality, we suppose that T is continuous, then we have

$$d(z, Tz) \le sd(z, Tx_n) + sd(Tx_n, Tz).$$

Letting $n \to \infty$, we get

$$d(z, Tz) \le s \lim_{n\to\infty} d(z, Tx_n) + s \lim_{n\to\infty} d(Tx_n, Tz) = 0.$$

Hence, we have $Tz = z$. Thus, z is a fixed point of T, since A and B are closed subsets of X, $z \in A \cap B$. By the first part of the proof, we conclude that z is a fixed point of S. □

In the following we provide an example which satisfies our result but dose not satisfy the result from most current literature. In fact Theorem 2.4 can not apply to this example.

Example 3.1. *Let $X_1 = [0, \frac{1}{3}]$ and $X_2 = \{\alpha, \beta, \gamma\}$ where $\alpha = \frac{1}{2} \le \gamma \le \beta = 1$. Let $X = X_1 \cup X_2$ and define $d : X \times X \to [0, \infty)$ by*
$d(\beta, \alpha) = \frac{1}{2}$, $d(\alpha, \gamma) = 1$, $d(\beta, \gamma) = 2$,
$d(x, y) = \frac{|x-y|}{100}$ if $x, y \in X_1$,
$d(x, y) = 1$ if $x \in X_1$, $y \in X_2$ or $x \in X_2$, $y \in X_1$
Since $d(\beta, \gamma) = 2 \not\le \frac{3}{2} = d(\beta, \alpha) + d(\alpha, \gamma)$, (X, d) is not a metric space. It is easy to see that

$$d(x, y) \le 2 \max\{d(x, z), d(z, y)\} \quad (x, y, z \in X)$$

Therefore (X, d) is 2-quasimetric space, which is also complete.
Let $S, T : X \to X$ be define by $Sx = Tx = \frac{x}{10}$ for all $x \in X$.
Take $\psi(t) = 2t$ and $\varphi(t) = \frac{1}{4}t$.
It follows from Theorem 3.2 that 0 is fixed point of T.
This example can not be applied for Theorem 2.4. In fact, in Theorem 2.4 for $s = 2, L = 0$ if we take $x = 0, y = \frac{1}{3}$, we have

$$\psi(2^4 d(Tx, Ty)) \not\le \psi(M_s(x, y)) - \varphi((M_s(x, y)))$$

$$\frac{32}{3000} \not\le \frac{7}{1200}$$

4. Conclusion

We investigate the s-quasimetric spaces and we introduce a new class of generalized contractive mappings to establish a fixed point theorem for this class of mappings in complete s-quasimetric spaces. Notice that there exist s-quasimetric spaces which are not metric, finally we provide an example to support our result.

References

[1] Alber, Ya. I. and Guerre-Delabriere, S., (1997), Principle of weakly contractive maps in Hilbert spaces, Advances and Appl., 98, pp. 7-22.

[2] Chrzaszcz, K., Jachymski, J. and Turobos, F., (2019), Two refinements of frink's metrization theorem and fixed point results for lipschitzian mappings on quasimetric spaces, Aequat. Math., 93, pp. 277-297.

[3] Czerwik, S., (1993), Contraction mappings in b-metric spaces, Acta Math. Inform. Univ. Ostraviensis., 1, pp. 5-11.

[4] Czerwik, S., (1998), Nonlinear set-valued contraction mappings in b-metric spaces, Atti Sem. Mat. Fis. Univ. Modena., 46 (2), pp. 263-276.

[5] Doric, D., (2009), Common fixed point for generalized (ψ-φ)-weak contractions, Applied Mathematics Letters., 22, pp. 1896-1900.

[6] Dutta, P. N. and Choudhury, B. S., (2008), A generalization of contraction principle in metric spaces, Fixed Point Theory and Applications., Article ID 406368.

[7] Frink, A. H., (1937), Distance functions and the metrization problem, Bull. Am. Math. Soc., 43, pp. 133-142.

[8] Hussain, N., Parvaneh, V., Rezaei Roshan J. and Kadelburg, Z., (2013), Fixed point of cyclic weakly (ψ, ϕ, L, A, B)- contractive b-metric spaces with applications, Fixed Point Theory and Applications.

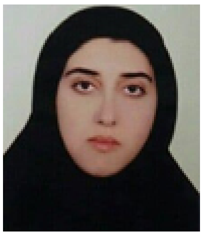
[9] Khan, MS. Swaleh, M. and Sessa, S., (1984), Fixed point theorems by altering distances between the points, Bull. Aust. Math. Soc., 30, pp. 1-9.

[10] Rhoades, B. E., (2001), Some theorems on weakly contractive maps, Nonlinear Anal., 47, pp. 2683-2693.

[11] Roshan, JR., Parvaneh, V., Sedghi, S., Shobkolaei, N. and Shatanawi, W., (2013), Common fixed points of almost generalized $(\psi, \phi)_s$-contractive mappings in ordered b-metric spaces, Fixed Point Theory and Applications., Article ID 159 (2013). doi: 10.1186/1687-1812-2013-159.

[12] Schroeder, V., (2006), Quasi-metric and metric spaces, Conform. Geom. Dyn., 10, pp. 355-360.

[13] Shatanawi, W. and Postolache, M., (2013), Common fixed point results of mappings for nonlinear contraction of cyclic form in ordered metric spaces, Fixed Point Theory and Applications., Article ID 60 (2013). doi: 10.1186/1687-1812-2013-60.



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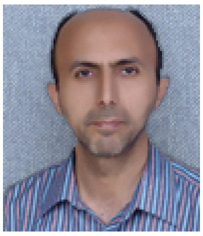

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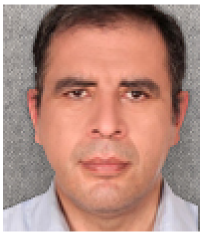

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