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HARMONIC MULTIVALENT FUNCTIONS ASSOCIATED WITH A (P,Q)-ANALOGUE OF RUSCHEWEYH OPERATOR

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Abstract. The aim of this paper is to introduce and investigate a new class of harmonic multivalent functions defined by (p,q)-analogue of Ruscheweyh operator for multivalent functions. For this new class, we obtain a (p,q)-coefficient inequality as a sufficient condition. Using this coefficient inequality, we establish sharp bounds of the real parts of the ratios of harmonic multivalent functions to its sequences of partial sums. We further consider a subclass of our new class and for which we obtain (p,q)-analogue of coefficient characterization which in fact helps us to determine its properties such as distortion bounds, extreme points, convolutions and convexity conditions. In the last section on conclusion, it is pointed out that the results obtained in this paper may also be extended to some generalized classes.

Keywords: (p,q)-calculus; (p,q)-Ruscheweyh operator; multivalent harmonic functions; a (p,q)-Ruscheweyh multivalent operator; partial sums.

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1. Introduction

Duren et al. in 1996 [6] introduced multivalent harmonic functions in the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ via the argument principle. For natural number m, let $\mathcal{H}(m)$ denote the class of all multivalent functions $f = h + \overline{g}$, where

$$h(z) = z^m + \sum_{k=m+1}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=m}^{\infty} b_k z^k.$$

Let $S_{\mathcal{H}}(m)$ be a subclass of $\mathcal{H}(m)$ of functions $f = h + \overline{g}$ that are sense-preserving in $\mathbb{D}$ and $S^0_{\mathcal{H}}(m)$ denotes a subclass of $S_{\mathcal{H}}(m)$ when $g^{(m)}(0) = 0$ for a given natural number m.

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Thus if $f = h + \overline{g} \in S^0_{\mathcal{H}}(m)$, then

$$h(z) = z^m + \sum_{k=m+1}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=m+1}^{\infty} b_k z^k. \tag{1}$$

We observe that if $g \equiv 0$ for each $f = h + \overline{g} \in S^0_{\mathcal{H}}(m)$, then $S^0_{\mathcal{H}}(m)$ reduces to $S(m)$, a well known subclass of analytic multivalent and normalized functions in $\mathbb{D}$. Note that $S^0_{\mathcal{H}}(1)$ is a subclass of $S_{\mathcal{H}}(1)$ of harmonic locally univalent and sense-preserving functions defined in $\mathbb{D}$ that was studied by Clunie and Sheil-Small [5].

We next recall a few notations and definitions of (p, q)-calculus that are needed in this paper. The theory of (p, q)-calculus (or post-quantum calculus) are used in various areas of science and mathematics; see for example [12]. Let $0 < q < p \leq 1$. The (p, q)-derivative operator of a function h is defined by

$$\partial_{p,q} h(z) = \begin{cases} \frac{h(pz)-h(qz)}{(p-q)z}, & z \neq 0, \\ h'(0), & z = 0. \end{cases}$$

It is clear that if h_1 and h_2 are two functions, then

$$\partial_{p,q}(h_1(z) + h_2(z)) = \partial_{p,q}(h_1(z)) + \partial_{p,q}(h_2(z))$$

and

$$\partial_{p,q}(ch(z)) = c\partial_{p,q}(h(z)),$$

where c is a constant. In particular, if $h(z) = z^k$, then

$$\partial_{p,q}(z^k) = [k]_{p,q} z^{k-1}, \tag{2}$$

where

$$[k]_{p,q} = \frac{p^k - q^k}{p - q} = p^{k-1} + p^{k-2}q + \cdots + pq^{k-2} + q^{k-1}, \quad k \in \mathbb{N}$$

is a (p, q)-bracket or twin number. The (p, q)-number factorial is defined by

$$[k]_{p,q}! = [1]_{p,q}[2]_{p,q}[3]_{p,q} \ldots [k]_{p,q}, \qquad [0]_{p,q}! = 1.$$

The (p, q)-shifted factorial is defined as

$$([k]_{p,q})_n = \begin{cases} [k]_{p,q}[k+1]_{p,q}[k+2]_{p,q} \ldots [k+n-1]_{p,q}, & \text{if } n \geq 1 \\ 1, & \text{if } n = 0. \end{cases}$$

Moreover, the (p, q)-gamma function is defined as

$$\Gamma_{p,q}(k+1) = [k]_{p,q}\Gamma_{p,q}(k) \qquad \text{and} \qquad \Gamma_{p,q}(1) = 1.$$

Some applications of (p, q)-calculus may be found in [20].

Note that, if $p = 1$, the (p, q)-calculus reduces to the q-calculus and the (p, q)-derivative $\partial_{p,q}$ reduces to the q-derivative. Using the q-derivative, in [22] authors introduced and studied various families of q-starlike functions $f \in S(1)$. It is mentioned in [22] that the results obtained may easily be extended into the corresponding results for (p, q)-analogue. A q-analogue of Noor integral operator is also introduced and studied by Arif et al. [3]. For more study of q-calculus, one may refer to [2, 7, 10, 11] and also some recent publications [15, 16, 19, 23, 24].

The convolution of two analytic functions

$$h_1(z) = \sum_{k=1}^{\infty} \alpha_k z^k \quad \text{and} \quad h_2(z) = \sum_{k=1}^{\infty} \beta_k z^k$$

is defined by

$$h_1(z) * h_2(z) = (h_1 * h_2)(z) = \sum_{k=1}^{\infty} \alpha_k \beta_k z^k.$$

A q-analogue of Ruscheweyh operator $R_q^{\lambda} : S(1) \to S(1)$ was introduced and studied by Kanas and Raducanu [14]. Recently, Arif et al. in [4] defined a q-analogue of Ruscheweyh operator for multivalent functions in the class $S(m)$.

Define a (p,q)-analogue of Ruscheweyh operator for multivalent functions: $L_{p,q}^{\delta+m-1}$: $S(m) \to S(m)$ by

$$L_{p,q}^{\delta+m-1} h(z) = h(z) * \phi_m(p,q,\delta;z), \tag{3}$$

where the function $\phi_m(p,q,\delta;z)$ is defined by

$$\phi_m(p,q,\delta;z) = z^m + \sum_{k=m+1}^{\infty} \frac{([\delta+m]_{p,q})_{k-m}}{[k-m]_{p,q}!} z^k \tag{4}$$

$$(\delta > -m, m \in \mathbb{N}, 0 < q < p \leq 1).$$

Note that the series in (4) converges absolutely in the unit disk $\mathbb{D}$.

For the case if $\delta + m \in \mathbb{N}$, the (p,q)-operator $\partial_{p,q}^{\delta+m-1}$ of order $\delta + m - 1$ is defined by

$$\partial_{p,q}^{\delta+m-1} = \underbrace{\partial_{p,q} \ldots \partial_{p,q}}_{\delta+m-1 \text{ times}}$$

and hence, for this case, the operator $L_{p,q}^{\delta+m-1}$ defined by (3) may also be given by

$$L_{p,q}^{\delta+m-1} h(z) = \frac{z^m \partial_{p,q}^{\delta+m-1}\left(z^{\delta-1} h(z)\right)}{[\delta+m-1]_{p,q}!}.$$

Making use of (3) and (4), for the functions h and g given by (1), we get

$$L_{p,q}^{\delta+m-1}\left(h(z)+g(z)\right) = z^m + \sum_{k=m+1}^{\infty} \psi_k(\delta)(a_k+b_k) z^k, \tag{5}$$

where

$$\psi_k(\delta) = \frac{([\delta+m]_{p,q})_{k-m}}{[k-m]_{p,q}!}. \tag{6}$$

Taking $p = 1$, we denote the operator $L_{p,q}^{\delta+m-1}$ by $L_q^{\delta+m-1}$ which was defined in [4, p. 1213]. Further, taking $p = 1$, $m = 1$, and replacing δ by λ, the operator $L_{p,q}^{\delta+m-1}$ reduces to R_q^{λ} which was considered in [14]. Some applications of the operator R_q^{λ} may be found in [1, 18]. Taking $p = 1$, $q \to 1^-$ and replacing m by p, δ by n, the operator $L_{p,q}^{\delta+m-1}$ reduces to the operator D^{n+p-1}, introduced by Goel and Sohi in [8, 9] for functions $h \in S(p)$, which generalizes the well known Ruscheweyh operator D^n [21] for univalent analytic functions $h \in S(1)$.

Clunie and Sheil-Small in [5] studied univalent harmonic functions through some geometric properties of related analytic functions by introducing a shearing technique. Using this concept of shearing and motivated with Karpuzoğullari et al. [13] and Li and Liu [17], involving the operator $L_{p,q}^{\delta+m-1}$, we define a class $S_{\mathcal{H}}^0(m,\delta,p,q,\alpha)$ as follows:

Definition 1.1. *A function $f = h + \overline{g} \in S^0_{\mathcal{H}}(m)$ is said to be in the class $S^0_{\mathcal{H}}(m, \delta, p, q, \alpha)$ if the function f satisfy the condition*

$$\Re e \left\{ \frac{\partial_{p,q} L^{\delta+m-1}_{p,q} \left(h(z) + g(z)\right)}{[m]_{p,q} z^{m-1}} \right\} > \alpha, \ 0 \leq \alpha < 1, \tag{7}$$

where $\delta > -m, m \in \mathbb{N}, 0 < q < p \leq 1$.

If $p = 1, q \to 1^-$, $m = 1$, $\delta = 0$ and replacing α to β, the class $S^0_{\mathcal{H}}(m, \delta, p, q, \alpha)$ reduces to the class $\mathcal{HP}(\beta)$ which was studied in [13]. If we take $p = 1, q \to 1^-$, $g \equiv 0$ and replacing m by p, δ by n in the condition (7), it reduces to the class condition that was studied by Goel and Sohi in [9] for analytic functions $h \in S(p)$.

In this paper, we obtain a sufficient coefficient condition in the form of a coefficient inequality for the functions $f = h + \overline{g}$ to be in the class $S^0_{\mathcal{H}}(m, \delta, p, q, \alpha)$. Results related to the partial sums are derived when the functions $f = h + \overline{g} \in S^0_{\mathcal{H}}(m)$ that satisfy the sufficient condition for the class $S^0_{\mathcal{H}}(m, \delta, p, q, \alpha)$. We further consider a subclass of the class $S^0_{\mathcal{H}}(m, \delta, p, q, \alpha)$ and then establish a (p, q)-coefficient characterization of the functions in this subclass. We also obtain distortion bounds, extreme points, convolutions and convexity for functions belonging to the subclass of $S^0_{\mathcal{H}}(m, \delta, p, q, \alpha)$.

2. Analogues of Partial Sums in (p,q)-Calculus

In order to obtain certain analogues of patial sums in (p, q)-calculus, we need the following result.

Lemma 2.1. *Let $f = h + \overline{g} \in S^0_{\mathcal{H}}(m)$, where h and g are given by* (1). *Then $f \in S^0_{\mathcal{H}}(m, \delta, p, q, \alpha)$ if*

$$\sum_{k=m+1}^{\infty} \frac{[k]_{p,q}}{[m]_{p,q}} \psi_k(\delta) \left(|a_k| + |b_k|\right) \leq 1 - \alpha, \tag{8}$$

where $\psi_k(\delta)$ is defined by (6). The inequality (8) *is sharp.*

Proof. Using the fact $\Re e(w) > \alpha$ if and only if $|1 - \alpha + w| > |1 + \alpha - w|$, to prove the lemma, it is sufficient to prove for $|z| = r$ $(0 < r < 1)$ that

$$\left| \frac{\partial_{p,q} L^{\delta+m-1}_{p,q} \left(h(z) + g(z)\right)}{[m]_{p,q} z^{m-1}} + 1 - \alpha \right| - \left| \frac{\partial_{p,q} L^{\delta+m-1}_{p,q} \left(h(z) + g(z)\right)}{[m]_{p,q} z^{m-1}} - (1 + \alpha) \right| > 0. \tag{9}$$

On using (5), the left-hand-side of (9) is given by

$$\begin{aligned} &\left| 2 - \alpha + \sum_{k=m+1}^{\infty} \frac{[k]_{p,q}}{[m]_{p,q}} \psi_k(\delta) \left(|a_k| + |b_k|\right) z^{k-m} \right| \\ &\quad - \left| -\alpha + \sum_{k=m+1}^{\infty} \frac{[k]_{p,q}}{[m]_{p,q}} \psi_k(\delta) \left(|a_k| + |b_k|\right) z^{k-m} \right| \\ &\geq 2 \left\{ 1 - \alpha - \sum_{k=m+1}^{\infty} \frac{[k]_{p,q}}{[m]_{p,q}} \psi_k(\delta) \left(|a_k| + |b_k|\right) r^{k-m} \right\} > 0 \end{aligned}$$

as $r \to 1^-$, if inequality (8) holds. The harmonic mappings

$$f(z) = z^m + \sum_{k=m+1}^{\infty} \frac{[m]_{p,q}(1 - \alpha)}{[k]_{p,q} \psi_k(\delta)} \left(x_k z^k + y_k \bar{z}^k \right) \tag{10}$$

where

$$\sum_{k=m+1}^{\infty}(|x_k|+|y_k|)=1,$$

shows that the coefficient inequality given by (8) is sharp. This completes the proof of Lemma 2.1. □

We now study partial sums of certain multivalent harmonic functions belonging to the class $S^0_{\mathcal{H}}(m,\delta,p,q,\alpha)$. We establish some new results giving the sharp bounds of the real parts of ratios of harmonic multivalent functions to its sequences of partial sums.

Let $f=h+\overline{g}\in S^0_{\mathcal{H}}(m)$, where h and g are of the form (1). Then the sequences of partial sum of functions f are defined by

$$S_s(f)=z^m+\sum_{k=m+1}^{s}a_kz^k+\sum_{k=m+1}^{\infty}b_k\overline{z}^k:=S_s(h)+\overline{g},$$

$$S_l(f)=z^m+\sum_{k=m+1}^{\infty}a_kz^k+\sum_{k=m+1}^{l}b_k\overline{z}^k:=h+S_l(\overline{g}),$$

and

$$S_{s,l}(f)=z^m+\sum_{k=m+1}^{s}a_kz^k+\sum_{k=m+1}^{l}b_k\overline{z}^k:=S_s(h)+S_l(\overline{g}).$$

In this section, we determine the sharp lower bounds for $\Re e\{\frac{f(z)}{S_s(f)}\}$, $\Re e\{\frac{S_s(f)}{f(z)}\}$, $\Re e\{\frac{f(z)}{S_l(f)}\}$, $\Re e\{\frac{S_l(f)}{f(z)}\}$, $\Re e\{\frac{f(z)}{S_{s,l}(f)}\}$, and $\Re e\{\frac{S_{s,l}(f)}{f(z)}\}$.

Theorem 2.1. *Let $f=h+\overline{g}$, where h and g are of the form* (1)*. If f satisfies the condition* (8)*, then*

$$i)\quad \Re e\left(\frac{f(z)}{S_s(f)}\right)\geq\frac{c_{s+1}-(1-\alpha)}{c_{s+1}}, \tag{11}$$

$$ii)\quad \Re e\left(\frac{S_s(f)}{f(z)}\right)\geq\frac{c_{s+1}}{c_{s+1}+1-\alpha}, \tag{12}$$

where

$$c_k=\frac{[k]_{p,q}\psi_k(\delta)}{[m]_{p,q}}\ \ and\ \ c_k\geq\begin{cases}1-\alpha, & k=m+1,m+2,...,s\\ c_{s+1}, & k=s+1,s+2,...\end{cases}. \tag{13}$$

These estimates are sharp for the function given by

$$f(z)=z^m+\frac{1-\alpha}{c_{s+1}}z^{s+m}, \tag{14}$$

where c_{s+1} is given by (13) for $k=s+1, s\geq m+1$ and $0\leq\alpha<1$.

Proof. i) In order to prove (11), we may write

$$\psi_1(z)=\frac{c_{s+1}}{1-\alpha}\left\{\frac{f(z)}{S_s(f)}-\left(1-\frac{1-\alpha}{c_{s+1}}\right)\right\}$$

$$=1+\frac{\frac{c_{s+1}}{1-\alpha}\sum\limits_{k=s+1}^{\infty}a_kz^k}{z^m+\sum\limits_{k=m+1}^{s}a_kz^k+\sum\limits_{k=m+1}^{\infty}b_k\overline{z^k}}.$$

It is now sufficient to show that $\Re e\psi_1(z) > 0$ or equivalently

$$\left|\frac{\psi_1(z)-1}{\psi_1(z)+1}\right| \leq 1.$$

On the other hand

$$\left|\frac{\psi_1(z)-1}{\psi_1(z)+1}\right| \leq \frac{\frac{c_{s+1}}{1-\alpha}\sum\limits_{k=s+1}^{\infty}|a_k|}{2-2\left(\sum\limits_{k=m+1}^{s}|a_k|+\sum\limits_{k=m+1}^{\infty}|b_k|\right)-\frac{c_{s+1}}{1-\alpha}\sum\limits_{k=s+1}^{\infty}|a_k|} \leq 1$$

if and only if

$$\sum_{k=m+1}^{s}|a_k|+\sum_{k=m+1}^{\infty}|b_k|+\frac{c_{s+1}}{1-\alpha}\sum_{k=s+1}^{\infty}|a_k| \leq 1. \tag{15}$$

In view of (8), it is enough to show that left side of (15) is bounded above by

$$\sum_{k=m+1}^{\infty}\frac{c_k}{1-\alpha}|a_k|+\sum_{k=m+1}^{\infty}\frac{c_k}{1-\alpha}|b_k|,$$

which is equivalent to

$$\sum_{k=m+1}^{s}\frac{c_k-(1-\alpha)}{1-\alpha}|a_k|+\sum_{k=m+1}^{\infty}\frac{c_k-(1-\alpha)}{1-\alpha}|b_k|+\sum_{k=s+1}^{\infty}\frac{c_k-c_{s+1}}{1-\alpha}|a_k| \geq 0;$$

but it is true because of (13). In order to show that $f(z)=z^m+\frac{1-\alpha}{c_{s+1}}z^{s+m}$ gives the sharp result, we observe for $z=re^{i\pi/s}$ that

$$\frac{f(z)}{S_s(f)}=1+\frac{1-\alpha}{c_{s+1}}z^s \to 1-\frac{1-\alpha}{c_{s+1}}r^s=\frac{c_{s+1}-(1-\alpha)}{c_{s+1}},$$

when $r \to 1^-$.

ii) In order to prove result (12), we write

$$\psi_2(z)=\frac{c_{s+1}+1-\alpha}{1-\alpha}\left\{\frac{S_s(f)}{f(z)}-\left(1-\frac{1-\alpha}{c_{s+1}+1-\alpha}\right)\right\}$$
$$=1-\frac{\frac{c_{s+1}+1-\alpha}{1-\alpha}\sum\limits_{k=s+1}^{\infty}a_kz^k}{z^m+\sum\limits_{k=m+1}^{\infty}a_kz^k+\sum\limits_{k=m+1}^{\infty}b_k\overline{z^k}}.$$

Therefore

$$\left|\frac{\psi_2(z)-1}{\psi_2(z)+1}\right| \leq \frac{\frac{c_{s+1}+1-\alpha}{1-\alpha}\sum\limits_{k=s+1}^{\infty}|a_k|}{2-2\left(\sum\limits_{k=m+1}^{s}|a_k|+\sum\limits_{k=m+1}^{\infty}|b_k|\right)-\frac{c_{s+1}-(1-\alpha)}{1-\alpha}\sum\limits_{k=s+1}^{\infty}|a_k|} \leq 1$$

if and only if

$$\sum_{k=m+1}^{s}|a_k|+\sum_{k=m+1}^{\infty}|b_k|+\frac{c_{s+1}}{1-\alpha}\sum_{k=s+1}^{\infty}|a_k| \leq 1. \tag{16}$$

Since, left side of (16) is bounded above by

$$\sum_{k=m+1}^{\infty}\frac{c_k}{1-\alpha}|a_k|+\sum_{k=m+1}^{\infty}\frac{c_k}{1-\alpha}|b_k|,$$

the proof is completed because of given condition (8). □

Theorem 2.2. *Let $f = h + \overline{g}$, where h and g are of the form (1). If f satisfies the condition (8), then*

$$i)\quad \Re e\left(\frac{f(z)}{S_l(f)}\right) \geq \frac{c_{l+1}-(1-\alpha)}{c_{l+1}}, \tag{17}$$

$$ii)\quad \Re e\left(\frac{S_l(f)}{f(z)}\right) \geq \frac{c_{l+1}}{c_{l+1}+1-\alpha}, \tag{18}$$

where

$$c_k = \frac{[k]_{p,q}\psi_k(\delta)}{[m]_{p,q}} \text{ and } c_k \geq \begin{cases} 1-\alpha, & k=m+1, m+2, ..., l \\ c_{l+1}, & k=l+1, l+2, ... \end{cases}. \tag{19}$$

These estimates are sharp for the function given by

$$f(z) = z^m + \frac{1-\alpha}{c_{l+1}}\overline{z^{l+m}}. \tag{20}$$

Proof. The proof of Theorem 2.2 is similar to the proof of Theorem 2.1, and therefore it is omitted. □

Theorem 2.3. *Let $f = h + \overline{g}$, where h and g are of the form (1). If f satisfies the condition (8), then*

$$i)\quad \Re e\left(\frac{f(z)}{S_{s,l}(f)}\right) \geq \frac{c_{s+1}-(1-\alpha)}{c_{s+1}}, \tag{21}$$

$$ii)\quad \Re e\left(\frac{S_{s,l}(f)}{f(z)}\right) \geq \frac{c_{s+1}}{c_{s+1}+1-\alpha}, \tag{22}$$

where c_k is given by (13). These estimates are sharp for the function given by (14).

Proof. i) We may write

$$\begin{aligned}\psi_3(z) &= \frac{c_{s+1}}{1-\alpha}\left\{\frac{f(z)}{S_{s,l}(z)} - \left(1-\frac{1-\alpha}{c_{s+1}}\right)\right\} \\ &= 1 + \frac{\frac{c_{s+1}}{1-\alpha}\left(\sum\limits_{k=s+1}^{\infty} a_k z^k + \sum\limits_{k=l+1}^{\infty} b_k \overline{z^k}\right)}{z^m + \sum\limits_{k=m+1}^{s} a_k z^k + \sum\limits_{k=m+1}^{l} b_k \overline{z^k}}.\end{aligned}$$

It is sufficient to show that $\Re e\psi_3(z) > 0$, or equivalently

$$\left|\frac{\psi_3(z)-1}{\psi_3(z)+1}\right| \leq \frac{\frac{c_{s+1}}{1-\alpha}\left(\sum\limits_{k=s+1}^{\infty}|a_k| + \sum\limits_{k=l+1}^{\infty} b_k\right)}{2-2\left(\sum\limits_{k=m+1}^{s}|a_k| + \sum\limits_{k=m+1}^{l}|b_k|\right) - \frac{c_{s+1}}{1-\alpha}\left(\sum\limits_{k=s+1}^{\infty}|a_k| + \sum\limits_{k=l+1}^{\infty} b_k\right)} \leq 1$$

if and only if

$$\sum_{k=m+1}^{s}|a_k|+\sum_{k=m+1}^{l}|b_k|+\frac{c_{s+1}}{1-\alpha}\left(\sum_{k=s+1}^{\infty}|a_k|+\sum_{k=l+1}^{\infty}b_k\right)\leq 1. \tag{23}$$

In view of (8), it suffices to show that left side of (23) is bounded above by

$$\sum_{k=m+1}^{\infty}\frac{c_k}{1-\alpha}|a_k|+\sum_{k=m+1}^{\infty}\frac{c_k}{1-\alpha}|b_k|,$$

which is equivalent to

$$\sum_{k=m+1}^{l}\frac{c_k-(1-\alpha)}{1-\alpha}|a_k|+\sum_{k=m+1}^{\infty}\frac{c_k-(1-\alpha)}{1-\alpha}|b_k|+\sum_{k=s+1}^{\infty}\frac{c_k-c_{s+1}}{1-\alpha}|a_k|+\sum_{k=l+1}^{\infty}\frac{c_k-c_{s+1}}{1-\alpha}|b_k|\geq 0.$$

To see that $f(z)=z^m+\frac{1-\alpha}{c_{s+1}}z^{s+m}$ gives the sharp result, let $z=re^{i\pi/s}$. Then

$$\frac{f(z)}{S_{s,l}(f)}=1+\frac{1-\alpha}{c_{s+1}}z^s\to 1-\frac{1-\alpha}{c_{s+1}}\quad (r\to 1^-).$$

This proves (21).

ii) Similarly, we obtain the assertion in (22). □

Theorem 2.4. *Let $f=h+\overline{g}$, where h and g are of the form* (1). *If f satisfies the condition* (8), *then*

$$i)\quad \Re e\left(\frac{f(z)}{S_{s,l}(f)}\right)\geq\frac{c_{l+1}-(1-\alpha)}{c_{l+1}}, \tag{24}$$

$$ii)\quad \Re e\left(\frac{S_{s,l}(f)}{f(z)}\right)\geq\frac{c_{l+1}}{c_{l+1}+1-\alpha}, \tag{25}$$

where c_k is given by (19). These estimates are sharp for the function given by (20).

Proof. The proof of Theorem 2.4 is similar to the proof of Theorem 2.1, and therefore it is omitted. □

3. (p,q)-Analogues of Certain Properties of a Subclass of $S^0_{\mathcal{H}}(m,\delta,p,q,\alpha)$

We now consider a subclass $\mathcal{T}S^0_{\mathcal{H}}(m,\delta,p,q,\alpha)$ of functions $f=h+\overline{g}\in S^0_{\mathcal{H}}(m,\delta,p,q,\alpha)$, where h and g are of the form

$$h(z)=z^m-\sum_{k=m+1}^{\infty}|a_k|z^k\quad\text{and}\quad g(z)=-\sum_{k=m+1}^{\infty}|b_k|z^k. \tag{26}$$

Lemma 3.1. *Let h and g be of the form* (26).*Then $f=h+\overline{g}\in\mathcal{T}S^0_{\mathcal{H}}(m,\delta,p,q,\alpha)$ if and only if the condition* (8) *holds.*

Proof. "If part" follows from Lemma 2.1 because $\mathcal{T}S^0_{\mathcal{H}}(m,\delta,p,q,\alpha)$ is a subset of $S^0_{\mathcal{H}}(m,\delta,p,q,\alpha)$. For "Only if part", assume that $f=h+\overline{g}\in\mathcal{T}S^0_{\mathcal{H}}(m,\delta,p,q,\alpha)$, where h and g are of the form (26). Then from (7) we have

$$\Re e\left\{\frac{\partial_{p,q}L^{\delta+m-1}_{p,q}\left(h(z)+g(z)\right)}{[m]_{p,q}z^{m-1}}\right\}>\alpha,\ \forall z\in\mathbb{D}$$

which on using the series expansion from (5), gives

$$\Re e \left\{ 1 - \sum_{k=m+1}^{\infty} \frac{[k]_{p,q}}{[m]_{p,q}} \psi_k(\delta)(|a_k| + |b_k|) z^{k-m} \right\} > \alpha \tag{27}$$

for all values of $z \in \mathbb{D}$. As for real values of $z \to 1^-$, the condition (27) proves the inequality (8). $\square$

Finally, using Lemma 3.1, we obtain (p,q)-analogues of certain properties for the class $\mathcal{T}S^0_{\mathcal{H}}(m,\delta,p,q,\alpha)$ such as distortion bounds, extreme points, convolutions and convexity.

Theorem 3.1. *If* $f \in \mathcal{T}S^0_{\mathcal{H}}(m,\delta,p,q,\alpha)$, *then*

$$|f(z)| \leq |z|^m + \frac{[m]_{p,q}(1-\alpha)}{[m+1]_{p,q}[\delta+m]_{p,q}} |z|^{m+1} \tag{28}$$

and

$$|f(z)| \geq |z|^m - \frac{[m]_{p,q}(1-\alpha)}{[m+1]_{p,q}[\delta+m]_{p,q}} |z|^{m+1}. \tag{29}$$

Proof. Let $f = h + \overline{g} \in \mathcal{T}S^0_{\mathcal{H}}(m,\delta,p,q,\alpha)$, where h and g are of the form (26). In view of Lemma 3.1 and letting $\beta = \frac{[m+1]_{p,q}[\delta+m]_{p,q}}{[m]_{p,q}}$, we have

$$\begin{aligned}
|f(z)| &\leq |z|^m + \sum_{k=m+1}^{\infty} (|a_k| + |b_k|)|z|^k \\
&\leq |z|^m + |z|^{m+1} \sum_{k=m+1}^{\infty} (|a_k| + |b_k|) \\
&= |z|^m + \frac{1}{\beta}|z|^{m+1} \sum_{k=m+1}^{\infty} \beta\,(|a_k| + |b_k|) \\
&\leq |z|^m + \frac{1}{\beta}|z|^{m+1} \sum_{k=m+1}^{\infty} \frac{[k]_{p,q}}{[m]_{p,q}} \psi_k(\delta)(|a_k| + |b_k|) \\
&\leq |z|^m + \frac{1-\alpha}{\beta}|z|^{m+1}.
\end{aligned}$$

This proves (28). The proof of (29) is similar to the proof of (28). $\square$

From the lower bound of $|f(z)|$ given in (29), we obtain following covering result.

Corollary 3.1. *If* $f \in \mathcal{T}S^0_{\mathcal{H}}(m,\delta,p,q,\alpha)$, *then*

$$\left\{ w \in \mathbb{C} : |w| < 1 - \frac{[m]_{p,q}(1-\alpha)}{[m+1]_{p,q}[\delta+m]_{p,q}} \right\} \subset f(\mathbb{D}).$$

Theorem 3.2. *A function* $f \in \mathcal{T}S^0_{\mathcal{H}}(m,\delta,p,q,\alpha)$ *if and only if*

$$f(z) = \sum_{k=m}^{\infty} \left[x_k h_k(z) + y_k g_k(z) \right], \tag{30}$$

where

$$h_m(z) = z^m, \quad h_k(z) = z^m - \frac{[m]_{p,q}(1-\alpha)}{[k]_{p,q}\psi_k(\delta)} z^k \quad (k = m+1, m+2, ...)$$

$$g_k(z) = z^m - \frac{[m]_{p,q}(1-\alpha)}{[k]_{p,q}\psi_k(\delta)} \overline{z}^k \quad (k = m+1, m+2, ...)$$

$$x_k, y_k \geq 0, k = m, m+1, ..., \quad x_m = 1 - \sum_{k=m+1}^{\infty} (x_k + y_k). \tag{31}$$

In particular, the points $\{h_k\}$ *and* $\{g_k\}$ *are called the extreme points of the closed convex hull of the class* $\mathcal{T}S^0_{\mathcal{H}}(m,\delta,p,q,\alpha)$ *denoted by* $clco\mathcal{T}S^0_{\mathcal{H}}(m,\delta,p,q,\alpha)$.

Proof. Let f be given by (30). Using (31), we can write

$$f(z) = z^m - \sum_{k=m+1}^{\infty} \frac{[m]_{p,q}(1-\alpha)}{[k]_{p,q}\psi_k(\delta)} \left(x_k z^k + y_k \overline{z}^k \right)$$

which by Lemma 3.1 proves that $f \in \mathcal{T}S^0_{\mathcal{H}}(m,\delta,p,q,\alpha)$, since for this function

$$\begin{aligned} &\sum_{k=m+1}^{\infty} \frac{[k]_{p,q}\psi_k(\delta)}{[m]_{p,q}(1-\alpha)} \frac{[m]_{p,q}(1-\alpha)}{[k]_{p,q}\psi_k(\delta)} (x_k + y_k) \\ &= \sum_{k=m+1}^{\infty} (x_k + y_k) = 1 - x_m \leq 1. \end{aligned}$$

Thus, $f \in clco\mathcal{T}S^0_{\mathcal{H}}(m,\delta,p,q,\alpha)$. Conversely, let $f = h + \overline{g} \in \mathcal{T}S^0_{\mathcal{H}}(m,\delta,p,q,\alpha)$, where h and g are of the form (26). Set

$$x_k = \frac{[k]_{p,q}\psi_k(\delta)}{[m]_{p,q}(1-\alpha)} |a_k|, \quad y_k = \frac{[k]_{p,q}\psi_k(\delta)}{[m]_{p,q}(1-\alpha)} |b_k|.$$

Then on using (31), we obtain

$$\begin{aligned} f(z) &= z^m - \sum_{k=m+1}^{\infty} |a_k| z^k - \sum_{k=m+1}^{\infty} |b_k| \overline{z}^k \\ &= z^m - \sum_{k=m+1}^{\infty} x_k \frac{[m]_{p,q}(1-\alpha)}{[k]_{p,q}\psi_k(\delta)} z^k - \sum_{k=m+1}^{\infty} y_k \frac{[m]_{p,q}(1-\alpha)}{[k]_{p,q}\psi_k(\delta)} \overline{z}^k \\ &= z^m - \sum_{k=m+1}^{\infty} x_k \{z^m - h_k(z)\} - \sum_{k=m+1}^{\infty} y_k \{z^m - g_k(z)\} \\ &= \left[1 - \sum_{k=m+1}^{\infty} (x_k + y_k) \right] z^m + \sum_{k=m+1}^{\infty} \{x_k h_k(z) + y_k g_k(z)\} \end{aligned}$$

which is of the form (30). This proves Theorem 3.2. $\square$

Theorem 3.3. *For* $0 \leq \beta < \alpha < 1$, *let* $f \in \mathcal{T}S^0_{\mathcal{H}}(m,\delta,p,q,\alpha)$ *and* $F \in \mathcal{T}S^0_{\mathcal{H}}(m,\delta,p,q,\alpha)$. *Then* $f * F \in S^0_{\mathcal{H}}(m,\delta,p,q,\alpha) \subset S^0_{\mathcal{H}}(m,\delta,p,q,\beta)$.

Proof. Let

$$f(z) = z^m - \sum_{k=m+1}^{\infty} |a_k| z^k - \sum_{k=m+1}^{\infty} |b_k| \overline{z}^k$$

and

$$F(z) = z^m - \sum_{k=m+1}^{\infty} |A_k| z^k - \sum_{k=m+1}^{\infty} |B_k| \overline{z}^k.$$

Then

$$(f * F) = z^m + \sum_{k=m+1}^{\infty} |a_k||A_k| z^n + \sum_{k=m+1}^{\infty} |b_k||B_k| \overline{z}^k$$

Since $F \in \mathcal{T}S^0_{\mathcal{H}}(m, \delta, p, q, \alpha)$, by Lemma 3.1, we have $|A_k| \leq 1$ and $|B_k| \leq 1$. Therefore,

$$\sum_{k=m+1}^{\infty} \left(\frac{[k]_{p,q}\psi_k(\delta)}{[m]_{p,q}(1-\alpha)} |a_k||A_k| + \frac{[k]_{p,q}\psi_k(\delta)}{[m]_{p,q}(1-\alpha)} |b_k||B_k| \right)$$
$$\leq \sum_{k=m+1}^{\infty} \left(\frac{[k]_{p,q}\psi_k(\delta)}{[m]_{p,q}(1-\alpha)} |a_k| + \frac{[k]_{p,q}\psi_k(\delta)}{[m]_{p,q}(1-\alpha)} |b_k| \right) \leq 1$$

By Lemma 3.1, it follows that $f * F \in \mathcal{T}S^0_{\mathcal{H}}(m, \delta, p, q, \alpha)$. Further, it is obvious that $\mathcal{T}S^0_{\mathcal{H}}(m, \delta, p, q, \alpha) \subset \mathcal{T}S^0_{\mathcal{H}}(m, \delta, p, q, \beta)$. □

Theorem 3.4. *The class $\mathcal{T}S^0_{\mathcal{H}}(m, \delta, p, q, \alpha)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3 \ldots$, let $f_i \in \mathcal{T}S^0_{\mathcal{H}}(m, \delta, p, q, \alpha)$ be given by

$$f_i(z) = z^m - \sum_{k=m+1}^{\infty} |a_{k,i}| z^k - \sum_{k=m+1}^{\infty} |b_{k,i}| \overline{z}^k.$$

Then by (8), we have

$$\sum_{k=m+1}^{\infty} \frac{[k]_{p,q}}{[m]_{p,q}} \psi_k(\delta) \{|a_{k,i}| + |b_{k,i}|\} \leq 1 - \alpha. \tag{32}$$

For $\sum_{i=1}^{\infty} t_i = 1, \quad 0 \leq t_i \leq 1$, the convex combination of f_i is

$$\sum_{i=1}^{\infty} t_i f_i(z) = z^m - \sum_{k=m+1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{k,i}| \right) z^k - \sum_{k=m+1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{k,i}| \right) \overline{z}^k.$$

Then on using Lemma 3.1, we see that

$$\sum_{k=m+1}^{\infty} \frac{[k]_{p,q}}{[m]_{p,q}} \psi_k(\delta) \left(\left| \sum_{i=1}^{\infty} t_i |a_{k,i}| \right| + \left| \sum_{i=1}^{\infty} t_i |b_{k,i}| \right| \right)$$
$$\leq \sum_{i=1}^{\infty} t_i \left\{ \sum_{k=m+1}^{\infty} \frac{[k]_{p,q}}{[m]_{p,q}} \psi_k(\delta) (|a_{k,i}| + |b_{k,i}|) \right\}$$
$$\leq 1 - \alpha,$$

and so $\sum_{i=1}^{\infty} t_i f_i(z) \in \mathcal{T}S^0_{\mathcal{H}}(m, \delta, p, q, \alpha)$. □

4. Conclusion

Recently, in Geometric Function Theory, q-calculus and (p, q)-calculus are being applied not only in defining several linear operators but also in defining various analogue of previously defined well known classes of analytic as well as harmonic functions. In this study, a (p, q)-analogue of Ruscheweyh operator $L^{\delta+m-1}_{p,q}$ for multivalent functions is being defind by (3). In view of the shearing technique, a class $S^0_{\mathcal{H}}(m, \delta, p, q, \alpha)$ of functions $f = h + \overline{g} \in S^0_{\mathcal{H}}(m)$ is defined in Definition 1.1 by involving the operator $L^{\delta+m-1}_{p,q}$. A sufficient coefficient condition for $f \in S^0_{\mathcal{H}}(m,\delta,p,q,\alpha)$ is obtained as Lemma 2.1. With the use of this coefficient condition, sharp bounds of the real parts of ratios of functions $f \in S^0_{\mathcal{H}}(m)$ to its partial sums $S_s(f), S_l(f)$ and $S_{s,l}(f)$ are obtained in Section 2. Further, in Section 3, it is proved that the coefficient condition (8) is necessary and sufficient for the functions in a subclass $\mathcal{T}S^0_{\mathcal{H}}(m,\delta,p,q,\alpha)$. Again in Section 3, for functions in this subclass, results based on bounds, convolution, extreme points and on convexity are derived.

There is a possibility of extension of the results obtained in this paper to some generalized classes associated with the Janowski type of functions as well as the Rønning class of functions. For this, we may define classes $S^0_{\mathcal{H}}(m,\delta,p,q,A,B)$ and k-$S^0_{\mathcal{H}}(m,\delta,p,q,\alpha)$ of functions $f = h + \overline{g} \in S^0_{\mathcal{H}}(m)$ satisfying, respectively, the conditions

$$\frac{\partial_{p,q} L^{\delta+m-1}_{p,q}\left(h(z)+g(z)\right)}{[m]_{p,q} z^{m-1}} \prec \frac{1+Az}{1+Bz} \;\left(-1 \le B < A \le 1\right),$$

where the notion $\prec$ denotes the familiar subordination and

$$\Re e \left\{ \frac{\partial_{p,q} L^{\delta+m-1}_{p,q}\left(h(z)+g(z)\right)}{[m]_{p,q} z^{m-1}} \right\} > k \left| \frac{\partial_{p,q} L^{\delta+m-1}_{p,q}\left(h(z)+g(z)\right)}{[m]_{p,q} z^{m-1}} - 1 \right| + \alpha, \;\; 0 \le \alpha < 1.$$

Obviously, if $A = 1-2\alpha, B = -1$, then $S^0_{\mathcal{H}}(m,\delta,p,q,A,B) = S^0_{\mathcal{H}}(m,\delta,p,q,\alpha)$ and if $k = 0$, then k-$S^0_{\mathcal{H}}(m,\delta,p,q,\alpha) = S^0_{\mathcal{H}}(m,\delta,p,q,\alpha)$.

References

[1] Aldweby H. and Darus M., (2014), Some subordination results on q-analogue of Ruscheweyh differential operator, Abstr. Appl. Anal., Art. ID 958563, 6 pp.

[2] Aral A., Gupta V. and Agarwal R. P., (2013), Applications of q-calculus in operator theory, Springer, New York.

[3] Arif M., Ul Haq M. and Liu J.-L., (2018), A subfamily of univalent functions associated with q-analogue of Noor integral operator, J. Funct. Spaces, Art. ID 3818915, 5 pp.

[4] Arif M., Srivastava H. M. and Umar S., (2019), Some applications of a q-analogue of the Ruscheweyh type operator for multivalent functions, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 113, no. 2, 1211–1221.

[5] Clunie J. and Sheil-Small T., (1984), Harmonic univalent functions, Ann. Acad. Sci. Fenn. Ser. A I Math., 9, 3–25.

[6] Duren P., Hengartner W. and Laugesen R. S, (1996), The argument principle for harmonic functions, Amer. Math. Monthly, 103, no. 5, 411–415.

[7] Gasper G. and Rahman M., (2004), Basic hypergeometric series, Cambridge University Press.

[8] Goel R. M. and Sohi N. S., (1980), A new criterion for p-valent functions, Proc. Amer. Math. Soc., 78, no. 3, 353–357.

[9] Goel R. M. and Sohi N. S., (1980), New criteria for p-valence, Indian J. Pure Appl. Math., 11, no. 10, 1356–1360.

[10] Ismail M. E. H., Merkes E. and Styer D., (1990), A generalization of starlike functions, Complex Variables Theory Appl., 14, no. 1-4, 77–84.

[11] Jackson F., (1910), On q-Definite Integrals, Quart. J. Pure Appl. Math., 41, 193–203.

[12] Jagannathan R. and Srivanasava K., (2006), Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series, Mathematics, Physics; arXiv:math/0602613v1 [math.NT].

[13] Karpuzoğullari S.Y., Öztürk M. and Yamankaradeniz M., (2003), A subclass of harmonic univalent functions with negative coefficients, Appl. Math. Comput., 142, no. 2-3, 469–476.

[14] Kanas S. and Răducanu D., (2014), Some class of analytic functions related to conic domains, Math. Slovaca, 64, no. 5, 1183–1196.

[15] Khan B., Srivastava H. M., Khan N., Darus M., Ahmad Q. Z. and Tahir M., (2021), Applications of Certain Conic Domains to a Subclass of q-Starlike Functions Associated with the Janowski Functions, Symmetry, 13, 574 doi.org/10.3390/sym130405740.

[16] Khan N., Shafiq M., Darus M., Khan B. and Ahmad Q. Z., (2020), Upper Bound of the Third Hankel Determinant for a Subclass of q-Starlike Functions Associated with Lemniscate of Bernoulli, Journal of Mathematical Inequalities (JMI), 14(1), 51-63.
[17] Li S. and Liu P., (2007), A new class of harmonic univalent functions by the generalized Salagean operator, Wuhan Univ. J. Nat. Sci., 12, no. 6, 965–970.
[18] Mahmood S. and Sokół J., (2017), New subclass of analytic functions in conical domain associated with Ruscheweyh q-differential operator, Results Math., 71, no. 3-4, 1345–1357.
[19] Sharma P. and Mishra O., (2020), A class of harmonic functions associated with a q-Sălăgean operator, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 82, no. 3, 3–12.
[20] Sharma P., Porwal S.and Mishra O., (2020), An application of (p, q)-calculus in harmonic univalent functions, TWMS J. App. and Eng. Math., 10, no. 3, 751–760
[21] Ruscheweyh S., (1975), New criteria for univalent functions, Proc. Amer. Math. Soc., 49, 109–115.
[22] Srivastava H. M., Muhammad T., Khan B., Ahmad Q. and Khan N., (2019), Some general families of q-starlike functions associated with the Janowski functions, Filomat, 33, no. 9, 2613–2626.
[23] Srivastava H. M., Khan B., Khan N., Tahir M., Ahmad S. and Khan Nasir, (2021), Upper bound of the third Hankel determinant for a subclass of q-starlike functions associated with the q-exponential function, Bull. Sci. Math., 167, 102942, doi.org/10.1016/j.bulsci.2020.102942.
[24] Srivastava H. M., Khan S., Ahmad Q. Z., Khan N. and Hussain S., (2018), The Faber polynomial expansion method and its application to the general coefficient problem for some subclasses of bi-univalent functions associated with a certain q-integral operator, Stud. Univ. Babes-Bolyai Math., 63(4), 419–436.

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