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TOPOLOGICAL INDICES OF TOTAL GRAPH OF THE RING $\mathbb{Z}_n \times \mathbb{Z}_m$

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ABSTRACT. In this paper, we compute some distance-based topological indices namely Wiener index, hyper Wiener index and reverse Wiener index of the total graph of the ring $\mathbb{Z}_n \times \mathbb{Z}_m$. We also compute some eccentricity-based topological indices namely first Zagreb eccentricity index, second Zagreb eccentricity index, eccentric connectivity index, connective eccentricity index and eccentric distance sum index of the total graph of the ring $\mathbb{Z}_n \times \mathbb{Z}_m$. Finally, we compute some degree-based topological indices namely second Zagreb index, product-connectivity index, sum connectivity index, atom-bond connectivity index and geometric arithmetic index of the total graph of the ring $\mathbb{Z}_n \times \mathbb{Z}_m$ when both n and m are even, when n or m is even, and the case when n and m are both prime numbers (not neccesarily distinct).

Keywords: Total graph; topological indices.

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1. Introduction

Chemical graph theory is an active area of research in recent times. In this field, researchers study the structures of chemical compounds with the aid of graph theory and mathematics. The most important concept in chemical graph theory is topological indices. Topological indices are numeric values that are associated with graphs and which remain invariant under graph automorphisms. They have a wide range of applications in the field of chemistry, biology and medicines. In particular, they help in the study of certain physical characteristics of chemical compounds such as boiling point, flash point, density, stability and many more. They also serve as powerful tools to execute biological network analysis, and also to determine the physical features and the chemical reactions associated with various drugs without having to carry out the actual experiment, for instance see [4], [9], [16].

A graph G consists of a pair $(V(G), E(G))$, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges. If uv is an edge, then u and v are said to be adjacent in G, we write

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$u \sim v$; otherwise $u \not\sim v$. The degree $deg(v)$ of $v \in V(G)$ is defined as the number of edges incident to v. The distance between two vertices $u, v \in V(G)$, denoted by $d(u, v)$, is the number of edges on the shortest path between them in G. The greatest distance between any two pair of vertices of a connected graph G is called the diameter of G and it is denoted by $diam(G)$. The eccentricity of $u \in V(G)$, denoted by $e(u)$, is the distance from u to a vertex v which is farthest from u in G.

Throughout this paper, R denotes a finite commutative ring with unity. Recently, Anderson and Badawi [2] introduced the total graph of R, denoted by $T_\Gamma(R)$. It is the graph with all elements of R as vertices and for distinct $u, v \in R$, the vertices u and v are adjacent if and only if $u + v \in Z(R)$, where $Z(R)$ is the set of zero-divisors. Dhorajia [7] studied certain fundamental properties of the total graph of the ring $\mathbb{Z}_n \times \mathbb{Z}_m$, such as independent number, clique number and traversibility property of $T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$. Nikmehr et al. [12] computed the topological indices of the total graph of $\mathbb{Z}_n$ and Suthar et al. [14] computed the energy and the Wiener index of the total graph of the ring $\mathbb{Z}_n$. The study of topological indices of the graph associated with algebraic structures is an interesting area of research works now a days. Many research works related to this field have been appearing recently. For instance see [1], [11], [12], [14]. Motivated by the large number of applications of topological indices, in this paper we devote ourselves to investigating and determining the topological indices of the total graph of the ring $\mathbb{Z}_n \times \mathbb{Z}_m$.

Throughout this paper, n, m, and k are positive integers, p and q are distinct primes and $\phi(m)$ is the Euler phi function. We denote the set of zero-divisors and the set of units of the ring $\mathbb{Z}_n \times \mathbb{Z}_m$ by $Z(\mathbb{Z}_n \times \mathbb{Z}_m)$ and $U(\mathbb{Z}_n \times \mathbb{Z}_m)$ respectively.

2. Preliminaries and Observations

We now recall the definitions of various topological indices which will be investigated in the next section. We start with distance-based topological indices:

(1) The *Wiener index* of G is given by $W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u, v)$.

(2) The *hyper-Wiener index* of G is given by $WW(G) = \frac{1}{2}W(G) + \frac{1}{2}\sum_{\{u,v\} \subseteq V(G)} d^2(u, v)$.

(3) The *reverse Wiener index* of a graph G with n vertices and diameter $diam(G)$ is given by $\Lambda(G) = \frac{1}{2}n(n-1)diam(G) - W(G)$.

The definitions of some eccentricity-based topological indices are:

(1) The *first Zagreb eccentricity index* is define as $E_1(G) = \sum_{u \in V(G)} (e(v))^2$.

(2) The *second Zagreb eccentricity index* is define as $E_2(G) = \sum_{(u,v) \in E(G)} e(u)e(v)$.

(3) The *eccentric connectivity index* is define as $\xi^c(G) = \sum_{u \in V(G)} e(u)deg(u)$.

(4) The *connective eccentricity index* is define as $C^\xi(G) = \sum_{u \in V(G)} \frac{deg(u)}{e(u)}$.

(5) The *eccentric distance sum index* is define as $\xi^d(G) = \sum_{u \in V(G)} e(u)D(u)$, where $D(u) = \sum_{v \in E(G)} d(u, v)$ is the total distance of the vertex u.

The degree-based topological indices are given by the following function:

$$T_d(G) = \sum \psi(deg(u), deg(v)),$$

where $\psi(deg(u), deg(v))$ is a real valued function. Based on the values of ψ, we have different degree based topological indices:

(1) If $\psi(deg(u), deg(v)) = deg(u)deg(v)$, then $T_d(G)$ is called the second Zagreb index, denoted by $M_2(G)$.

(2) If $\psi(deg(u), deg(v)) = \dfrac{1}{\sqrt{deg(u)deg(v)}}$, then $T_d(G)$ is called the product connectivity index, denoted by $\chi(G)$.

(3) If $\psi(deg(u), deg(v)) = \dfrac{1}{\sqrt{deg(u)+deg(v)}}$, then $T_d(G)$ is called the sum connectivity index, denoted by $SCI(G)$.

(4) If $\psi(deg(u), deg(v)) = \sqrt{\dfrac{deg(u)+deg(v)-2}{deg(u)deg(v)}}$, then $T_d(G)$ is called the atom-bond connectivity index, denoted by $ABC(G)$.

(5) If $\psi(deg(u), deg(v)) = \dfrac{2\sqrt{deg(u)deg(v)}}{deg(u)+deg(v)}$, then $T_d(G)$ is called the geometric arithmetic index, denoted by $GA(G)$.

Now in the following, we will discuss several graph theoretic properties of $T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ which will assist us in the next section.

Lemma 2.1. *Let $n, m > 1$ be any two integers. If $(a, b) \neq (0, 0) \in \mathbb{Z}_n \times \mathbb{Z}_m$, then $(a, b) \in Z(\mathbb{Z}_n \times \mathbb{Z}_m)$ if and only if $a \in Z(\mathbb{Z}_n)$ or $b \in Z(\mathbb{Z}_m)$, and $(a, b) \in U(\mathbb{Z}_n \times \mathbb{Z}_m)$ if and only if $a \in U(\mathbb{Z}_n)$ and $b \in U(\mathbb{Z}_m)$.*

Lemma 2.2. *Let $n, m > 1$ be any two integers. Then the number of zero-divisors in $\mathbb{Z}_n \times \mathbb{Z}_m$ is $nm - \phi(n)\phi(m)$ and the number of units is $\phi(n)\phi(m)$.*

Remark 2.1. *Let $n, m > 1$ be any two integers. Then by Anderson and Badawi* [2] *Theorem 3.4 and Corollary 3.7, $T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ is a connected graph for all $n, m \geq 2$.*

We recall the following Lemma to compute the number of edges in the graph $T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$.

Lemma 2.3 ([7], Lemma 2.12)**.** *Let $n, m > 1$, be any two integers, then the following conditions hold for $v \in V(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$:*

(1) *If either n or m is even, then $\deg(v) = nm - \phi(n)\phi(m) - 1$. In this case, $\Delta = \delta = nm - \phi(n)\phi(m) - 1$.*

(2) *If n and m are both odd, then*

$$\deg(v) = \begin{cases} nm - \phi(n)\phi(m) - 1 \text{ } if \text{ } v \in Z(\mathbb{Z}_n \times \mathbb{Z}_m) \\ nm - \phi(n)\phi(m) \text{ } if \text{ } v \notin Z(\mathbb{Z}_n \times \mathbb{Z}_m). \end{cases}$$

In this case, $\Delta = nm - \phi(n)\phi(m)$ and $\delta = nm - \phi(n)\phi(m) - 1$.

Lemma 2.4. *For integers $n, m > 1$, the total number of edges in $T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ is given as follows:*

$$|E(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))| = \begin{cases} \dfrac{nm[nm - \phi(n)\phi(m) - 1]}{2} \text{ } if \text{ either } n \text{ or } m \text{ is even} \\ \dfrac{(nm - 1)[nm - \phi(n)\phi(m)]}{2} \text{ } if \text{ both } n \text{ and } m \text{ are odd.} \end{cases}$$

Proof. The following two cases complete the proof of the Theorem:

Case 1. If either n or m is even, then by Lemma 2.3 $deg(v) = nm - \phi(n)\phi(m) - 1$ and

by handshaking Lemma we have:
$|E(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))| = \sum_{|V(T_\Gamma(\mathbb{Z}_n\times\mathbb{Z}_m))|} \frac{nm - \phi(n)\phi(m) - 1}{2} = \frac{nm[nm - \phi(n)\phi(m) - 1]}{2}$.

Case 2. If n and m are both odd, then by Lemma 2.3 $deg(v) = nm - \phi(n)\phi(m) - 1$ if $v \in Z(\mathbb{Z}_n \times \mathbb{Z}_m)$ and $deg(v) = nm - \phi(n)\phi(m)$ if $v \notin Z(\mathbb{Z}_n \times \mathbb{Z}_m)$. Therefore, by handshaking Lemma and Lemma 2.2 we have:

$$\begin{aligned}|E(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))| &= \frac{1}{2}[\sum_{v\in Z(\mathbb{Z}_n\times\mathbb{Z}_m)} deg(v) + \sum_{v\notin Z(\mathbb{Z}_n\times\mathbb{Z}_m)} deg(v)] \\ &= \frac{(nm\ -\ 1)\ [nm\ - \phi(n)\phi(m)]}{2}.\end{aligned}$$

□

3. Topological indices of $T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$

In this section, we determine various topological indices of $T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$. We start with the Wiener index of $T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$.

Theorem 3.1. *The Wiener index of $T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ is as follows:*

$$W(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = \begin{cases} \frac{nm[nm + \phi(n)\phi(m) - 1]}{2} & if\ either\ n\ or\ m\ is\ even \\ \frac{(nm\ -\ 1)[nm\ +\ \phi(n)\phi(m)]}{2} & if\ both\ n\ and\ m\ are\ odd. \end{cases}$$

Proof. The graph $T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ has nm vertices. Let us divide the set of vertices into two partition sets according to the distances between them.
$V_1 = \{u \in V(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))|\ d(u, v) = 1\ \forall\ v \in V(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))\}$
$V_2 = \{u \in V(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))|\ d(u, v) = 2\ \forall\ v \in V(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))\}$
Then, $|V_1| = |E(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))|$ and $|V_2| = \frac{nm(nm-1)}{2} - |E(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))|$. Hence, by Lemma 2.4 we see that,

$$|V_1| = \begin{cases} \frac{nm[nm - \phi(n)\phi(m) - 1]}{2} & if\ either\ n\ or\ m\ is\ even \\ \frac{(nm-1)[nm - \phi(n)\phi(m)]}{2} & if\ both\ n\ and\ m\ are\ odd. \end{cases}$$

and

$$|V_2| = \begin{cases} \frac{nm\phi(n)\phi(m)}{2} & if\ either\ n\ or\ m\ is\ even \\ \frac{(nm-1)\phi(n)\phi(m)}{2} & if\ both\ n\ and\ m\ are\ odd. \end{cases}$$

Therefore,
$W(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = \sum_{\{u,v\}\subseteq V(T_\Gamma(\mathbb{Z}_n\times\mathbb{Z}_m))} d(u, v) = \sum_{\{u,v\}\subseteq V_1} d(u, v) + \sum_{\{u,v\}\subseteq V_2} d(u, v) = |V_1| + 2|V_2|$.
Hence,

$$W(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = \begin{cases} \frac{nm[nm + \phi(n)\phi(m) - 1]}{2} & if\ either\ n\ or\ m\ is\ even \\ \frac{(nm\ -\ 1)[nm\ +\ \phi(n)\phi(m)]}{2} & if\ both\ n\ and\ m\ are\ odd. \end{cases}$$

□

Theorem 3.2. *The hyper-Wiener index of $T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ is as follows:*

$$WW(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = \begin{cases} \dfrac{nm[nm + 2\phi(n)\phi(m) - 1]}{2} & if\ either\ n\ or\ m\ is\ even \\ \dfrac{(nm - 1)[nm + 2\phi(n)\phi(m)]}{2} & if\ both\ n\ and\ m\ are\ odd. \end{cases}$$

Proof. The proof follows directly from the definition of hyper-Wiener index and Theorem 3.1. □

Theorem 3.3. *The reverse Wiener-index of $T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ is as follows:*

$$\Lambda(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = \begin{cases} \dfrac{nm[nm - \phi(n)\phi(m) - 1]}{2} & if\ either\ n\ or\ m\ is\ even \\ \dfrac{(nm - 1)[nm - \phi(n)\phi(m)]}{2} & if\ both\ n\ and\ m\ are\ odd. \end{cases}$$

Proof. The proof follows directly from the definition of reverse Wiener index, Theorem 3.1 and Remark 3.2. □

Based on the above Theorems, we give examples of Wiener index, hyper Wiener index and reverse Wiener index of $T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ in the following table:

$T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$	$W(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$	$WW(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$	$\Lambda(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$
$T_\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_4)$	90	114	42
$T_\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)$	52	68	20

Next, we compute some of the eccentricity-topological indices of $T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$.

Remark 3.1. *If $v \in V(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$, then the eccentricity of v is given by $e(v) = 2$.*

Theorem 3.4. *The first Zagreb eccentricity index of $T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ is:*

$$E_1(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = 4nm.$$

Proof. If $u \in V(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$, then by Remark 3.1 $e(u) = 2$. Now, by the definition of the first Zagreb eccentricity index:
$E_1(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = \sum_{u \in V(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))} e(u)^2 = \sum_{|V(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))|} 2^2 = 4nm.$ □

Theorem 3.5. *The second Zagreb eccentricity index of $T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ is:*

$$E_2(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = \begin{cases} 2nm[nm - \phi(n)\phi(m) - 1] & if\ either\ n\ or\ m\ is\ even \\ 2(nm - 1)(nm - \phi(n)\phi(m)) & if\ both\ n\ and\ m\ are\ odd. \end{cases}$$

Proof. If $u \in V(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$, then by Remark 3.1 $e(u) = 2$. Now, by the definition of the second Zagreb eccentricity index:

$$E_2(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = \sum_{uv \in E(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))} e(u)e(v) = 4|E(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))|.$$

Now, two cases arise:
Case 1. If either n or m is even, then by Lemma 2.4
$|E(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))| = \dfrac{nm[nm - \phi(n)\phi(m) - 1]}{2}$.
Thus, $E_2(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = 2nm[nm - \phi(n)\phi(m) - 1]$.

Case 2. If n and m are both odd, then by Lemma 2.4

$$|E(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))| = \frac{(nm-1)[nm-\phi(n)\phi(m)]}{2}.$$

Thus, $E_2(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = 2(nm-1)[nm-\phi(n)\phi(m)]$. □

Theorem 3.6. *The eccentric connectivity of $T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ is:*

$$\xi^c(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = \begin{cases} 2nm[nm-\phi(n)\phi(m)-1] & if\ either\ n\ or\ m\ is\ even \\ 2(nm-1)[nm-\phi(n)\phi(m)] & if\ both\ n\ and\ m\ are\ odd. \end{cases}$$

Proof. If $u \in V(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$, then by Remark 3.1 $e(u) = 2$. Now, by the definition of the eccentric connectivity:
$\xi^c(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = \sum_{u\in V(T_\Gamma(\mathbb{Z}_n\times\mathbb{Z}_m))} e(u)deg(u) = \sum_{u\in V(T_\Gamma(\mathbb{Z}_n\times\mathbb{Z}_m))} 2deg(u)$.
Therefore, two cases arise:
Case 1. If either n or m is even, then by Lemma 2.3 we have:
$\xi^c(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = 2nm[nm-\phi(n)\phi(m)-1]$.
Case 2. If n and m are odd, then by Lemma 2.3 we have:

$$\begin{aligned}\xi^c(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) &= \sum_{|Z(\mathbb{Z}_n\times\mathbb{Z}_m)|} 2(nm-\phi(n)\phi(m)-1) + \sum_{|U(\mathbb{Z}_n\times\mathbb{Z}_m)|} 2[nm-\phi(n)\phi(m)] \\ &= 2(nm-\phi(n)\phi(m)-1)(nm-\phi(n)\phi(m)) + 2[nm-\phi(n)\phi(m)]\phi(n)\phi(m) \\ &= 2(nm-1)(nm-\phi(n)\phi(m)).\end{aligned}$$

□

Theorem 3.7. *The connective eccentricity index of $T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ is:*

$$C^\xi(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = \begin{cases} \dfrac{nm[nm-\phi(n)\phi(m)-1]}{2} & if\ either\ n\ or\ m\ is\ even \\ \dfrac{(nm-1)[nm-\phi(n)\phi(m)]}{2} & if\ both\ n\ and\ m\ are\ odd. \end{cases}$$

Proof. If $u \in V(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$, then by Remark 3.1 $e(u) = 2$. Now, by the definition of the connective eccentricity index:
$C^\xi(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = \sum_{u\in V(T_\Gamma(\mathbb{Z}_n\times\mathbb{Z}_m))} \frac{deg(u)}{e(u)} = \sum_{u\in V(T_\Gamma(\mathbb{Z}_n\times\mathbb{Z}_m))} \frac{deg(u)}{2}$.
Therefore, two cases arise:
Case 1. If either n or m is even, then by Lemma 2.3 we have:
$C^\xi(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = \sum_{|V(T_\Gamma(\mathbb{Z}_n\times\mathbb{Z}_m))|} \frac{(nm-\phi(n)\phi(m)-1)}{2} = \frac{nm[nm-\phi(n)\phi(m)-1]}{2}$.

Case 2. If both n and m are odd, then by Lemma 2.3, we have:

$$\begin{aligned}\xi^c(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) &= \sum_{|Z(\mathbb{Z}_n\times\mathbb{Z}_m)|} \frac{(nm-\phi(n)\phi(m)-1)}{2} + \sum_{|U(\mathbb{Z}_n\times\mathbb{Z}_m)|} \frac{(nm-\phi(n)\phi(m))}{2} \\ &= \frac{(nm-\phi(n)\phi(m)-1)(nm-\phi(n)\phi(m))}{2} + \frac{[nm-\phi(n)\phi(m)]\phi(n)\phi(m)}{2} \\ &= \frac{(nm-1)[nm-\phi(n)\phi(m)]}{2}.\end{aligned}$$

□

In order to compute the eccentric distance sum index we determine the total distance of a vertex u of $T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ in the following Lemma.

Lemma 3.1. *The total distance of $u \in V(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$ is as follows:*

(1) *If either n or m is even, then* $D(u) = nm + \phi(n)\phi(m) - 1$.
(2) *If both n and m are odd, then*

$$D(u) = \begin{cases} nm + \phi(n)\phi(m) - 1 \; if \; u \in Z(\mathbb{Z}_n \times \mathbb{Z}_m) \\ nm + \phi(n)\phi(m) - 2 \; if \; u \in U(\mathbb{Z}_n \times \mathbb{Z}_m). \end{cases}$$

Proof. By the definition of total distance of a vertex, we have $D(u) = \sum_{v \in V(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))} d(u,v)$. Now, $d(u,v) = 1$ if u is adjacent to v, and $d(u,v) = 2$ if u is not adjacent to v. The number of vertices which are adjacent to u is equal to the degree of u, and the number of vertices which are not adjacent to u is equal to $nm - deg(u) - 1$. Since $T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ is a simple graph, so two cases arise:

Case 1. If either n or m is even, then $deg(u) = nm - \phi(n)\phi(m) - 1, \forall\, u \in V(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$. Therefore, total number of $d(u,v)$ with length 1 is $nm - \phi(n)\phi(m) - 1$ and total number of $d(u,v)$ of length 2 is $\phi(n)\phi(m)$. Hence, $D(u) = nm - \phi(n)\phi(m) - 1 + 2\phi(n)\phi(m) = nm + \phi(n)\phi(m) - 1$.

Case 2. If n and m are both odd, then we have two subcases:

Subcase 1. If $u \in Z(\mathbb{Z}_n \times \mathbb{Z}_m)$, then $deg(u) = nm - \phi(n)\phi(m) - 1$. Therefore, total number of $d(u,v)$ with length 1 is $nm - \phi(n)\phi(m) - 1$ and total number of $d(u,v)$ of length 2 is $\phi(n)\phi(m)$. Hence, $D(u) = nm - \phi(n)\phi(m) - 1 + 2\phi(n)\phi(m) = nm + \phi(n)\phi(m) - 1$.

Subcase 2. If $u \in U(\mathbb{Z}_n \times \mathbb{Z}_m)$, then $deg(u) = nm - \phi(n)\phi(m)$. Therefore, total number of $d(u,v)$ with length 1 is $nm - \phi(n)\phi(m)$ and total number of $d(u,v)$ of length 2 is $\phi(n)\phi(m) - 1$. Hence, $D(u) = nm - \phi(n)\phi(m) + 2[\phi(n)\phi(m) - 1] = nm + \phi(n)\phi(m) - 2$. □

Theorem 3.8. *The eccentric distance sum index of* $T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ *is:*

$$\xi^d(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = \begin{cases} 2nm[nm + \phi(n)\phi(m) - 1] \; if \; either \; n \; or \; m \; is \; even \\ 2[n^2m^2 + 2\phi(n)^2\phi(m)^2 + 3nm\phi(n)\phi(m) - nm - 3\phi(n)\phi(m)] \\ \; if \; both \; n \; and \; m \; are \; both \; odd. \end{cases}$$

Proof. By the definition of eccentric distance sum index and by Lemma 3.1 we have: $\xi^d(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = \sum_{u \in V(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))} 2D(u)$. Then, two cases arise:

Case 1. If either n or m is even, then by Lemma 3.1 we have:
$\xi^d(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = \sum_{u \in V(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))} 2(nm + \phi(n)\phi(m) - 1) = 2nm(nm + \phi(n)\phi(m) - 1)$.

Case 2. If n and m are both odd, then by Lemma 3.1 we have:

$$\begin{aligned} \xi^d(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) &= \sum_{|Z(\mathbb{Z}_n \times \mathbb{Z}_m)|} 2(nm + \phi(n)\phi(m) - 1) + \sum_{|U(\mathbb{Z}_n \times \mathbb{Z}_m)|} 2(nm + \phi(n)\phi(m) - 2) \\ &= 2[n^2m^2 + 2\phi(n)^2\phi(m)^2 + 3nm\phi(n)\phi(m) - nm - 3\phi(n)\phi(m)]. \end{aligned}$$

□

Based on the above Theorems, we give examples of the first Zagreb eccentricity index, the second Zagreb eccentricity index, the eccentric connectivity index, the connective eccentricity index and the eccentric distance sum index of $T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ in the following table:

$T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$	$E_1(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$	$E_2(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$	$\xi^c(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$	$C^\xi(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$	$\xi^d(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$
$T_\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_4)$	48	168	168	42	360
$T_\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)$	36	80	80	20	400

In the following, we compute some of the degree-based topological indices.

Remark 3.2. *Let $n,\ m\ > 1$. Then $diam(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = 2$.*

Lemma 3.2 ([12], Lemma 3.4)**.** *For any graph G, G is r-regular if and only if one of the following holds:*

(1) $M_2 = r^2|E(G)|$.

(2) $\chi(G) = \frac{1}{r}|E(G)|$.

(3) $SCI(G) = \frac{1}{\sqrt{2r}}|E(G)|$.

(4) $ABC(G) = \frac{\sqrt{2(r-1)}}{r}|E(G)|$.

(5) $GA(G) = |E(G)|$.

By using the above Lemma, we obtain the following results:

Theorem 3.9. *If n and m are both even or if either n or m is even, then:*

(1) $M_2(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = \frac{nm[nm-\phi(n)\phi(m)-1]^3}{2}$.

(2) $\chi(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = \frac{nm}{2}$.

(3) $SCI(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = \frac{nm\sqrt{nm-\phi(n)\phi(m)-1}}{2\sqrt{2}}$.

(4) $ABC(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = nm\sqrt{\frac{nm-\phi(n)\phi(m)-2}{2}}$.

(5) $GA(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = \frac{nm(nm-\phi(n)\phi(m)-1)}{2}$.

Lemma 3.3. *Let p, q denote two prime numbers (not neccessarily distinct), then:*
$T_d(T_\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)) = [\frac{p(p-1)}{2} + \frac{q(q-1)}{2}]\psi(p+q-2, p+q-2) + 2(p-1)(q-1)\psi(p+q-2, p+q-1) + \frac{p^2q+pq^2-p^2-q^2-5pq+4p+4q-3}{2}\psi(p+q-1, p+q-1)$.

Proof. The graph $T_\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ has pq vertices and $(p+q-2)(pq-1)$ edges. Since every non-zero element of $\mathbb{Z}_p \times \mathbb{Z}_q$ is either a unit or a zero-divisor, therefore we divide the edges of $T_\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ into 3 partition sets.

$$E_{zz} = \{xy \mid x \in Z(\mathbb{Z}_p \times \mathbb{Z}_q) \text{ and } y \in Z(\mathbb{Z}_p \times \mathbb{Z}_q)\}$$
$$E_{zu} = \{xy \mid x \in Z(\mathbb{Z}_p \times \mathbb{Z}_q) \text{ and } y \in U(\mathbb{Z}_p \times \mathbb{Z}_q)\}$$
$$E_{uu} = \{xy \mid x \in U(\mathbb{Z}_p \times \mathbb{Z}_q) \text{ and } y \in U(\mathbb{Z}_p \times \mathbb{Z}_q)\}$$

We note that the set of zero-divisors of $\mathbb{Z}_p \times \mathbb{Z}_q$ is $Z(\mathbb{Z}_p \times \mathbb{Z}_q) = I \cup J$, where $I = \{(0,0),(1,0),(2,0),...,(p-1,0)\}$ and $J = \{(0,0),(0,1),(0,2),...,(0,q-1)\}$. If $xy \in E_{zz}$ then, x and y are either in I or J. Again, if $x \in I$ and $y \in J$ then $x \not\sim y$. Hence, $|E_{zz}| = \frac{p(p-1)}{2} + \frac{q(q-1)}{2}$.
Next, let $xy \in E_{zu}$. This implies that $x \in Z(\mathbb{Z}_p \times \mathbb{Z}_q)$ and $y \in U(\mathbb{Z}_p \times \mathbb{Z}_q)$. Since, $deg(x) = p+q-2$, therefore number of units which are adjacent to x is $(p+q-2)-(p-1) = (q-1)$ if $x \in I$, and $(p+q-2)-(q-1) = p-1$ if $x \in J$. Hence, $|E_{zu}| = 2(p-1)(q-1)$.
Finally, $|E_{uu}| = |E(Z(\mathbb{Z}_p \times \mathbb{Z}_q))| - (|E_{zz}| + |E_{zu}|) = \frac{p^2q+pq^2-p^2-q^2-5pq+4p+4q-3}{2}$.

Hence,

$$
\begin{aligned}
T_d(\mathbb{Z}_p \times \mathbb{Z}_q) &= \sum_{xy\in E_{zz}} \psi(deg(x), deg(y)) + \sum_{xy\in E_{zu}} \psi(deg(x), deg(y)) + \sum_{xy\in E_{uu}} \psi(deg(x), deg(y)) \\
&= [\frac{p(p-1)}{2} + \frac{q(q-1)}{2}]\psi(p+q-2, p+q-2) + 2(p-1)(q-1)\psi(p+q-2, \\
&\quad p+q-1) + \frac{p^2q + pq^2 - p^2 - q^2 - 5pq + 4p + 4q - 3}{2}\psi(p+q-1, p+q-1).
\end{aligned}
$$

□

Now in the following, we determine some degree-based topological indices of $T_\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$, where p and q are two primes. Prior to that, we assign $A = [\frac{p(p-1)}{2} + \frac{q(q-1)}{2}]$, $B = 2(p-1)(q-1)$ and $C = \frac{p^2q + pq^2 - p^2 - q^2 - 5pq + 4p + 4q - 3}{2}$ as per our convenience for the calculations.

Theorem 3.10. *If p and q are two primes (not necessarily distinct), then the following results hold:*

(1) *The second Zagreb index is given by:*
$M_2(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = A(p+q-2)^2 + B(p+q-2)(p+q-1) + C(p+q-1)^2$.

(2) *The product connectivity index is given by:*
$\chi(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = \frac{A}{(p+q-2)} + \frac{B}{\sqrt{(p+q-1)(p+q-2)}} + \frac{C}{(p+q-1)}$.

(3) *The sum connectivity index is given by:*
$SCI(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = \frac{A}{\sqrt{(2p+2q-4)}} + \frac{B}{\sqrt{(2p+2q-3)}} + \frac{C}{\sqrt{2p+2q-2}}$.

(4) *The atom-bond connectivity index is given by:*
$ABC(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = A\frac{\sqrt{2(p+q)-6}}{p+q-2} + B\sqrt{\frac{2(p+q)-5}{(p+q-1)(p+q-2)}} + C\frac{\sqrt{2(p+q)-4}}{p+q-1}$.

(5) *The geometric arithmetic index is given by:*
$GA(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = A + B\frac{2\sqrt{(p+q-2)(p+q-1)}}{2p+2q-3} + C$.

Proof. (1) For computation of the second Zagreb index we have $\psi(p+q-2, p+q-2) = (p+q-2)^2$, $\psi(p+q-2, p+q-1) = (p+q-2)(p+q-1)$ and $\psi(p+q-1, p+q-1) = (p+q-1)^2$. Hence, by Lemma 3.3 we have
$M_2(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = A(p+q-2)^2 + B(p+q-2)(p+q-1) + C(p+q-1)^2$.

(2) For computation of the product connectivity index we have $\psi(p+q-2, p+q-2) = \frac{1}{(p+q-2)}$, $\psi(p+q-2, p+q-1) = \frac{1}{\sqrt{(p+q-1)(p+q-2)}}$ and $\psi(p+q-1, p+q-1) = \frac{1}{(p+q-1)}$. Hence, by Lemma 3.3 we have
$\chi(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = \frac{A}{(p+q-2)} + \frac{B}{\sqrt{(p+q-1)(p+q-2)}} + \frac{C}{(p+q-1)}$.

(3) For computation of the sum connectivity index we have $\psi(p+q-2, p+q-2) = \frac{1}{\sqrt{(2p+2q-4)}}$, $\psi(p+q-2, p+q-1) = \frac{1}{\sqrt{(2p+2q-3)}}$ and $\psi(p+q-1, p+q-1) =$

$\frac{1}{\sqrt{2p+2q-2}}$. Hence, by Lemma 3.3 we have

$SCI(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = \frac{A}{\sqrt{(2p+2q-4)}} + \frac{B}{\sqrt{(2p+2q-3)}} + \frac{C}{\sqrt{2p+2q-2}}$.

(4) For computation of the atom-bond connectivity index we have $\psi(p+q-2, p+q-2) = \frac{\sqrt{2(p+q)-6}}{p+q-2}$, $\psi(p+q-2, p+q-1) = \sqrt{\frac{2(p+q)-5}{(p+q-1)(p+q-2)}}$ and $\psi(p+q-1, p+q-1) = \frac{\sqrt{2(p+q)-4}}{p+q-1}$. Hence, by Lemma 3.3 we have

$ABC(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = A\frac{\sqrt{2(p+q)-6}}{p+q-2} + B\sqrt{\frac{2(p+q)-5}{(p+q-1)(p+q-2)}} + C\frac{\sqrt{2(p+q)-4}}{p+q-1}$.

(5) For computation of the geometric arithmetic index we have $\psi(p+q-2, p+q-2) = 1$, $\psi(p+q-2, p+q-1) = \frac{2\sqrt{(p+q-2)(p+q-1)}}{2p+2q-3}$ and $\psi(p+q-1, p+q-1) = 1$. Hence, by Lemma 3.3 we have

$GA(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) = A + B\frac{2\sqrt{(p+q-2)(p+q-1)}}{2p+2q-3} + C$.

□

Based on the above Theorems, we give the examples of the second Zagreb index, the product-connectivity index, the sum connectivity index, the atom-bond connectivity index and the geometric arithmetic index of $T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ in the following table:

$T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$	$M_2(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$	$\chi(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$	$SCI(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$	$ABC(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$	$GA(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$
$T_\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4)$	500	4	6.32	11.31	20
$T_\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_4)$	2058	6	11.22	20.78	42
$T_\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)$	406	4.48	6.68	11.80	19.95
$T_\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_5)$	2120	7.49	13.53	24.93	48.95

4. Conclusions

This study will help pharmacists and druggists while designing new chemical compounds and drugs. They can make use of the above topological indices and study the various physical structures of new chemical compounds, protein structures and etc. Moreover, there are still a large number of topological indices which can be computed for the total graph over $\mathbb{Z}_n \times \mathbb{Z}_m$ and other graph associated with rings in near future.

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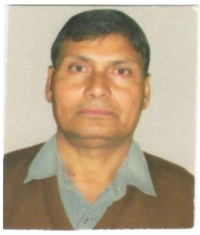

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