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UNIFORM STABILIZATION OF THE PETROVSKY-WAVE NONLINEAR COUPLED SYSTEM WITH STRONG DAMPING

AKRAM BEN AISSA[1], §

Abstract. This paper concerns the well-posedness and uniform stabilization of the Petrovsky-Wave Nonlinear coupled system with strong damping. Existence of global weak solutions for this problem is established by using the Galerkin method. Meanwhile, under a clever use of the multiplier method, we estimate the total energy decay rate.

Keywords: Coupled systems, strong damping, Well-posedness, Faedo-Galerkin method.

AMS Subject Classification: 35D30, 93D15, 74J30.

For simplicity reasons, we omit the space variable x of $u(x,t), u_t(x,t)$ and we denote $u(x,t) = u, u_t(x,t) = u'$ and $u_{tt}(x,t) = u''$. In addition, when no confusion arises, the functions considered are all real valued.
Our main interest lies in the following system of the coupled Petrovsky-wave system of the type

$$\begin{cases} u_1'' + \Delta^2 u_1 - a(x)\Delta u_2 - g_1(\Delta u_1') = 0, & x \in \Omega, t \geq 0 \\ u_2'' - \Delta u_2 - a(x)\Delta u_1 - g_2(\Delta u_2') = 0 & x \in \Omega, t \geq 0 \\ \Delta u_1 = u_1 = u_2 = 0, & x \in \Gamma, t \geq 0 \\ u_i(x,0) = u_i^0(x), \;\; u_i'(x,0) = u_i^1(x), & x \in \Omega, \quad i = 1,2. \end{cases} \tag{1}$$

Here Ω is a bounded domain of $\mathbb{R}^n$ with regular boundary Γ.
When $a(x) = 0$, the Petrovsky equation was treated by Komornik [8], where he used the semigroup approach for setting the well-possedness, then he studied the strong stability of such system by introducing a multiplier method combined with a nonlinear integral inequalities. Recently, Bahlil et al. [6], studied the system

$$\begin{cases} u_1'' + a(x)u_2 + \Delta^2 u_1 - g_1(u_1'(x,t)) = f_1(u_1,u_2), & in \quad \Omega \times \mathbb{R}_+, \\ u_2'' + a(x)u_1 - \Delta u_2 - g_2(u_2'(x,t)) = f_2(u_1,u_2), & in \quad \Omega \times \mathbb{R}_+, \\ \partial_\nu u_1 = u_1 = v = u_2 = 0 & on \quad \Gamma \times \mathbb{R}_+, \end{cases} \tag{2}$$

for g_i $(i = 1,2)$ do not necessarily having a polynomial growth near the origin, by using Faedo-Galerkin method to prove the existence and uniqueness of solution and established energy decay results depending on g_i. Guesmia [7] consider the problem (2) without source Terms f_1 and f_2.

[1] UR Analysis and Control of PDE's, UR 13ES64, Higher Institute of Transport and Logistics of Sousse, University of Sousse, Tunisia.
e-mail: akram.benaissa@fsm.rnu.tn, issaakram26@gmail.com,
ORCID: https://orcid.org/0000-0002-8598-9238.

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He deals with global existence and uniform decay of solutions.

In this paper, we prove the global existence of weak solutions of the problem (1) by using the Galerkin method (see Lions [12]) we use some technique from [6] to establish an explicit and general decay result, depending on g_i. The proof is based on a powerful tool which is the multiplier method [13, 9] and makes use of some properties of convex functions, and general Jensen and Young's inequalities. These convexity arguments were introduced and developed by Lasiecka and co-workers ([11],[10]) and exploited later on, with appropriate modifications, by Liu and Zuazua [14], Alabau-Boussouira [4] and others.

The paper is organized as follows. In section 2 we introduce our functional framework and state the main results. Section 3 is devoted to prove the existence and uniqueness of a global solution. In the last section we prove the energy estimates.

1. Functional setting and statement of main results

Let us introduce for brevity the following Hilbert spaces

$$H = L^2(\Omega) \times L^2(\Omega)$$
$$W = H_0^1(\Omega) \times H_0^1(\Omega)$$
$$H_\Delta^3(\Omega) = \{v \in H^3(\Omega) | v = \Delta v = 0 \text{ on } \Gamma\}, \quad \|v\|^2_{H_\Delta^3(\Omega)} = \int_\Omega |\nabla \Delta v|^2 dx$$
$$V = H_\Delta^3(\Omega) \cap H^2(\Omega) \times H^2(\Omega)$$
$$\widetilde{V} = (H^4(\Omega) \cap H_\Delta^3(\Omega)) \times (H_\Delta^3(\Omega) \cap H^2(\Omega)).$$

Identifying H with its dual, we obtain the diagram

$$\widetilde{V} \subset V \subset W \subset H = H' \subset W' \subset V' \subset \widetilde{V}'.$$

We impose the following assumptions on a and g_i

► The function $a : \Omega \to \mathbb{R}$ is nonnegative and bounded such that

$$a(x) \in W^{1,\infty}(\Omega).$$
$$\|a\|_{L^\infty(\Omega)} < \min\left\{\frac{1}{c'}, 1\right\} \tag{3}$$

where $c' > 0$ (depending only on the geometry of Ω) is the constant

$$\|\Delta v\| \leq c' \|\nabla \Delta v\|, \quad \forall v \in H_\Delta^3(\Omega).$$
$$\|\nabla v\| \leq c \|\Delta v\|, \quad \forall v \in H_0^2(\Omega).$$

► $g_i : \mathbb{R} \to \mathbb{R}$ be non decreasing convex function of class $\mathcal{C}^1$ such that there exists ϵ (sufficiently small), c_i, $\tau_i > 0$, $(i = 1, 2)$, and $G : \mathbb{R}_+ \to \mathbb{R}_+$ is convex, increasing and of class $\mathcal{C}^1(\mathbb{R}_+) \cap \mathcal{C}^2(]0, +\infty[)$ satisfying

$$\begin{gathered} G(0) = 0 \text{ and } G \text{ is linear on } [0, \epsilon] \text{ or} \\ G'(0) = 0 \text{ and } G'' > 0 \text{ on }]0, \epsilon] \text{ such that} \\ c_1|s| \leq |g_i(s)| \leq c_2|s| \quad \text{if } |s| > \epsilon \\ s^2 + g_i^2(s) \leq G^{-1}(s g_i(s)) \quad \text{if } |s| \leq \epsilon, \\ \exists \tau_1,\ \tau_2 > 0,\ \tau_1 \leq g_i'(s) \leq \tau_2,\ \forall s \in \mathbb{R}. \end{gathered} \tag{4}$$

We are now in a position to state our main results.

Theorem 1.1. *Let $(u_1^0, u_2^0) \in \widetilde{V}$ and $(u_1^1, u_2^1) \in V$ arbitrarily. Assume that (3) and (4) hold. Then, system (1) has a unique weak solution satisfying*

$$(u_1, u_2) \in L^\infty(\mathbb{R}_+, \widetilde{V}), \quad (u_1', u_2') \in L^\infty(\mathbb{R}_+, V)$$

and

$$(u_1'', u_2'') \in L^\infty(\mathbb{R}_+, W).$$

Theorem 1.2. *Let* $(u_1^0, u_2^0) \in \widetilde{V}$ *and* $(u_1^1, u_2^1) \in V$*. Assume that (3) and (4) hold. The energy of the unique solution of system* (1)*, given by (6) decays as*

$$E(t) \leq \psi^{-1}\Big(h(t) + \psi(E(0))\Big), \ \forall t \geq 0 \tag{5}$$

where $\psi(t) = \displaystyle\int_t^1 \frac{1}{\omega\varphi(s)}\, ds$ *for* $t > 0$, $\ h(t) = 0$ *for* $0 \leq t \leq \frac{E(0)}{\omega\varphi(E(0))}$ *and*

$$h^{-1}(t) = t + \frac{\psi^{-1}(t + \psi(E(0)))}{\varphi(\psi^{-1}(t + \psi(E(0))))}, \ \forall t \geq \frac{E(0)}{\varphi(E(0))}$$

$$\varphi(t) = \begin{cases} t & \text{if } G \text{ is linear on } [0,\varepsilon] \\ tG'(\varepsilon_0 t) & \text{if } G'(0) = 0 \text{ and } G'' > 0 \text{ on }]0,\varepsilon], \end{cases}$$

where ω *and* ε_0 *are positive constants.*

Lemma 1.1. *The energy functional associated to the solution of the problem (1) given by the following formula*

$$E(t) = \frac{1}{2}\int_\Omega |\nabla u_1'|^2 + |\nabla u_2'|^2 + |\nabla\Delta u_1|^2 + |\Delta u_2|^2\, dx + \int_\Omega a(x)\Delta u_1 \Delta u_2 dx, \tag{6}$$

is nonnegative.

Proof. Multiplying the first equation in (1) by $-\Delta u_1'$ and the second equation by $-\Delta u_2'$, integrating (by parts) over Ω, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[\int_\Omega |\nabla u_1'|^2 + |\nabla u_2'|^2 + |\nabla\Delta u_1|^2 + |\Delta u_2|^2\, dx + 2\int_\Omega a(x)\Delta u_1\Delta u_2\, dx\Big] \\
&\quad = -\int_\Omega \Delta u_1' g_1(\Delta u_1') + \Delta u_2' g_2(\Delta u_2')\, dx.
\end{aligned}$$

Using Hölder's inequality, Sobolev embedding and the condition (3), we get

$$\begin{aligned}
\int_\Omega a(x)\Delta u_1\Delta u_2 dx &\geq -\frac{1}{2}\|a\|_{L^\infty(\Omega)}\frac{\sqrt{c'}}{\sqrt{c'}}\int_\Omega |\Delta u_1 \Delta u_2|\, dx \\
&\geq -\frac{1}{2}\|a\|_{L^\infty(\Omega)}\int_\Omega \frac{1}{c'}|\Delta u_1|^2 + c'|\Delta u_2|^2\, dx \\
&\geq -\frac{1}{2}\|a\|_{L^\infty(\Omega)}\int_\Omega \frac{c'^2}{c'}|\nabla\Delta u_1|^2 + c'|\Delta u_2|^2\, dx \\
&\geq -\frac{c'}{2}\|a\|_{L^\infty(\Omega)}\int_\Omega |\nabla\Delta u_1|^2 + |\Delta u_2|^2\, dx
\end{aligned}$$

then

$$\begin{aligned}
E(t) &\geq \frac{1}{2}\int_\Omega |\nabla u_1'|^2 + |\nabla u_2'|^2 + (1 - c'\|a\|_{L^\infty(\Omega)})(|\nabla\Delta u_1|^2 + |\Delta u_2|^2)\, dx \\
&\geq 0.
\end{aligned}$$

Hence, E is a nonnegative function and its derivative is

$$E'(t) = -\int_\Omega \Delta u_1' g_1(\Delta u_1') + \Delta u_2' g_2(\Delta u_2')\, dx. \tag{7}$$

□

The following result is due to Nakao [15] and will be needed later.

Lemma 1.2. *Let $E:\mathbb{R}_+\to\mathbb{R}_+$ be a non-increasing differentiable function, $\lambda\in\mathbb{R}_+$ and $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ a convex and increasing function such that $\varphi(0)=0$. Assume that*

$$\begin{cases} \displaystyle\int_s^{+\infty}\varphi(E(t))\,dt\le E(s), & \forall s\ge 0\\ E'(t)\le\lambda E(t) & \forall t\ge 0.\end{cases}$$

Then E satisfies the following estimate:

$$E(t)\le e^{\tau_0\lambda}d^{-1}\Big(e^{\lambda(t-h(t))}\varphi\Big(\psi^{-1}\Big(h(t)+\psi(E(0))\Big)\Big),\ \forall t\ge 0 \tag{8}$$

where

$$\psi(t)=\int_t^1\frac{1}{\varphi(s)}\,ds,\quad \forall t\ge 0$$

$$d(t)=\begin{cases}\psi(t), & \text{if } \lambda=0\\ \displaystyle\int_0^t\frac{\varphi(s)}{s}\,ds & \text{if } \lambda>0\end{cases}$$

$$h(t)=\begin{cases}K^{-1}(D(t)), & \forall t>T_0\\ 0 & \forall t\in[0,T_0]\end{cases}$$

$$K(t)=D(t)+\frac{\psi^{-1}(t+\psi(E(0)))}{\varphi(\psi^{-1}(t+\psi(E(0))))}e^{\lambda t},\ \forall t\ge 0$$

$$D(t)=\int_0^t e^{\lambda s}\,ds\quad \forall t\ge 0$$

$$T_0=D^{-1}\Big(\frac{E(0)}{\varphi(E(0))}\Big),\quad \tau_0=\begin{cases}0 & \forall t>T_0\\ T_0 & \forall t\in[0,T_0]\end{cases}$$

2. Proof of Theorem 1.1

We will use the Faedo-Galerkin method [12] to prove the existence of a global solutions. Let $T>0$ be fixed and denote by V^k the space generated by $\{w_i^1,w_i^2,...,w_i^k\}$, where the set $\{w_i^k,\ k\in\mathbb{N}\}$ is a basis of $\widetilde{V}$.
We construct approximate solution $u_i^k,\ k=1,2,3,.....$ in the form

$$u_i^k(x,t)=\sum_{j=1}^k c^{jk}(t)w_i^j(x),$$

where c^{jk} $(j=1,2,...,k)$ are determined by the following ordinary differential equations

$$\begin{cases}(\ddot u_1^k+\Delta^2u_1^k-a(x)\Delta u_2^k-g_1(\Delta\dot u_1^k),w_1^j)=0 & \forall w_j^1\in V^k\\ (\ddot u_2^k-\Delta u_2^k-a(x)\Delta u_1^k-g_2(\Delta\dot u_2^k),w_2^j)=0 & \forall w_j^2\in V^k\\ u_i^k(0)=u_i^{0k},\ \ \dot u_i^k(0)=u_i^{1k}, & x\in\Omega,\quad i=1,2\end{cases} \tag{9}$$

with initial conditions

$$u_1^k(0)=u_1^{0k}=\sum_{j=1}^k\langle u_1^0,w_1^j\rangle w_1^j\to u_1^0,\quad \text{in } H^4(\Omega)\cap H^3_\Delta(\Omega)\ \text{ as } k\to+\infty, \tag{10}$$

$$u_2^k(0)=u_2^{0k}=\sum_{j=1}^k\langle u_2^0,w_2^j\rangle w_2^j\to u_2^0,\quad \text{in } H^3_\Delta(\Omega)\cap H^2(\Omega)\ \text{as } k\to+\infty, \tag{11}$$

$$\dot u_1^k(0)=u_1^{1k}=\sum_{j=1}^k\langle u_1^1,w_1^j\rangle w_1^j\to u_1^1,\quad \text{in } H^3_\Delta(\Omega)\cap H^2(\Omega)\ \text{as } k\to+\infty. \tag{12}$$

$$\dot{u}_2^k(0) = u_2^{1k} = \sum_{j=1}^{k} \langle u_2^1, w_2^j \rangle w_2^j \to u_2^1, \quad \text{in } H^2(\Omega) \text{ as } k \to +\infty. \tag{13}$$

$$-\Delta^2 u_1^{0k} + a(x)\Delta u_2^{0k} + g_1(\Delta u_1^{1k}) \longrightarrow -\Delta^2 u_1^0 + a(x)\Delta u_2^0 + g_1(\Delta u_1^1), \quad \text{in } L^2(\Omega) \text{ as } k \to +\infty. \tag{14}$$

$$\Delta u_2^{0k} + a(x)\Delta u_1^{0k} + g_2(\Delta u_2^{1k}) \longrightarrow \Delta u_2^0 + a(x)\Delta u_1^0 + g_2(\Delta u_2^1), \quad \text{in } H_0^1(\Omega) \text{ as } k \to +\infty. \tag{15}$$

First, we are going to use some a priori estimates to show that $t_k = \infty$. Then, we will show that the sequence of solutions to (9) converges to a solution of (1) with the claimed smoothness. Choosing $w_i^j = -2\Delta \dot{u}_i^k$ in (9), we obtain

$$\begin{aligned} &\frac{d}{dt}\int_\Omega |\nabla \dot{u}_1^k|^2 + |\nabla \dot{u}_2^k|^2 + |\nabla \Delta u_1^k|^2 + |\Delta u_2^k|^2\,dx + 2a(x)\Delta u_1^k \Delta u_2^k\,dx \\ &+ 2\int_\Omega \Delta \dot{u}_1^k g_1(\Delta \dot{u}_1^k)\,dx + 2\int_\Omega \Delta \dot{u}_2^k g_2(\Delta \dot{u}_2^k)\,dx = 0, \end{aligned} \tag{16}$$

and choosing $w_i^j = \Delta^2 \dot{u}_i^k$ in (9), implies

$$\begin{aligned} &\frac{d}{dt}\int_\Omega |\Delta \dot{u}_1^k|^2 + |\Delta \dot{u}_2^k|^2 + |\Delta^2 u_1^k|^2 + |\nabla \Delta u_2^k|^2 + 2a(x)\nabla \Delta u_1^k \nabla \Delta u_2^k\,dx \\ &+ 2\int_\Omega \nabla a(x)\Delta u_2^k \nabla \Delta \dot{u}_1^k\,dx + 2\int_\Omega \nabla a(x)\Delta u_1^k \nabla \Delta \dot{u}_2^k\,dx \\ &+ 2\int_\Omega |\nabla \Delta \dot{u}_1^k|^2 g_1'(\Delta \dot{u}_1^k)\,dx + 2\int_\Omega |\nabla \Delta \dot{u}_2^k|^2 g_2'(\Delta \dot{u}_2^k)\,dx = 0. \end{aligned} \tag{17}$$

Summing (16) and (17), we obtain

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \{|\Delta \dot{u}_1^k|^2 + |\Delta \dot{u}_2^k|^2 + |\nabla \dot{u}_1^k|^2 + |\nabla \dot{u}_2^k|^2 + |\Delta^2 u_1^k|^2 + |\nabla \Delta u_2^k|^2| + \nabla \Delta u_1^k|^2 + |\Delta u_2^k|^2\}\,dx \\ &+ 2\frac{d}{dt}\int_\Omega \{a(x)\Delta u_1^k \Delta u_2^k + a(x)\nabla \Delta u_1^k \nabla \Delta u_2^k\}\,dx + 2\int_\Omega \Delta \dot{u}_1^k g_1(\Delta \dot{u}_1^k)\,dx + 2\int_\Omega \Delta \dot{u}_2^k g_2(\Delta \dot{u}_2^k)\,dx \\ &+ 2\int_\Omega \nabla a(x)\Delta u_2^k \nabla \Delta \dot{u}_1^k\,dx + 2\int_\Omega \nabla a(x)\Delta u_1^k \nabla \Delta \dot{u}_2^k\,dx \\ &+ 2\int_\Omega |\nabla \Delta \dot{u}_1^k|^2 g_1'(\Delta \dot{u}_1^k)\,dx + 2\int_\Omega |\nabla \Delta \dot{u}_2^k|^2 g_2'(\Delta \dot{u}_2^k)\,dx = 0. \end{aligned} \tag{18}$$

Using Hölder's inequality and Sobolev embedding, we have

$$\begin{aligned} &2\Big|\int_\Omega a(x)\Delta u_2^k \Delta u_1^k\,dx\Big| \le 2\frac{\sqrt{c'}}{\sqrt{c'}}\int_\Omega |a(x)||\Delta u_2^k||\Delta u_1^k|\,dx \\ &\le c'\|a\|\int_\Omega |\nabla \Delta u_1^k(x,t)|^2\,dx + c'\|a\|\int_\Omega |\Delta u_2^k(x,t)|^2\,dx \end{aligned} \tag{19}$$

and

$$\begin{aligned} &\Big|2\int_\Omega a(x)\nabla \Delta u_1^k \nabla \Delta u_2^k\,dx\Big| \\ &\le 2\|a\|\int_\Omega |\nabla \Delta u_1^k||\nabla \Delta u_2^k|\,dx \\ &\le \|a\|\int_\Omega |\nabla \Delta u_1^k|^2\,dx + \|a\|\int_\Omega |\nabla \Delta u_2^k|^2\,dx. \end{aligned} \tag{20}$$

By Hölder's inequality, Sobolev embedding and the condition (4), we get

$$\begin{aligned}
&2\Big|\int_\Omega \nabla a(x)\Delta u_2^k \nabla\Delta \dot u_1^k\,dx\Big| \le 2\int_\Omega |\nabla a(x)||\Delta u_2^k||\nabla\Delta\dot u_1^k|\,dx\\
&\le 2\int_\Omega |\nabla a(x)||\Delta u_2^k||\nabla\Delta\dot u_1^k|\frac{\sqrt{g_1'(\Delta\dot u_1^k)}}{\sqrt{\tau_1}}\,dx \qquad (21)\\
&\le \int_\Omega |\nabla\Delta\dot u_1^k|^2 g_1'(\Delta\dot u_1^k)\,dx + \frac{1}{\tau_1}\|\nabla a\|^2\int_\Omega |\Delta u_2^k|^2\,dx.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
2\Big|\int_\Omega \nabla a(x)\Delta u_1^k\nabla\Delta\dot u_2^k\,dx\Big| &\le \int_\Omega |\nabla\Delta\dot u_2^k|^2 g_2'(\Delta\dot u_2^k)\,dx + \frac{1}{\tau_1}\|\nabla a\|^2\int_\Omega |\Delta u_1^k|^2\,dx\\
&\le \int_\Omega |\nabla\Delta\dot u_2^k|^2 g_2'(\Delta\dot u_2^k)\,dx + \frac{c'}{\tau_1}\|\nabla a\|^2\int_\Omega |\nabla\Delta u_1^k|^2\,dx.
\end{aligned} \qquad (22)$$

Reporting (19)-(22), into (18) and integrating over $(0,t)$, we find

$$\begin{aligned}
&F^k(t) + 2\int_0^t\int_\Omega \Delta\dot u_1^k(s) g_1(\Delta\dot u_1^k(s))\,dx\,dt + 2\int_0^t\int_\Omega \Delta\dot u_2^k(s) g_2(\Delta\dot u_2^k(s))\,dx\,dt\\
&+\int_0^t\int_\Omega |\nabla\Delta\dot u_1^k(s)|^2 g_1'(\Delta\dot u_1^k(s))\,dx\,dt + \int_0^t\int_\Omega |\nabla\Delta\dot u_2^k(s)|^2 g_2'(\Delta\dot u_2^k(s))\,dx\,dt\\
&\le F^k(0) + C_1\int_0^t F^k(s)\,dx\,ds, \quad \forall t\in[0,t_k)
\end{aligned}$$

where

$$\begin{aligned}
F^k(t) =& \int_\Omega |\Delta\dot u_1^k(t)|^2 + |\Delta\dot u_2^k(t)|^2 + |\nabla\dot u_1^k(t)|^2 + |\nabla\dot u_2^k(t)|^2 + |\Delta^2 u_1^k(t)|^2\,dx\\
&+(1-c'\|a\|-|a\|)\int_\Omega |\nabla\Delta u_1^k(t)|^2\,dx + (1-c'\|a\|)\int_\Omega |\Delta u_2^k(t)|^2\,dx + (1-\|a\|)\int_\Omega |\nabla\Delta u_2^k(t)|^2\,dx
\end{aligned}$$

and C_1 is a positive constant depending only on $\|a\|$, $\|\nabla a\|$ and τ_1.
So that, thanks to the monotonicity condition on the function g_i and using Gronwall's lemma, we conclude that

$$u_1^k \quad \text{is bounded in} \quad L^\infty(0,T;H^4(\Omega)\cap H^3_\Delta(\Omega)) \qquad (23)$$

$$u_2^k \quad \text{is bounded in} \quad L^\infty(0,T;H^3_\Delta(\Omega)\cap H^2(\Omega)) \qquad (24)$$

$$\dot u_1^k \quad \text{is bounded in} \quad L^\infty(0,T;H^2(\Omega)\cap H^1_0(\Omega)) \qquad (25)$$

$$\dot u_2^k \quad \text{is bounded in} \quad L^\infty(0,T;H^2(\Omega)\cap H^1_0(\Omega)) \qquad (26)$$

$$\Delta\dot u_i^k g_i(\Delta\dot u_i^k) \quad \text{is bounded in} \quad L^1(\mathcal{A}), \qquad (27)$$

where $\mathcal{A} = \Omega\times(0,T)$.
We assume first $t<T$ and let $0<\xi<T-t$. Set

$$u_i^{k\xi}(x,t) = u_i^k(x,t+\xi),$$

$$U^{k\xi} = u_1^k(x,t+\xi) - u_1^k(x,t),$$

and

$$D^{k\xi} = u_2^k(x,t+\xi) - u_2^k(x,t).$$

Then, $U^{k\xi}$ solves the differential equation

$$(\ddot U^{k\xi} + \Delta^2 U^{k\xi} - a(x)\Delta D^{k\xi} - (g_1(\Delta\dot u_1^{k\xi}) - g_1(\Delta\dot u_1^k)), w_1^j) = 0, \quad \forall w_1^j\in V^k. \qquad (28)$$

and $D^{k\xi}$ solves

$$(\ddot D^{k\xi} - \Delta D^{k\xi} - a(x)\Delta U^{k\xi} - (g_2(\Delta\dot u_2^{k\xi}) - g_2(\Delta\dot u_2^k)), w_2^j) = 0, \quad \forall w_2^j\in V^k. \qquad (29)$$

Choosing $w_1^j = -\Delta\dot{U}^{k\xi}$ in (28) and $w_2^j = \Delta\dot{D}^{k\xi}$ in (29), and using the fact that g_i is nondecreasing, we find

$$\frac{d}{dt}\int_\Omega \{|\nabla\dot{U}^{k\xi}(x,t)|^2 + |\nabla\dot{D}^{k\xi}(x,t)|^2 + |\nabla\Delta U^{k\xi}(x,t)|^2 + |\Delta D^{k\xi}(x,t)|^2\}\,dx + 2\frac{d}{dt}\int_\Omega a(x)\Delta D^{k\xi}(x,t)\Delta U^{k\xi}(x,t)\,dx \le 0 \quad \forall t \ge 0.$$

Integrating it over $[0,t]$, to get

$$\int_\Omega |\nabla\dot{U}^{k\xi}(t)|^2 + |\nabla\dot{D}^{k\xi}(t)|^2\,dx + (1 - c'\|a\|)\int_\Omega |\nabla\Delta U^{k\xi}(t)|^2 + |\Delta D^{k\xi}(t)|^2\,dx \le C_2\int_\Omega \{|\nabla\dot{U}^{k\xi}(0)|^2 + |\nabla\dot{D}^{k\xi}(0)|^2 + \int_\Omega |\nabla\Delta U^{k\xi}(0)|^2 + |\Delta D^{k\xi}(0)|^2\}\,dx$$

and C_2 is a positive constant depending only on $\|a\|$ and c'.
Dividing by ξ^2, and letting $\xi \to 0$, we find

$$\int_\Omega \{|\nabla\ddot{u}_1^k(t)|^2 + |\nabla\ddot{u}_2^k(t)|^2 + |\nabla\Delta\dot{u}_1^k(t)|^2 + |\Delta\dot{u}_2^k(t)|^2\}\,dx \le C_2'\int_\Omega \{|\nabla\ddot{u}_1^k(0)|^2 + |\nabla\ddot{u}_2^k(0)|^2 + |\nabla\Delta u_1^{1k}|^2 + |\Delta u_2^{1k}|^2\}\,dx.$$

We estimate $\|\nabla\ddot{u}_i^k(0)\|$. Choosing $w_{ij} = -\Delta\ddot{u}_i^k$ and $t = 0$ in (9), we obtain that

$$\|\nabla\ddot{u}_1^k(0)\|^2 = \int_\Omega \nabla\ddot{u}_1^k(0)\nabla(-\Delta^2 u_1^{0k} - a(x)\Delta u_2^{0k} + g_1(\Delta u_1^{1k}))\,dx.$$

and

$$\|\nabla\ddot{u}_2^k(0)\|^2 = \int_\Omega \nabla\ddot{u}_2^k(0)\nabla(\Delta u_2^{0k} - a(x)\Delta u_1^{0k} + g_2(\Delta u_2^{1k}))\,dx.$$

Using Cauchy-Schwarz inequality, we have

$$\|\nabla\ddot{u}_1^k(0)\| \le \Big(\int_\Omega |\nabla(-\Delta^2 u_1^{0k} - a(x)u_2^{0k} + g_1(\Delta u_1^{1k}))|^2\,dx\Big)^{\frac{1}{2}}.$$

and

$$\|\nabla\ddot{u}_2^k(0)\| \le \Big(\int_\Omega |\nabla(\Delta u_2^{0k} - a(x)u_1^{0k} + g_2(\Delta u_2^{1k}))|^2\,dx\Big)^{\frac{1}{2}}.$$

By (14) and (15) yields

$$(\ddot{u}_1^k(0), \ddot{u}_2^k(0)) \quad \text{are bounded in} \quad W \times W \tag{30}$$

By (12), (13) and (30), we deduce that

$$\int_\Omega \{|\nabla\ddot{u}_1^k(t)|^2 + |\nabla\ddot{u}_2^k(t)|^2 + |\nabla\Delta\dot{u}_1^k(t)|^2 + |\Delta\dot{u}_2^k(t)|^2\}\,dx \le C_3 \quad \forall t \ge 0,$$

where C_3 is a positive constant independent of $k \in \mathbb{N}$. Therefore, we conclude that

$$\dot{u}_1^k \quad \text{is bounded in} \quad L^\infty(0,T;H^3_\Delta(\Omega)) \tag{31}$$
$$\dot{u}_2^k \quad \text{is bounded in} \quad L^\infty(0,T;H^2(\Omega)) \tag{32}$$
$$\ddot{u}_1^k \quad \text{is bounded in} \quad L^\infty(0,T;H^1_0(\Omega)) \tag{33}$$
$$\ddot{u}_2^k \quad \text{is bounded in} \quad L^\infty(0,T;H^1_0(\Omega)). \tag{34}$$

Applying Dunford-Pettis and Banach-Alaoglu-Bourbaki theorems, we conclude from (23)-(27) and (31)-(34) that there exists a subsequence $\{u_i^m\}$ of $\{u_i^k\}$ such that

$$(u_1^m, u_2^m) \rightharpoonup (u_1, u_2), \text{ weak-star in } L^\infty(0,T;\widetilde{V}), \tag{35}$$
$$(\dot{u}_1^m, \dot{u}_2^m) \rightharpoonup (u_1', u_2') \text{ weak-star in } L^\infty(0,T;V), \tag{36}$$
$$(\ddot{u}_1^m, \ddot{u}_2^m) \rightharpoonup (u_1'', u_2'') \text{ weak-star in } L^\infty(0,T;W), \tag{37}$$
$$(\dot{u}_1^m, \dot{u}_2^m) \longrightarrow (u_1', u_2'), \text{ almost everywhere in } \Omega \times [0,+\infty) \tag{38}$$
$$g_i(\Delta\dot{u}_i^m) \rightharpoonup \chi_i \text{ weak-star in } L^2(\mathcal{A}). \tag{39}$$

As (u_1^m, u_2^m) is bounded in $L^\infty(0,T;\widetilde{V})$ (by (35)) and the injection of $\widetilde{V}$ in H is compact, we have

$$(u_1^m, u_2^m) \longrightarrow (u_1, u_2), \text{ strong in } L^2(0,T;H). \tag{40}$$

On the other hand, using (35), (37) and (40), we have

$$\begin{aligned} &\int_0^T \int_\Omega \Big(\ddot{u}_1^m(x,t) + \Delta^2 u_1^m(x,t) - a(x)\Delta u_2^k(x,t)\Big) w \; dx \; dt \longrightarrow \\ &\qquad \int_0^T \int_\Omega \Big(u_1''(x,t) + \Delta^2 u_1(x,t) - a(x)\Delta u_2(x,t)\Big) w \; dx \; dt, \end{aligned} \tag{41}$$

and

$$\begin{aligned} &\int_0^T \int_\Omega \Big(\ddot{u}_2^m(x,t) - \Delta u_2^m(x,t) - a(x)\Delta u_1^m(x,t)\Big) w \; dx \; dt \longrightarrow \\ &\qquad \int_0^T \int_\Omega \Big(u_2''(x,t) - \Delta u_2(x,t) - a(x)\Delta u_1(x,t)\Big) w \; dx \; dt, \end{aligned} \tag{42}$$

for all $w \in L^2(0,T;L^2(\Omega))$.
It remains to show the convergence

$$\int_0^T \int_\Omega g_i(\Delta \dot{u}_i^m) \; w \, dx \; dt \longrightarrow \int_0^T \int_\Omega g_i(\Delta u_i') \; w \, dx \; dt,$$

when $m \to +\infty$.

Lemma 2.1. *For each $T > 0$, $g_i(\Delta u_i') \in L^1(\mathcal{A})$, $\|g_i(\Delta u_i')\|_{L^1(\mathcal{A})} \leq K$, where K is a constant independent of t and $g_i(\Delta \dot{u}_i^k) \to g_i(\Delta u_i')$ in $L^1(\mathcal{A})$.*

Proof. We claim that

$$g(\Delta u') \in L^1(\mathcal{A}).$$

Indeed, since g_i is continuous, we deduce from (38)

$$g_i(\Delta \dot{u}_i^k) \longrightarrow g_i(\Delta u_i') \quad \text{almost everywhere in} \quad \mathcal{A}. \tag{43}$$

$$\Delta \dot{u}_i^k g_i(\Delta \dot{u}_i^k) \longrightarrow \Delta u_i' g_i(\Delta u_i') \quad \text{almost everywhere in} \quad \mathcal{A}.$$

Hence, by (27) and Fatou's Lemma, we have

$$\int_0^T \int_\Omega \Delta u_i'(x,t) g_i(\Delta u_i'(x,t)) \, dx \, dt \leq K_1, \quad \text{for } T > 0. \tag{44}$$

Now, we can estimate $\int_0^T \int_\Omega |g_i(\Delta u_i'(x,t))| \, dx \, dt$. By Cauchy-Schwarz inequality, we have

$$\int_0^T \int_\Omega |g_i(\Delta u_i'(x,t))| \, dx \, dt \leq c|\mathcal{A}|^{1/2} \Big(\int_0^T \int_\Omega |g_i(\Delta u_i'(x,t))|^2 \, dx \, dt \Big)^{1/2}.$$

Using (4) and (44), we obtain

$$\begin{aligned} \int_0^T \int_\Omega |g_i(\Delta u_i'(x,t))|^2 \, dx \, dt &\leq \int_0^T \int_{|\Delta u_i'| > \varepsilon} \Delta u_i' g_i(\Delta u_i') \, dx \, dt + \int_0^T \int_{|\Delta u_i'| \leq \varepsilon} G^{-1}(\Delta u_i' g_i(\Delta u_i')) \, dx \, dt \\ &\leq c \int_0^T \int_\Omega \Delta u_i' g_i(\Delta u_i') \, dx \, dt + c G^{-1} \Big(\int_{\mathcal{A}} \Delta u_i' g_i(\Delta u_i') \, dx \, dt \Big) \\ &\leq c \int_0^T \int_\Omega \Delta u_i' g_i(\Delta u_i') \, dx \, dt + c' G^*(1) + c'' \int_\Omega \Delta u_i' g(\Delta u_i') \, dx \, dt \\ &\leq c K_1 + c' G^*(1), \quad \text{for } T > 0. \end{aligned}$$

Then

$$\int_0^T \int_{\mathcal{A}} |g_i(\Delta u_i'(x,t))| \, dx \, d \leq K, \quad \text{for } T > 0.$$

Let $E \subset \Omega \times [0,T]$ and set

$$E_1 = \Big\{(x,t) \in E : |g_i(\Delta \dot{u}_i^m(x,t))| \leq \frac{1}{\sqrt{|E|}}\Big\}, \quad E_2 = E \backslash E_1,$$

where $|E|$ is the measure of E. If $M(r) = \inf\{|s| : s \in \mathbb{R} \text{ and } |g_i(s)| \geq r\}$

$$\int_E |g_i(\Delta \dot{u}_i^m)|\, dx\, dt \leq c\sqrt{|E|} + \Big(M\Big(\frac{1}{\sqrt{|E|}}\Big)\Big)^{-1} \int_{E_2} |\Delta \dot{u}_i^m g_i(\Delta \dot{u}_i^m)|\, dx\, dt.$$

By applying (27) we deduce that

$$\sup_m \int_E g_i(\Delta \dot{u}_i^m)\ dx\ dt \longrightarrow 0, \text{ when } |E| \longrightarrow 0.$$

From Vitali's convergence theorem we deduce that

$$g_i(\Delta \dot{u}_i^m) \to g_i(\Delta u_i') \quad \text{in } L^1(\mathcal{A}).$$

This completes the proof. □

Then (39) implies that

$$g_i(\Delta \dot{u}_i^m) \rightharpoonup g_i(\Delta u_i'), \text{ weak-star in } L^2([0,T] \times \Omega).$$

We deduce, for all $v \in L^2([0,T] \times L^2(\Omega)$, that

$$\int_0^T \int_\Omega g_i(\Delta \dot{u}_i^m) w\, dx\, dt \longrightarrow \int_0^T \int_\Omega g_i(\Delta u_i') w\, dx\, dt.$$

Finally we have shown that, for all $w \in L^2([0,T] \times L^2(\Omega))$:

$$\int_0^T \int_\Omega \Big(u_1''(x,t) + \Delta^2 u_1(x,t) - a(x)\Delta u_2(x,t) - g_1(\Delta u_1'(x,t))\Big) w\, dx\, dt = 0.$$

and

$$\int_0^T \int_\Omega \Big(u_2''(x,t) - \Delta u_2(x,t) - a(x)\Delta u_1(x,t) - g_2(\Delta u_2'(x,t))\Big) w\, dx\, dt = 0.$$

Therefore, (u_1, u_2) are a solutions for the problem (1).

3. Proof of Theorem 1.2

From now on, we denote by c various positive constants which may be different on different occurrences. Multiplying the first equation of (1) by $-\frac{\varphi(E)}{E}\Delta u_1$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is convex, increasing and of class $\mathcal{C}^1$ on $]0, +\infty[$ such that $\varphi(0) = 0$. Thus, we obtain

$$\begin{aligned}
0 &= \int_S^T -\frac{\varphi(E)}{E} \int_\Omega \Delta u_1 (u_1'' + \Delta^2 u_1 - a(x)\Delta u_2 - g_1(\Delta u_1'))\, dx\, dt \\
&= -\Big[\frac{\varphi(E)}{E} \int_\Omega u_1' \Delta u_1\, dx\Big]_S^T + \int_S^T \Big(\frac{\varphi(E)}{E}\Big)' \int_\Omega \Delta u_1 u_1'\, dx\, dt \\
&- 2\int_S^T \frac{\varphi(E)}{E} \int_\Omega |\nabla u_1'|^2\, dx\, dt + \int_S^T \frac{\varphi(E)}{E} \int_\Omega (|\nabla u_1'|^2 + |\nabla \Delta u_1|^2)\, dx\, dt \\
&+ \int_S^T \frac{\varphi(E)}{E} \int_\Omega a(x)\Delta u_1 \Delta u_2\, dx\, dt + \int_S^T \frac{\varphi(E)}{E} \int_\Omega \Delta u_1 . g_1(\Delta u_1')\, dx\, dt
\end{aligned}$$

Similarly, we have

$$\begin{aligned} 0 &= \int_S^T -\frac{\varphi(E)}{E}\int_\Omega \Delta u_2(u_2''+\Delta u_2 - a(x)\Delta u_1 - g_2(\Delta u_2'))\,dx\,dt \\ &= -\Big[\frac{\varphi(E)}{E}\int_\Omega u_2'\Delta u_2\,dx\Big]_S^T + \int_S^T \Big(\frac{\varphi(E)}{E}\Big)'\int_\Omega \Delta u_2 u_2'\,dx\,dt \\ &- 2\int_S^T \frac{\varphi(E)}{E}\int_\Omega |\nabla u_2'|^2\,dx\,dt + \int_S^T \frac{\varphi(E)}{E}\int_\Omega (|\nabla u_2'|^2+|\Delta u_2|^2)\,dx\,dt \\ &+ \int_S^T \frac{\varphi(E)}{E}\int_\Omega a(x)\Delta u_2\Delta u_1\,dx\,dt + \int_S^T \frac{\varphi(E)}{E}\int_\Omega \Delta u_2.g_2(\Delta u_2')\,dx\,dt \end{aligned}$$

Taking their sum, we obtain

$$\begin{aligned} \int_S^T \varphi(E)\,dt \le & \Big[\frac{\varphi(E)}{E}\int_\Omega u_1'\Delta u_1 + u_2'\Delta u_2\,dx\Big]_S^T \\ & - \int_S^T \Big(\frac{\varphi(E)}{E}\Big)'\int_\Omega \Delta u_1 u_1' + \Delta u_2 u_2'\,dx\,dt \\ & + 2\int_S^T \frac{\varphi(E)}{E}\int_\Omega |\nabla u_1'|^2 + |\nabla u_2'|^2\,dx\,dt \\ & + \int_S^T \frac{\varphi(E)}{E}\int_\Omega \Delta u_1.g_1(\Delta u_1') + \Delta u_2.g_2(\Delta u_2')\,dx\,dt \end{aligned} \tag{45}$$

Since E is non-increasing, we find that

$$\Big[\frac{\varphi(E)}{E}\int_\Omega u_1'\Delta u_1 + u_2'\Delta u_2\,dx\Big]_S^T \le c\varphi(E(S))$$

$$\Big|\int_S^T \Big(\frac{\varphi(E)}{E}\Big)'\int_\Omega \Delta u_1 u_1' + \Delta u_2 u_2'\,dx\,dt\Big| \le c\varphi(E(S))$$

Using these estimates, we conclude from (45) that

$$\begin{aligned} \int_S^T \varphi(E)\,dt \le & C\varphi(E(S)) + 2\int_S^T \frac{\varphi(E)}{E}\int_\Omega |\nabla u_1'|^2 + |\nabla u_2'|^2\,dx\,dt \\ & + \int_S^T \frac{\varphi(E)}{E}\int_\Omega |\Delta u_1|.|g_1(\Delta u_1')| + |\Delta u_2|.|g_2(\Delta u_2')|\,dx\,dt \end{aligned} \tag{46}$$

Now, we estimate the terms of the right-hand side of (46) in order to apply the results of Lemma 1.2.
As in Komornik [8], we consider the following partition of Ω,

$$\Omega^+ = \{x\in\Omega : |\Delta u_i'| > \epsilon\}, \quad \Omega^- = \{x\in\Omega : |\Delta u_i'| \le \epsilon\}.$$

We distinguish two cases:
►**Case 1.** G **is linear on** $[0,\epsilon]$. By using Sobolev embedding and Young's inequality, we obtain

$$\begin{aligned} & \int_S^T \frac{\varphi(E)}{E}\int_{\Omega^+} |\Delta u_1|.|g_1(\Delta u_1')|\,dx\,dt + \int_S^T \frac{\varphi(E)}{E}\int_{\Omega^+} |\nabla u_1'|^2\,dx\,dt \\ & \le \varepsilon\int_S^T \frac{\varphi(E)}{E}\int_{\Omega^+} |\Delta u_1|^2\,dx\,dt + C(\varepsilon)\int_S^T \frac{\varphi(E)}{E}\int_{\Omega^+} |g_1(\Delta u_1')|^2\,dx\,dt + c\int_S^T \frac{\varphi(E)}{E}\int_{\Omega^+} |\Delta u_1'|^2 \\ & \le \varepsilon c'\int_S^T \frac{\varphi(E)}{E}\int_\Omega |\nabla\Delta u_1|^2\,dx\,dt + (C(\varepsilon)c_2 + \frac{c}{c_1})\int_S^T \frac{\varphi(E)}{E}\int_\Omega \Delta u_1' g_1(\Delta u_1')\,dx\,dt \\ & \le \varepsilon C\int_S^T \varphi(E)\,dt + C_1(\varepsilon)\int_S^T \frac{\varphi(E)}{E}\int_\Omega \Delta u_1' g_1(\Delta u_1')\,dx\,dt, \end{aligned} \tag{47}$$

Similarly, we have

$$\begin{aligned}&\int_S^T \frac{\varphi(E)}{E}\int_{\Omega^+}|\Delta u_2|.|g_2(\Delta u_2')|\,dx\,dt+\int_S^T \frac{\varphi(E)}{E}\int_{\Omega^+}|\nabla u_2'|^2\,dx\,dt\\&\le \varepsilon C\int_S^T\varphi(E)\,dt+C_2(\varepsilon)\int_S^T\frac{\varphi(E)}{E}\int_\Omega \Delta u_2' g_2(\Delta u_2')\,dx\,dt.\end{aligned}\tag{48}$$

Summing (47) and (48), and noting that $s\mapsto\frac{\varphi(s)}{s}$ is non-decreasing, we obtain

$$\begin{aligned}\int_S^T \frac{\varphi(E)}{E}\int_{\Omega^+}&|\Delta u_1|.|g_1(\Delta u_1')|+|\Delta u_2|.|g_2(\Delta u_2')|\,dx\,dt\\&+\int_S^T\frac{\varphi(E)}{E}\int_{\Omega^+}|\nabla u_1'|^2+|\nabla u_2'|^2\,dx\,dt\\&\le\varepsilon C\int_S^T\varphi(E)\,dt+C'(\varepsilon)\int_S^T\frac{\varphi(E)}{E}(-E'(t))\,dt\\&\le\varepsilon C\int_S^T\varphi(E)\,dt+C'(\varepsilon)\varphi(E(S))\end{aligned}\tag{49}$$

and

$$\begin{aligned}\int_S^T \frac{\varphi(E)}{E}\int_{\Omega^-}&|\Delta u_1|.|g(\Delta u_1')|+|\Delta u_2|.|g(\Delta u_2')|\,dx\,dt\\&+\int_S^T\frac{\varphi(E)}{E}\int_{\Omega^-}|\nabla u_1'|^2+|\nabla u_2'|^2\,dx\,dt\\&\le\varepsilon C\int_S^T\varphi(E)\,dt+C'(\varepsilon)\int_S^T\frac{\varphi(E)}{E}(-E'(t))\,dt\\&\le\varepsilon C\int_S^T\varphi(E)\,dt+C'(\varepsilon)\varphi(E(S)).\end{aligned}$$

Inserting these two inequalities into (46) and choosing $\varepsilon>0$ small enough, we deduce that

$$\int_S^T\varphi(E(t))\,dt\le c\varphi(E(S))$$

Since, choosing $\varphi(s)=s$, we deduce from (8) that

$$E(t)\le ce^{-\omega t}.$$

▶ **Case 2.** $G'(0)=0,\ \ G''>0$ **on** $]0,\epsilon]$
Using (4) and the fact that $s\mapsto\frac{\varphi(s)}{s}$ is non-decreasing, we obtain

$$\begin{aligned}&\int_S^T\frac{\varphi(E)}{E}\int_{\Omega^+}|\nabla u_1'|^2+|\Delta u_1|.|g(\Delta u_1')|\,dx\,dt\\&\le\varepsilon C\int_S^T\varphi(E)\,dt+\int_S^T\frac{\varphi(E)}{E}\int_{\Omega^+}\Delta u_1' g_1(\Delta u_1')\,dx\,dt\end{aligned}$$

and

$$\begin{aligned}&\int_S^T\frac{\varphi(E)}{E}\int_{\Omega^+}|\nabla u_2'|^2+|\Delta u_2|.|g(\Delta u_2')|\,dx\,dt\\&\le\varepsilon C\int_S^T\varphi(E)\,dt+\int_S^T\frac{\varphi(E)}{E}\int_{\Omega^+}\Delta u_2' g_2(\Delta u_2')\,dx\,dt.\end{aligned}$$

Summing these two inequalities, we have

$$\begin{aligned}&\int_S^T \frac{\varphi(E)}{E}\int_{\Omega^+}|\nabla u_1'|^2+|\Delta u_1|.|g_1(\Delta u_1')|\,dx\,dt\\&+\int_S^T \frac{\varphi(E)}{E}\int_{\Omega^+}|\nabla u_2'|^2+|\Delta u_2|.|g_2(\Delta u_2')|\,dx\,dt\\&\le \varepsilon C\int_S^T \varphi(E)\,dt+C\varphi(E(S)),\end{aligned}\tag{50}$$

and exploit Jensen's inequality and the concavity of G^{-1} to obtain

$$\begin{aligned}&\int_S^T \frac{\varphi(E)}{E}\int_{\Omega^-}|\Delta u_1|.|g_1(\Delta u_1')|+|\nabla u_1'|^2\,dx\,dt\\&\le \varepsilon c'\int_S^T \frac{\varphi(E)}{E}\int_\Omega|\nabla\Delta u_1|^2\,dx\,dt+C(\varepsilon)\int_S^T\frac{\varphi(E)}{E}\int_\Omega(|\Delta u_1'|^2+|g_1(\Delta u_1')|^2)\,dx\,dt\\&\le \varepsilon c'\int_S^T \frac{\varphi(E)}{E}\int_\Omega|\nabla\Delta u_1|^2\,dx\,dt+C(\varepsilon)\int_S^T\frac{\varphi(E)}{E}|\Omega|G^{-1}\Big(\frac{1}{|\Omega|}\int_\Omega \Delta u_1' g_1(\Delta u_1')\,dx\Big)\,dt\\&\le \varepsilon C\int_S^T\varphi(E)\,dt+C(\varepsilon)\int_S^T\frac{\varphi(E)}{E}|\Omega|G^{-1}\Big(\frac{1}{|\Omega|}\int_\Omega \Delta u_1' g_1(\Delta u_1')\,dx\Big)\,dt\end{aligned}\tag{51}$$

Similarly, we have

$$\begin{aligned}&\int_S^T \frac{\varphi(E)}{E}\int_{\Omega^-}|\Delta u_2|.|g_2(\Delta u_2')|+|\nabla u_2'|^2\,dx\,dt\\&\le \varepsilon\int_S^T \frac{\varphi(E)}{E}\int_\Omega|\Delta u_2|^2\,dx\,dt+C(\varepsilon)\int_S^T\frac{\varphi(E)}{E}|\Omega|G^{-1}\Big(\frac{1}{|\Omega|}\int_\Omega \Delta u_2' g_2(\Delta u_2')\,dx\Big)\,dt\\&\le \varepsilon C\int_S^T\varphi(E)\,dt+C(\varepsilon)\int_S^T\frac{\varphi(E)}{E}|\Omega|G^{-1}\Big(\frac{1}{|\Omega|}\int_\Omega \Delta u_2' g_2(\Delta u_2')\,dx\Big)\,dt\end{aligned}\tag{52}$$

Let G^* denote the dual function of the convex function G in the sense of Young (see Arnold [5, p. 64]). Then G^* is the Legendre transform of G, which is given by (see Arnold [5, p. 61-62]) i.e.,

$$G^*(s)=\sup_{t\in\mathbb{R}_+}(st-G(t)).$$

Then G^* is given by

$$G^*(s)=s(G')^{-1}(s)-G[(G')^{-1}(s)],\quad \forall s\ge 0$$

and satisfies the following inequality

$$st\le G^*(s)+G(t)\quad \forall s,\,t\ge 0.\tag{53}$$

Choosing $\varphi(s)=sG'(\epsilon s)$, we obtain

$$G^*\Big(\frac{\varphi(s)}{s}\Big)=s\epsilon G'(\epsilon s)=\epsilon sG'(\epsilon s)-G(\epsilon s)\le\epsilon\varphi(s).\tag{54}$$

Making use of (53) and (54), we have

$$\begin{aligned}&\int_S^T\frac{\varphi(E)}{E}|\Omega|G^{-1}\Big(\frac{1}{|\Omega|}\int_\Omega\Delta u_i' g_i(\Delta u_i')\Big)\,dx\,dt\\&\le c\int_S^T G^*(\frac{\varphi(E)}{E})\,dt+c\int_S^T\int_\Omega \Delta u_i' g_i(\Delta u_i')\Big)\,dx\,dt\\&\le \epsilon\int_S^T\varphi(E)\,dt+c\int_S^T\int_\Omega\Delta u_i' g_i(\Delta u_i')\Big)\,dx\,dt.\end{aligned}\tag{55}$$

Summing (51) and (52) and using (55), we obtain

$$\int_S^T \frac{\varphi(E)}{E} \int_{\Omega^-} |\Delta u_1|.|g_1(\Delta u_1')| + |\nabla u_1'|^2 \, dx \, dt + \int_S^T \frac{\varphi(E)}{E} \int_{\Omega^-} |\Delta u_2|.|g_2(\Delta u_2')| + |\nabla u_2'|^2 \, dx \, dt$$
$$\leq \varepsilon C \int_S^T \varphi(E) \, dt + C(\varepsilon)E(S) \tag{56}$$

Then, choosing $\varepsilon > 0$ small enough and substitution of (50) and (56) into (46) gives

$$\int_S^T \varphi(E(t)) \, dt \leq c(E(S) + \varphi(E(S)))$$
$$\leq c\Big(1 + \frac{\varphi(E(S))}{E(S)}\Big)E(S) \leq cE(S), \ \ \forall S \geq 0$$

Using Lemma 1.2 in the particular case where $\Psi(s) = \omega\varphi(s)$ we deduce from (5) our estimate (46). The proof of Theorem 1.2 is now complete.

Example 3.1. *Let g_i be given by $g_i(s) = s^p(-\ln s)^q$ where $p \geq 1$ and $q \in \mathbb{R}$ on $]0, \epsilon]$. Then $g_i'(s) = s^{p-1}(-\ln s)^{q-1}(p(-\ln s) - q)$ which is an increasing function in the right neighborhood of* 0 *(if $q = 0$ we can take $\epsilon = 1$). The function G is defined in the neighborhood of* 0 *by*

$$G(s) = cs^{\frac{p+1}{2}}(-\ln \sqrt{s})^q$$

and we have

$$G'(s) = cs^{\frac{p-1}{2}}(-\ln \sqrt{s})^{q-1}\Big(\frac{p+1}{2}(-\ln \sqrt{s}) - \frac{q}{2}\Big), \quad \text{when } s \text{ is near } 0$$

Thus

$$\varphi(s) = cs^{\frac{p+1}{2}}(-\ln \sqrt{s})^{q-1}\Big(\frac{p+1}{2}(-\ln \sqrt{s}) - \frac{q}{2}\Big), \quad \text{when } s \text{ is near } 0$$

and

$$\psi(t) = c\int_t^1 \frac{1}{s^{\frac{p+1}{2}}(-\ln \sqrt{s})^{q-1}\Big(\frac{p+1}{2}(-\ln \sqrt{s}) - \frac{q}{2}\Big)} ds$$
$$= c\int_1^{\frac{1}{\sqrt{t}}} (\ln z)^{q-1}\Big(\frac{p+1}{2}\ln z - \frac{q}{2}\Big) dz, \quad \text{when } t \text{ is near } 0$$

We obtain in the neighborhood of 0

$$\psi(t) = \begin{cases} c\dfrac{1}{t^{\frac{p-1}{2}}(-\ln t)^q} & \text{if } q > 1 \\ c(-\ln t)^{q-1} & \text{if } p = 1, \ q < 1 \\ c(\ln(-\ln t)) & \text{if } p = 1, \ q = 1 \end{cases}$$

and then in the neighborhood of $+\infty$

$$\psi^{-1}(t) = \begin{cases} ct^{-\frac{2}{p-1}}(\ln t)^{-\frac{2q}{p-1}} & \text{if } q > 1 \\ ce^{-t^{\frac{1}{1-q}}} & \text{if } p = 1, \ q < 1 \\ ce^{-e^t} & \text{if } p = 1, \ q = 1 \end{cases}$$

Since $h(t) = t$ as t tends to infinity, we obtain

$$E(t) \leq \begin{cases} ct^{-\frac{2}{p-1}}(\ln t)^{-\frac{2q}{p-1}} & \text{if } q > 1 \\ ce^{-t^{\frac{1}{1-q}}} & \text{if } p = 1, \ q < 1 \\ ce^{-e^t} & \text{if } p = 1, \ q = 1 \end{cases}$$

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Akram Ben Aissa is currently an associate professor of mathematics in Higher institute of Transport and Logistics, university of Sousse, Tunisia. He is a member of Laboratory Analysis and Control of PDE's, UR 13ES64. His research of interest are Control theory of Partial Differential Equations and Functional Analysis With Application to Linear and Nonlinear PDE's.