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# FACE ANTIMGAIC LABELING OF DOUBLE DUPLICATION FOR SOME SPECIAL GRAPHS

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Abstract. The main objective of the paper is to determine $(a, d)-$ face antimagic labeling for the double duplication of all vertices by edges of ladder, tadpole and m-copies of path graphs. Also, if a graph $G$ is $(a, d)-$ face antimagic except for 3-sided faces then the double duplication of all vertices by edges of a graph $G$ is also (a,d)-face antimagic.

Keywords: Double duplication of graphs, function, labeling, $(a, d)-$ face antimagic.

AMS Subject Classification: 05C78.

## 1. Introduction

Let $G(V, E, F)$ be a finite plane graph where $V$, $E$ and $F$ are its vertex set, edge set and set of interior faces with $|V| = p$, $|E| = q$ and $|F| = f$ . A labeling of type $(a, b, c)$ of a graph $G$ assigns labels from the set $\{1, 2, 3, ..., ap + bq + cf\}$ to the vertices, edges and faces of a graph $G$ such that each vertex gets $a-$ label, each edge gets $b-$ label and each face gets $c-$ label and each label is used exactly once. The values of $a$, $b$ and $c$ are restricted to 0 and 1. Labeling of type $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are called vertex labeling, edge labeling and face labeling respectively. The weight of a face $w(f)$ under a labeling is the sum of labels of face together with labels of vertices and edges forming that face. The notion of graph labeling was introduced by Rosa in 1967 [4]. A graph labeling is an assignment of integers to the vertices or edges or both subject to the certain conditions. If the domain of the mapping is the set of vertices (or edges) then the labeling is called a vertex labeling (or an edge labeling) and if the domain is $V(G) \cup E(G)$ then the labeling is called total labeling. The various types of labeling are investigated by Gallian [2]. Among the various types of labeling, our focus is on $(a, d)-$ face antimagic labeling. In [1], Baca defines a connected plane graph $G(V, E, F)$ with vertex set $V(G)$, edge set $E(G)$ and face set $F(G)$ to be $(a, d)-$ face antimagic, if there exists positive

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integers $a$ and $d$, a bijection $g : E(G) \to \{1, 2, \ldots, |E(G)|\}$ such that the induced mapping $\psi_g : F(G) \to \{a, a + d, \ldots, (|F(G)| - 1)d\}$, where for a face $f$, $\psi_g(f)$ is the sum of all $g(e)$ for all the edges $e$ surrounding $f$ is also a bijection.

The following definitions are used in our study.

**Definition 1.1.** [5] *The double duplication of a vertex by an edge of a graph is defined as a duplication of a vertex* $v_k$ *by an edge* $e = v'_k v''_k$ *in a graph* $G$ *produces a graph* $G'$*, in which* $N(v'_k) = \{v_k, v''_k\}$ *and* $N(v''_k) = \{v_k, v'_k\}$*. Again duplication of vertices* $v_k, v'_k$ *and* $v''_k$ *by edges* $e' = u_k w_k$*,* $e'' = u'_k w'_k$ *and* $e''' = u''_k w''_k$ *respectively in* $G'$ *produces a new graph* $G''$ *such that,* $N(u_k) = \{w_k, v_k\}$*,* $N(w_k) = \{u_k, v_k\}$*,* $N(u'_k) = \{w'_k, v'_k\}$*,* $N(w'_k) = \{u'_k, v'_k\}$*,* $N(u''_k) = \{w''_k, v''_k\}$ *and* $N(w''_k) = \{u''_k, v''_k\}$*. Double duplication of vertices by edges respectively of a graph* $G$ *is denoted by* $DD_{VV}(G)$*.*

**Definition 1.2.** [3] *The tadpole graph* $T_{n,k}$ *is the graph created by concatenating* $C_n$ *and* $P_k$ *with an edge from any vertex of* $C_n$ *to a pendant of* $P_k$ *for integers* $n \geq 3$ *and* $k \geq 1$*.*

**Definition 1.3.** [2] *The ladder graph* $L_n$ *is the product graph* $P_2 \times P_n$*, which contains* $2n$ *vertices and* $3n - 2$ *edges.*

**Definition 1.4.** [2] *The graph* $mP_n$ *is defined as union of* $m-$ *copies of path graph* $P_n$ *with* $V(mP_n) = \{v_j{}^i : 1 \leq i \leq m, 1 \leq j \leq n\}$ *and* $E(mP_n) = \{e_j{}^i : 1 \leq i \leq m, 1 \leq j \leq n - 1\}$*.*

## 2. Main results

**Theorem 2.1.** *The graph* $DD_{VV}(L_n)$ *of types* $(1, 0, 1)$*,* $(0, 1, 0)$*,* $(0, 1, 1)$*,* $(1, 1, 0)$ *and* $(1, 1, 1)$ *is (a,d)-face antimagic.*

*Proof.* Let $G(V, E, F)$ be a ladder graph $L_n$ with vertex set $V = \{s_y; 1 \leq y \leq 2n\}$ and edge set $E = \{s_y s_{y+1}, s_{2n+1-y} s_{2n-y}; 1 \leq y \leq n - 1\} \cup \{s_y s_{2n+1-y}; 1 \leq y \leq n\}$ and face set $F = \{f_* : s_y s_{y+1} s_{2n+1-y} s_{2n-y}; 1 \leq y \leq n - 1\}$, where $f_*$ denotes the interior faces of $L_n$. Let $G^{'}(V^{'}, E^{'}, F^{'})$ be a duplication of all vertices by edges of $G$ with $V^{'} = \{b_y, a_y; 1 \leq y \leq 2n\} \cup V$, $E^{'} = \{s_y b_y, s_y a_y, b_y a_y; 1 \leq y \leq 2n\} \cup E$ and $F^{'} = \{f_1 : s_y b_y a_y; 1 \leq y \leq 2n\} \cup F$. Let $G^{''}(V^{''}, E^{''}, F^{''})$ be a duplication of all vertices by edges of $G^{'}$ with $V^{''} = \{t_y, k_y, v_y, r_y, p_y, q_y; 1 \leq y \leq 2n\} \cup V^{'}$, $E^{''} = \{s_y t_y, s_y k_y, t_y k_y, b_y v_y, b_y r_y, v_y r_y, a_y p_y, a_y q_y, p_y q_y; 1 \leq y \leq 2n\} \cup E^{'}$ and $F^{''} = \{f_2 : b_y v_y r_y, f_3 : a_y p_y q_y, f_4 : s_y t_y k_y; 1 \leq y \leq 2n\} \cup F^{'}$. The following are the various types for which $(a, d)-$ face antimagic labeling is proved.

**i.** Type (1, 0, 1)

Consider $\phi : V^{''} \cup F^{''} \to \{1, 2, \ldots, 27n - 1\}$. The vertex and the face labeling pattern is defined by $\phi(s_y) = y; 1 \leq y \leq 2n$, $\phi(b_y) = 18n + 1 - y; 1 \leq y \leq 2n$, $\phi(a_y) = 14n + 1 - y; 1 \leq y \leq 2n$, $\phi(t_y) = 12n + 1 - y; 1 \leq y \leq 2n$, $\phi(k_y) = 16n + 1 - y; 1 \leq y \leq 2n$, $\phi(v_y) = 2n + y; 1 \leq y \leq 2n$, $\phi(r_y) = 10n + 1 - y; 1 \leq y \leq 2n$, $\phi(p_y) = 4n + y; 1 \leq y \leq 2n$, $\phi(q_y) = 8n + 1 - y; 1 \leq y \leq 2n$, $\phi(f_{1y}) = 26n + 1 - y; 1 \leq y \leq 2n$, $\phi(f_{2y}) = 24n + 1 - y; 1 \leq y \leq 2n$, $\phi(f_{3y}) = 20n + 1 - y; 1 \leq y \leq 2n$, $\phi(f_{4y}) = 22n + 1 - y; 1 \leq y \leq 2n$, $\phi(f_{*y}) = 26n + y; 1 \leq y \leq n - 1$ with the following $3-$ sided face weights $w(f_1) = \phi(s_y) + \phi(b_y) + \phi(a_y) + \phi(f_{1y}) = 58n + 3 - 2y; 1 \leq y \leq 2n$, $w(f_2) = \phi(b_y) + \phi(v_y) + \phi(r_y) + \phi(f_{2y}) = 54n + 3 - 2y; 1 \leq y \leq 2n$, $w(f_3) = \phi(a_y) + \phi(p_y) + \phi(q_y) + \phi(f_{3y}) = 46n + 3 - 2y; 1 \leq y \leq 2n$ and $w(f_4) = \phi(s_y) + \phi(t_y) + \phi(k_y) + \phi(f_{4y}) = 50n + 3 - 2y; 1 \leq y \leq 2n$, which forms an arithmetic progression $\{42n + 3, 42n + 3 + 1 \times 2, \ldots, 42n + 3 + (8n - 1)2\}$. The $4-$ sided face weights are given by $w(f_*) = \phi(s_y) + \phi(s_{y+1}) + \phi(s_{2n+1-y}) + \phi(s_{2n-y}) + \phi(f_{*y}) = 30n + y + 2; 1 \leq y \leq n - 1$,

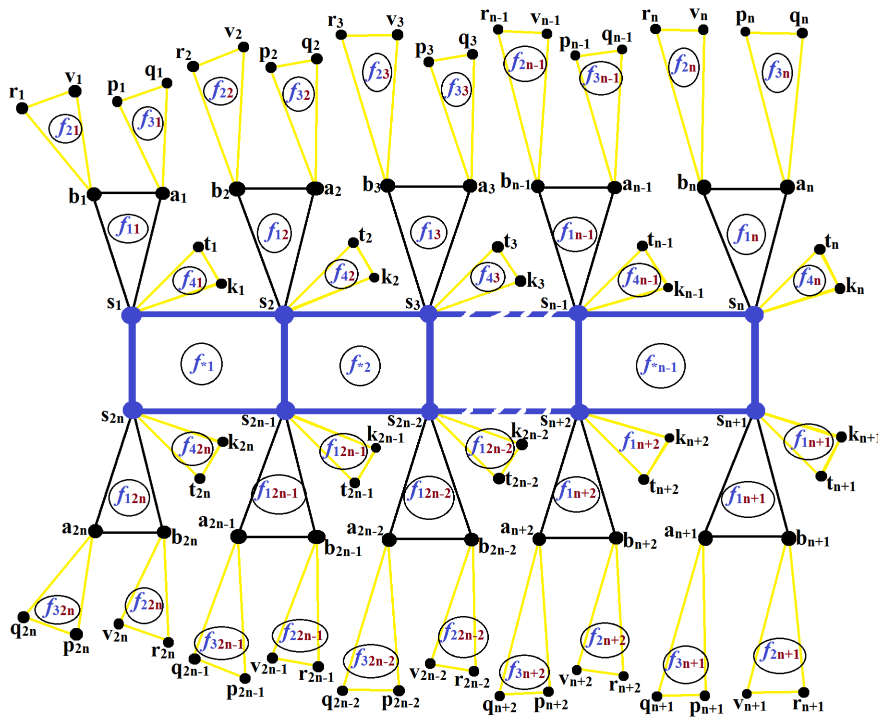


FIGURE 1. $DD_{VV}(L_n)$

where $\phi(f_{*y})$ denotes the face labeling for $4-$ sided faces in $L_n$, which forms an arithmetic progression $\{30n+3, 30n+3+1\times 1, \ldots, 30n+3+((n-1)-1)1\}$.

**ii.** Type (1, 1, 0)

Consider $\phi : V'' \cup E'' \rightarrow \{1, 2, \ldots, 45n-2\}$. The vertex labeling pattern is same as type $(i)$. The edge labeling pattern is defined by $\phi(s_y b_y) = 26n+1-y; 1 \le y \le 2n$, $\phi(s_y a_y) = 26n+y; 1 \le y \le 2n$, $\phi(b_y a_y) = 42n+1-y; 1 \le y \le 2n$, $\phi(b_y v_y) = 24n+1-y; 1 \le y \le 2n$, $\phi(b_y r_y) = 28n+y; 1 \le y \le 2n$, $\phi(v_y r_y) = 40n+1-y; 1 \le y \le 2n$, $\phi(a_y p_y) = 20n+1-y; 1 \le y \le 2n$, $\phi(a_y q_y) = 32n+y; 1 \le y \le 2n$, $\phi(p_y q_y) = 36n+1-y; 1 \le y \le 2n$, $\phi(s_y t_y) = 22n+1-y; 1 \le y \le 2n$, $\phi(s_y k_y) = 30n+y; 1 \le y \le 2n$, $\phi(t_y k_y) = 38n+1-y; 1 \le y \le 2n$, $\phi(s_y s_{2n+1-y}) = 44n-2+y; 1 \le y \le n$, $\phi(s_y s_{y+1}) = 42n+y; 1 \le y \le n-1$, $\phi(s_{2n+1-y} s_{2n-y}) = 44n-1-y; 1 \le y \le n-1$. The $3-$ sided face weights are as follows $w(f_1) = \phi(s_y) + \phi(b_y) + \phi(a_y) + \phi(s_y b_y) + \phi(b_y a_y) + \phi(s_y a_y) = 126n+4-2y; 1 \le y \le 2n$, $w(f_2) = \phi(b_y) + \phi(v_y) + \phi(r_y) + \phi(b_y v_y) + \phi(v_y r_y) + \phi(b_y r_y) = 122n+4-2y; 1 \le y \le 2n$, $w(f_3) = \phi(a_y) + \phi(p_y) + \phi(q_y) + \phi(a_y p_y) + \phi(p_y q_y) + \phi(a_y q_y) = 114n+4-2y; 1 \le y \le 2n$ and $w(f_4) = \phi(s_y) + \phi(t_y) + \phi(k_y) + \phi(s_y t_y) + \phi(t_y k_y) + \phi(s_y k_y) = 118n+4-2y; 1 \le y \le 2n$, which forms an arithmetic progression $\{110n+4, 110+4+1\times 2, \ldots, 110n+4+(8n-1)2)\}$ and the $4-$ sided face weights $w(f_*) = \phi(s_y) + \phi(s_{y+1}) + \phi(s_{2n+1-y}) + \phi(s_{2n-y}) + \phi(s_y s_{2n+1-y}) + \phi(s_{y+1} s_{2n-y}) + \phi(s_y s_{y+1}) + \phi(s_{2n+1-y} s_{2n-y}) = 178n-2+2y; 1 \le y \le n-1$, forms an arithmetic progression $\{178n, 178n+1\times 2, \ldots, 178n+((n-1)-1)2\}$.

**iii.** Type (1, 1, 1)

Consider $\phi : V'' \cup E'' \cup F'' \rightarrow \{1, 2, \ldots, 54n-3\}$. The vertex and edge labeling pattern is same as type $(ii)$. The face labeling pattern is defined by $\phi(f_{1y}) = 45n-2+y; 1 \le y \le 2n$, $\phi(f_{2y}) = 47n-2+y; 1 \le y \le 2n$, $\phi(f_{3y}) = 51n-2+y; 1 \le y \le 2n$, $\phi(f_{4y}) = 49n-2+y; 1 \le y \le 2n$, $\phi(f_{*y}) = 54n-2-y; 1 \le y \le n-1$. The following are the $3-$ sided face weights

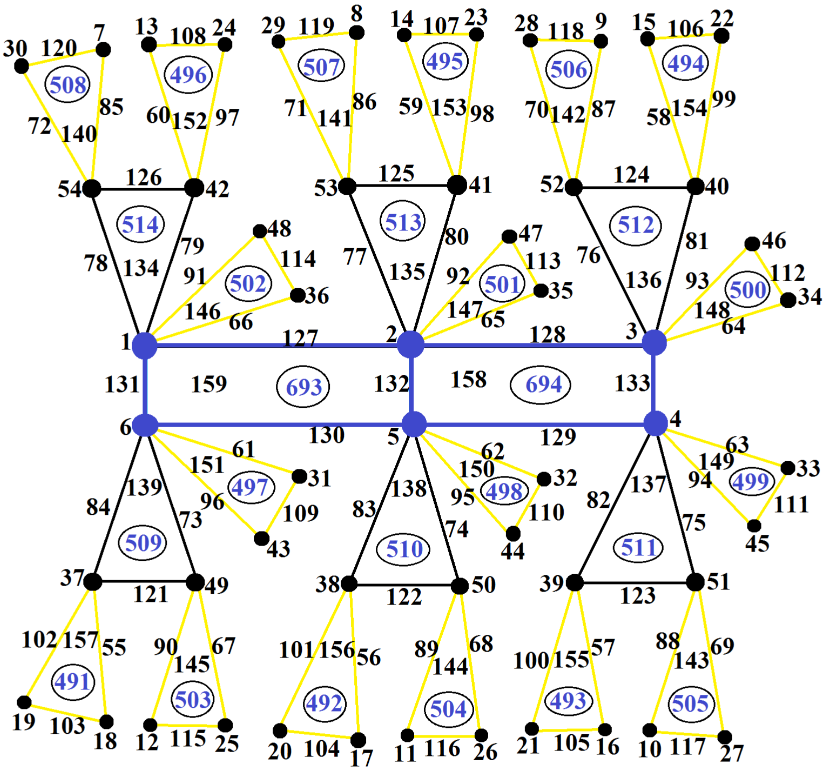


FIGURE 2. $(491, 1)-$ face antimagic labeling of $DD_{VV}$ $(L_3)$

$w(f_1) = \phi(s_y) + \phi(b_y) + \phi(a_y) + \phi(s_yb_y) + \phi(b_ya_y) + \phi(s_ya_y) + \phi(f_{1y}) = 171n + 2 - y; 1 \leq y \leq 2n$, $w(f_2) = \phi(b_y) + \phi(v_y) + \phi(r_y) + \phi(b_yv_y) + \phi(v_yr_y) + \phi(b_yr_y) + \phi(f_{2y}) = 169n + 2 - y; 1 \leq y \leq 2n$, $w(f_3) = \phi(a_y) + \phi(p_y) + \phi(q_y) + \phi(a_yp_y) + \phi(p_yq_y) + \phi(a_yq_y) + \phi(f_{3y}) = 165n + 2 - y; 1 \leq y \leq 2n$ and $w(f_4) = \phi(s_y) + \phi(t_y) + \phi(k_y) + \phi(s_yt_y) + \phi(t_yk_y) + \phi(s_yk_y) + \phi(f_{4y}) = 167n + 2 - y; 1 \leq y \leq 2n$, which forms an arithmetic progression $\{163n + 2, 163n + 2 + 1 \times 1, \ldots, 163n + 2 + (8n - 1)1\}$ and $4-$ sided face weights $w(f_*) = \phi(s_y) + \phi(s_{y+1}) + \phi(s_{2n+1-y}) + \phi(s_{2n-y}) + \phi(s_ys_{2n+1-y}) + \phi(s_{y+1}s_{2n-y}) + \phi(s_ys_{y+1}) + \phi(s_{2n+1-y}s_{2n-y}) + \phi(f_{*y}) = 232n - 4 + y; 1 \leq y \leq n - 1$, forms an arithmetic progression $\{232n - 3, 232n - 3 + 1 \times 1, \ldots, 232n - 3 + ((n - 1) - 1)1\}$.

**iv.** Type (0, 1, 0)

Consider $\phi : E'' \to \{1, 2, \ldots, 27n - 2\}$. The edge labeling pattern are as follows: $\phi(s_yb_y) = y; 1 \leq y \leq 2n$, $\phi(s_ya_y) = 16n + 1 - y; 1 \leq y \leq 2n$, $\phi(b_ya_y) = 16n + y; 1 \leq y \leq 2n$, $\phi(b_yv_y) = 2n + y; 1 \leq y \leq 2n$, $\phi(b_yr_y) = 14n + 1 - y; 1 \leq y \leq 2n$, $\phi(v_yr_y) = 18n + y; 1 \leq y \leq 2n$, $\phi(a_yp_y) = 4n + y; 1 \leq y \leq 2n$, $\phi(a_yq_y) = 12n + 1 - y; 1 \leq y \leq 2n$, $\phi(p_yq_y) = 20n + y; 1 \leq y \leq 2n$, $\phi(s_yt_y) = 6n + y; 1 \leq y \leq 2n$, $\phi(s_yk_y) = 22n + y; 1 \leq y \leq 2n$, $\phi(t_yk_y) = 10n + 1 - y; 1 \leq y \leq 2n$, $\phi(s_ys_{2n+1-y}) = 26n + y - 2; 1 \leq y \leq n$, $\phi(s_ys_{y+1}) = 24n + y; 1 \leq y \leq n - 1$, $\phi(s_{2n+1-y}s_{2n-y}) = 26n - y - 1; 1 \leq y \leq n - 1$. The following $3-$ sided face weights $w(f_1) = \phi(s_yb_y) + \phi(s_ya_y) + \phi(b_ya_y) = 32n + 1 + y; 1 \leq y \leq 2n$, $w(f_2) = \phi(b_yv_y) + \phi(b_yr_y) + \phi(v_yr_y) = 34n + 1 + y; 1 \leq y \leq 2n$, $w(f_3) = \phi(a_yp_y) + \phi(p_yq_y) + \phi(a_yq_y) = 36n + 1 + y; 1 \leq y \leq 2n$ and $w(f_4) = \phi(s_yt_y) + \phi(s_yk_y) + \phi(t_yk_y) = 38 + 1 + y; 1 \leq y \leq 2n$, forms an arithmetic progression $\{32n + 2, 32n + 2 + 1 \times 1, \ldots, 32n + 2 + (8n - 1)1\}$. The following $4-$ sided face weights $w(f_*) = \phi(s_ys_{2n+1-y}) + \phi(s_{y+1}s_{2n-y}) + \phi(s_ys_{y+1}) +$

$\phi(s_{2n+1-y}s_{2n-y}) = 102n + 2y - 4; 1 \leq y \leq n-1$, forms an arithmetic progression $\{102n - 2, 102n - 2 + 1 \times 2, \ldots, 102n - 2 + ((n-1)-1)2\}$.

**v.** Type (0, 1, 1)

Consider $\phi : E^{''} \cup F^{''} \rightarrow \{1, 2, \ldots, 36n-3\}$ defined as follows: The edge labeling pattern is same as type $(iv)$. The face labeling pattern is defined as $\phi(f_{1y}) = 28n - 3 + y; 1 \leq y \leq 2n$, $\phi(f_{2y}) = 30n - 3 + y; 1 \leq y \leq 2n$, $\phi(f_{3y}) = 32n - 3 + y; 1 \leq y \leq 2n$, $\phi(f_{4y}) = 34n - 3 + y; 1 \leq y \leq 2n$, $\phi(f_{*y}) = 28n - y - 2; 1 \leq y \leq n-1$ with the following face weights $w(f_1) = \phi(s_y b_y) + \phi(s_y a_y) + \phi(b_y a_y) + \phi(f_{1y}) = 60n - 2 + 2y; 1 \leq y \leq 2n$, $w(f_2) = \phi(b_y v_y) + \phi(b_y r_y) + \phi(v_y r_y) + \phi(f_{2y}) = 64n - 2 + 2y; 1 \leq y \leq 2n$, $w(f_3) = \phi(a_y p_y) + \phi(p_y q_y) + \phi(a_y q_y) + \phi(f_{3y}) = 68n - 2 + 2y; 1 \leq y \leq 2n$ and $w(f_4) = \phi(s_y t_y) + \phi(s_y k_y) + \phi(t_y k_y) + \phi(f_{4y}) = 72n - 2 + 2y; 1 \leq y \leq 2n$, forming an arithmetic progression $\{60n, 60n + 1 \times 2, \ldots, 60n + (8n-1)2)\}$. The $4-$ sided face weights $w(f_*) = \phi(s_y s_{2n+1-y}) + \phi(s_{y+1}s_{2n-y}) + \phi(s_y s_{y+1}) + \phi(s_{2n+1-y}s_{2n-y}) + \phi(f_{*y}) = 130n - 6 + y; 1 \leq y \leq n-1$, forms an arithmetic progression $\{130n-5, 130n-5+1\times 1, \ldots, 130n-5+((n-1)-1)1\}$. □

**Theorem 2.2.** *The graph $DD_{VV}(T_{r,s}); r \geq 4, s \geq 1$ of types* $(1,0,0)$, $(1,0,1)$, $(0,1,0)$, $(0,1,1)$, $(1,1,0)$ *and* $(1,1,1)$ *is (a,d)-face antimagic.*

*Proof.* Let $G(V, E, F)$ be a tadpole graph $T_{r,s}; r \geq 4, s \geq 1$ with vertex set $V = \{s^{'}_y; 1 \leq y \leq r+s\}$, edge set $E = \{s^{'}_y s^{'}_{y+1}, s^{'}_1 s^{'}_r; 1 \leq y \leq r+s-1\}$ and face set $F = \{f_* : s^{'}_1 s^{'}_2 \ldots s^{'}_r\}$, where $f_*$ denotes the face of $C_r$ in $T_{r,s}$. Let $G^{'}(V^{'}, E^{'}, F^{'})$ be a duplication of all vertices by edges of $G$ with $V^{'} = \{b_y, a_y; 1 \leq y \leq r+s\} \cup V$, $E^{'} = \{s^{'}_y b_y, s^{'}_y a_y, b_y a_y; 1 \leq y \leq r+s\} \cup E$ and $F^{'} = \{f_1 : s^{'}_y b_y a_y; 1 \leq y \leq r+s\} \cup F$. Let $G^{''}(V^{''}, E^{''}, F^{''})$ be a duplication of all vertices by edges of $G^{'}$ with $V^{''} = \{t_y, k_y, v_y, r^{'}_y, p_y, q_y; 1 \leq y \leq r+s\} \cup V^{'}$ and $E^{''} = \{s^{'}_y t_y, s^{'}_y k_y, t_y k_y, b_u v_y, b_y r^{'}_y, v_y r^{'}_y, a_y p_y, a_y q_y, p_y q_y; 1 \leq y \leq r+s\} \cup E^{'}$ and $F^{''} = \{f_2 : b_y v_y r^{'}_y,\ f_3 : a_y p_y q_y,\ f_4 : s^{'}_y t_y k_y; 1 \leq y \leq r+s\} \cup F^{'}$. The following provides the $(a,d)-$ face antimagic labeling for various types.

**i.** Type (1, 0, 0)

Consider $\phi : V^{''} \rightarrow \{1, 2, \ldots, 9(r+s)\}$. The vertex labeling pattern is as follows: $\phi(s^{'}_{r+s+1-y}) = r+s+1-y;\ 1 \leq y \leq r+s$, $\phi(b_{r+s+1-y}) = 8(r+s)+y;\ 1 \leq y \leq r+s$, $\phi(a_{r+s+1-y}) = 7(r+s)+1-y;\ 1 \leq y \leq r+s$, $\phi(v_{r+s+1-y}) = 2(r+s)+1-y;\ 1 \leq y \leq r+s$, $\phi(r^{'}_{r+s+1-y}) = 4(r+s)+y;\ 1 \leq y \leq r+s$, $\phi(p_{r+s+1-y}) = 3(r+s)+1-y;\ 1 \leq y \leq r+s$, $\phi(q_{r+s+1-y}) = 3(r+s)+y;\ 1 \leq y \leq r+s$, $\phi(t_{r+s+1-y}) = 5(r+s)+y;\ 1 \leq y \leq r+s$, $\phi(k_{r+s+1-y}) = 7(r+s)+y;\ 1 \leq y \leq r+s$ with the following $3-$ sided face weights $w(f_1)=\phi(s^{'}_{r+s+1-y}) + \phi(b_{r+s+1-y}) + \phi(a_{r+s+1-y}) = 16(r+s)+2-y;\ 1 \leq y \leq r+s$, $w(f_2)=\phi(b_{r+s+1-y}) + \phi(v_{r+s+1-y}) + \phi(r^{'}_{r+s+1-y}) = 14(r+s)+1+y;\ 1 \leq y \leq r+s$, $w(f_3)=\phi(a_{r+s+1-y}) + \phi(p_{r+s+1-y}) + \phi(q_{r+s+1-y}) = 13(r+s)+2-y;\ 1 \leq y \leq r+s$ and $w(f_4)=\phi(s^{'}_{r+s+1-y}) + \phi(t_{r+s+1-y}) + \phi(k_{r+s+1-y}) = 13(r+s)+1+y;\ 1 \leq y \leq r+s$, which forms an arithmetic progression $\{12(r+s)+2, 12(r+s)+2+1 \times 1, \ldots, 12(r+s)+2+(4(r+s)-1)1\}$ and the face weight of $r-$ sided of $T_{(r,s)} = \frac{r^2+r}{2}$.

**ii.** Type (1, 0, 1)

Consider $\phi : V^{''} \cup F^{''} \rightarrow \{1, 2, \ldots, 13(r+s)+1\}$ The vertex labeling pattern is same as type $(i)$. The face labeling pattern is defined by $\phi(f_{1(r+s+1-y)}) = 13(r+s)+1-y;\ 1 \leq y \leq r+s$, $\phi(f_{2(r+s+1-y)}) = 11(r+s)+y$ ; $1 \leq y \leq r+s$, $\phi(f_{3(r+s+1-y)}) = 10(r+s)+1-y;\ 1 \leq y \leq r+s$, $\phi(f_{4(r+s+1-y)}) = 10(r+s)+y;\ 1 \leq y \leq r+s$, $\phi(f_{*1}) = 13(r+s)+1$ with the following 3-sided face weights $w(f_1)=\phi(s^{'}_{r+s+1-y}) + \phi(b_{r+s+1-y}) + \phi(a_{r+s+1-y}) +$

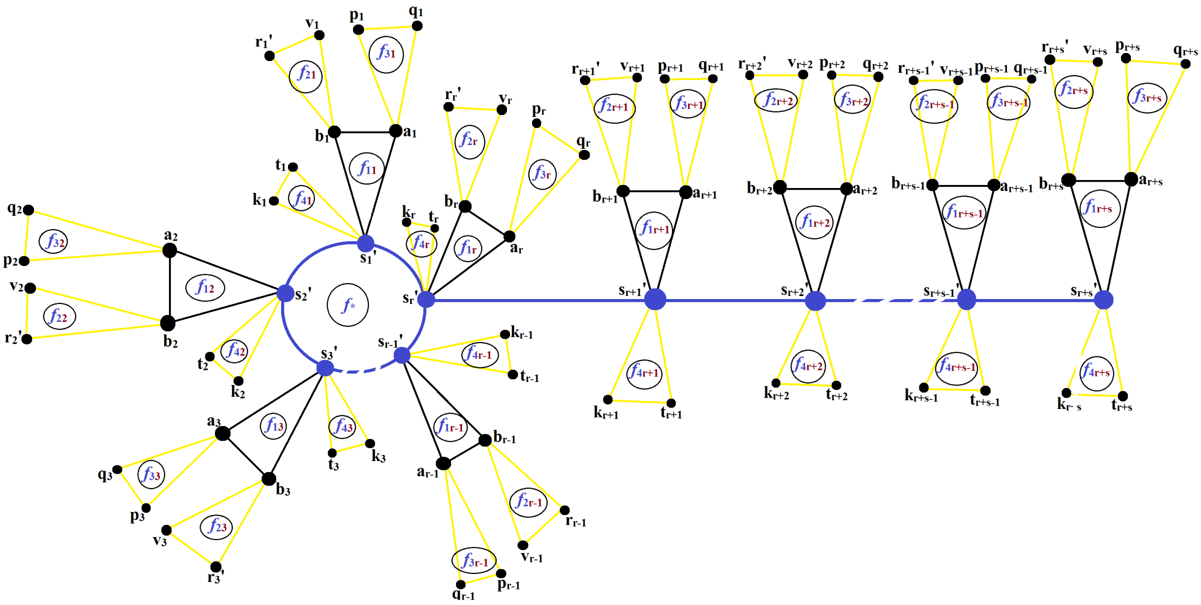

FIGURE 3. $DD_{VV}$ $(T_{r,s})$

$\phi(f_{1(r+s+1-y)})= 29(r+s)+3-2y$; $1 \le y \le r+s$, $w(f_2)= \phi(b_{r+s+1-y}) + \phi(v_{r+s+1-y}) + \phi(r'_{r+s+1-y}) + \phi(f_{2(r+s+1-y)})= 25(r+s)+1+2y$; $1 \le y \le r+s$, $w(f_3)= \phi(a_{r+s+1-y}) + \phi(p_{r+s+1-y}) + \phi(q_{r+s+1-y}) + \phi(f_{3(r+s+1-y)})= 23(r+s)+3-2y$; $1 \le y \le r+s$ and $w(f_4)= \phi(s'_{r+s+1-y}) + \phi(t_{r+s+1-y}) + \phi(k_{r+s+1-y}) + \phi(f_{4(r+s+1-y)}) = 23(r+s)+1+2y$; $1 \le y \le r+s$, which forms an arithmetic progression $\{21(r+s)+3, 21(r+s)+3+1\times 2, \ldots, 21(r+s)+3+(4(r+s)-1)2\}$ and the face weight of $r-$ sided of $T_{(r,s)} = \frac{r^2+27r+26s+2}{2}$.

**iii.** Type (0, 1, 0)

Consider $\phi : E'' \to \{1, 2, \ldots, 13(r+s)\}$. The edge labeling pattern is as follows: $\phi(s'_{r+s+1-y}b_{r+s+1-y}) = r+s+1-y$; $1 \le y \le r+s$, $\phi(s'_{r+s+1-y}a_{r+s+1-y})=7(r+s)+y$; $1 \le y \le r+s$, $\phi(b_{r+s+1-y}a_{r+s+1-y})=9(r+s)+1-y$; $1 \le y \le r+s$, $\phi(b_{r+s+1-y}v_{r+s+1-y})= 2(r+s)+1-y$; $1 \le y \le r+s$, $\phi(b_{r+s+1-y}r'_{r+s+1-y}) = 6(r+s)+y$; $1 \le y \le r+s$, $\phi(v_{r+s+1-y}r'_{r+s+1-y})= 10(r+s)+1-y$; $1 \le y \le r+s$, $\phi(a_{r+s+1-y}p_{r+s+1-y})= 3(r+s)+1-y$; $1 \le y \le r+s$, $\phi(a_{r+s+1-y}q_{r+s+1-y})=5(r+s)+y$; $1 \le y \le r+s$, $\phi(p_{r+s+1-y}q_{r+s+1-y})= 11(r+s)+1-y$; $1 \le y \le r+s$, $\phi(s'_{r+s+1-y}t_{r+s+1-y})= 4(r+s)+1-y$; $1 \le y \le r+s$, $\phi(s'_{r+s+1-y}k_{r+s+1-y})=4(r+s)+y$; $1 \le y \le r+s$, $\phi(t_{r+s+1-y}k_{r+s+1-y})= 12(r+s)+1-y$; $1 \le y \le r+s$, $\phi(s'_1s'_r)= 13(r+s)-1$, $\phi(s'_ys'_{y+1}) = 12(r+s)+y$; $1 \le y \le r-1$, $\phi(s'_ys'_{y+1})= 13(r+s)-1+y$ ; $r \le y \le (r+s)-1$ with the following 3-sided face weights $w(f_1)= \phi(s'_{r+s+1-y}b_{r+s+1-y})+ \phi(s'_{r+s+1-y}a_{r+s+1-y}) + \phi(b_{r+s+1-y}a_{r+s+1-y}) = 17(r+s)+2-y$; $1 \le y \le r+s$, $w(f_2) = \phi(b_{r+s+1-y}v_{r+s+1-y}) + \phi(b_{r+s+1-y}r'_{r+s+1-y}) + \phi(v_{r+s+1-y}r'_{r+s+1-y}) = 18(r+s)+2-y$; $1 \le y \le r+s$, $w(f_3) = \phi(a_{r+s+1-y}p_{r+s+1-y}) + \phi(p_{r+s+1-y}q_{r+s+1-y}) + \phi(a_{r+s+1-y}q_{r+s+1-y})=19(r+s)+2-y$; $1 \le y \le r+s$ and $w(f_4)= \phi(s'_{r+s+1-y}t_{r+s+1-y})+ \phi(s'_{r+s+1-y}k_{r+s+1-y})+ \phi(t_{r+s+1-y}k_{r+s+1-y}) = 20(r+s)+2-y$; $1 \le y \le r+s$, which forms an arithmetic progression $\{16(r+s)+2, 16(r+s)+2+1\times 1, \ldots, 16(r+s)+2+(4(r+s)-1)1\}$ and the face weight of $r-$ sided of $T_{(r,s)} = \frac{25r^2+24rs+r}{2}$.

**iv.** Type (0, 1, 1)

Consider $\phi : E^{''} \cup F^{''} \rightarrow \{1, 2, \ldots, 17(r+s)+1\}$. The edge labeling pattern is same as type $(iii)$. The face labeling pattern is as follows $\phi(f_{1(r+s+1-y)})= 14(r+s)+1-y$; $1 \le y \le r+s$, $\phi(f_{2(r+s+1-y)})= 15(r+s)+1-y$; $1 \le y \le r+s$, $\phi(f_{3(r+s+1-y)}) = 16(r+s)+1-y$; $1 \le y \le r+s$, $\phi(f_{4(r+s+1-y)})= 17(r+s)+1-y$; $1 \le y \le r+s$, $\phi(f_{*1})= 17(r+s)+1$ with the following 3− sided face weights $w(f_1)$=$\phi(s^{'}_{r+s+1-y}b_{r+s+1-y})$ + $\phi(s^{'}_{r+s+1-y}a_{r+s+1-y})$+ $\phi(b_{r+s+1-y}a_{r+s+1-y})$ + $\phi(f_{1(r+s+1-y)})$= $31(r+s)+3-2y$; $1 \le y \le r+s$, $w(f_2)$= $\phi(b_{r+s+1-y}v_{r+s+1-y})$+$\phi(b_{r+s+1-y}r^{'}_{r+s+1-})$ + $\phi(v_{r+s+1-y}r^{'}_{r+s+1-y})$ + $\phi(f_{2(r+s+1-y)}) = 33(r+s)+3-2y$; $1 \le y \le r+s$, $w(f_3)$= $\phi(a_{r+s+1-y}p_{r+s+1-y})$ + $\phi(p_{r+s+1-y}q_{r+s+1-y})$+ $\phi(a_{r+s+1-y}q_{r+s+1-y})$ + $\phi(f_{3(r+s+1-y)}) = 35(r+s)+3-2y$; $1 \le y \le r+s$ and $w(f_4)$= $\phi(s^{'}_{r+s+1-y}t_{r+s+1-y})$+ $\phi(s^{'}_{r+s+1-y}k_{r+s+1-y})$+ $\phi(t_{r+s+1-y}k_{r+s+1-y})$+ $\phi(f_{4(r+s+1-y)})$= $37(r+s)+3-2y$; $1 \le y \le r+s$, which forms an arithmetic progression $\{29(r+s)+3, 29(r+s)+3+1\times 2, \ldots, 29(r+s)+3+(4(r+s)-1)2\}$ and the face weight of $r-$ sided of $T_{(r,s)} = \frac{25r^2+24rs+35r+34s+2}{2}$.

**v.** Type (1, 1, 0)

Consider $\phi : V^{''} \cup E^{''} \rightarrow \{1, 2, \ldots, 22(r+s)\}$. The vertex labeling pattern is same as type $(i)$. The edge labeling pattern is as follows $\phi(s^{'}_{r+s+1-y}b_{r+s+1-y}) = 13(r+s)+1-y$; $1 \le y \le r+s$, $\phi(s^{'}_{r+s+1-y}a_{r+s+1-y}) = 13(r+s)+y$; $1 \le y \le r+s$, $\phi(b_{r+s+1-y}a_{r+s+1-y})$ =$21(r+s)+1-y$; $1 \le y \le r+s$, $\phi(b_{r+s+1-y}v_{r+s+1-y})$= $11(r+s)+y$; $1 \le y \le r+s$, $\phi(b_{r+s+1-y}r^{'}_{r+s+1-y})$= $15(r+s)+1-y$; $1 \le y \le r+s$, $\phi(v_{r+s+1-y}r^{'}_{r+s+1-y})$= $19(r+s)+y$; $1 \le y \le r+s$, $\phi(a_{r+s+1-y}p_{r+s+1-y})$= $10(r+s)+1-y$; $1 \le y \le r+s$, $\phi(a_{r+s+1-y}q_{r+s+1-y})$= $16(r+s)+y$; $1 \le y \le r+s$, $\phi(p_{r+s+1-y}q_{r+s+1-y}) = 18(r+s)+1-y$; $1 \le y \le r+s$, $\phi(s^{'}_{r+s+1-y}t_{r+s+1-y})$= $10(r+s)+y$; $1 \le y \le r+s$, $\phi(s^{'}_{r+s+1-y}k_{r+s+1-y})$= $16(r+s)+1-y$; $1 \le y \le r+s$, $\phi(t_{r+s+1-y}k_{r+s+1-y})$= $18(r+s)+y$; $1 \le y \le r+s$, $\phi(s^{'}_1 s^{'}_r)$= $22(r+s)-1$, $\phi(s^{'}_y s^{'}_{y+1})$= $21(r+s)+y$; $1 \le y \le r-1$, $\phi(s^{'}_y s^{'}_{y+1})$= $21(r+s)+y+1$; $r \le y \le (r+s)-1$ with the following 3− sided face weights $w(f_1)$=$\phi(s^{'}_{r+s+1-y})$+ $\phi(b_{r+s+1-y})$+ $\phi(a_{r+s+1-y})$+ $\phi(s^{'}_{r+s+1-y}\ b_{r+s+1-y})$+ $\phi(s^{'}_{r+s+1-y}\ a_{r+s+1-y})$+ $\phi(b_{r+s+1-y}\ a_{r+s+1-y})$ = $63(r+s)+4-2y$; $1 \le y \le r+s$, $w(f_2)$= $\phi(b_{r+s+1-y})$+ $\phi(v_{r+s+1-y})$+ $\phi(r^{'}_{r+s+1-y})$ + $\phi(b_{r+s+1-y}\ v_{r+s+1-y})$+ $\phi(b_{r+s+1-y}\ r^{'}_{r+s+1-y})$+ $\phi(v_{r+s+1-y}\ r^{'}_{r+s+1-y})$= $59(r+s)+2+2y$; $1 \le y \le r+s$, $w(f_3)$= $\phi(a_{r+s+1-y})$+ $\phi(p_{r+s+1-y})$+ $\phi(q_{r+s+1-y})$+ $\phi(a_{r+s+1-y}\ p_{r+s+1-y})$ + $\phi(p_{r+s+1-y}\ q_{r+s+1-y})$ + $\phi(a_{r+s+1-y}\ q_{r+s+1-y})$= $57(r+s)+4-2y$; $1 \le y \le r+s$ and $w(f_4)$ =$\phi(s^{'}_{r+s+1-y})$+ $\phi(t_{r+s+1-y})$ + $\phi(k_{r+s+1-y})$+ $\phi(s^{'}_{r+s+1-y}\ t_{r+s+1-y})$+ $\phi(t_{r+s+1-y}\ k_{r+s+1-y})$+ $\phi(s^{'}_{r+s+1-y}\ k_{r+s+1-y})$= $57(r+s)+2+2y$; $1 \le y \le r+s$, which forms an arithmetic progression $\{55(r+s)+4, 55(r+s)+4+1\times 2, \ldots, 55(r+s)+4+(4(r+s)-1)2\}$ and the weight of $r-$ sided face of $T_{(r,s)} = \frac{44r^2+42rs+2r}{2}$.

**vi.** Type $(1, 1, 1)$

Consider $\phi : V^{''} \cup E^{''} \cup F^{''} \rightarrow \{1, 2, \ldots, 26(r+s)+1\}$. The vertex and edge labeling pattern is same as type $(v)$. The face labeling pattern is defined by $\phi(f_{1(r+s+1-y)})$ = $22(r+s)+y$; $1 \le y \le r+s$, $\phi(f_{2(r+s+1-y)})$= $24(r+s)+1-y$; $1 \le y \le r+s$, $\phi(f_{3(r+s+1-y)})$=$25(r+s)+y$; $1 \le y \le r+s$, $\phi(f_{4(r+s+1-y)})$= $25(r+s)+1-y$; $1 \le y \le r+s$, $\phi(f_{*1})$= $26(r+s)+1$ with the following 3-sided face weights $w(f_1)$= $\phi(s^{'}_{r+s+1-y})$+ $\phi(b_{r+s+1-y})$+ $\phi(a_{r+s+1-y})$+ $\phi(s^{'}_{r+s+1-y}\ b_{r+s+1-y})$+ $\phi(s^{'}_{r+s+1-y}\ a_{r+s+1-y})$+ $\phi(b_{r+s+1-y}$ $a_{r+s+1-y})$+ $\phi(f_{1(r+s+1-y)})$= $85(r+s)+4-y$; $1 \le y \le r+s$, $w(f_2)$= $\phi(b_{r+s+1-y})$+ $\phi(v_{r+s+1-y})$+ $\phi(r^{'}_{r+s+1-y})$ +$\phi(b_{r+s+1-y}\ v_{r+s+1-y})$ + $\phi(b_{r+s+1-y}\ r^{'}_{r+s+1-y})$ + $\phi(v_{r+s+1-y}$

$r'_{r+s+1-y}$)+ $\phi(f_{2(r+s+1-y)}) = 83(r+s)+3+y$; $1 \leq y \leq r+s$, $w(f_3)= \phi(a_{r+s+1-y})+ \phi(p_{r+s+1-y})+ \phi(q_{r+s+1-y}) + \phi(a_{r+s+1-y}\ p_{r+s+1-y})+\phi(p_{r+s+1-y}\ q_{r+s+1-y})+ \phi(a_{r+s+1-y}\ q_{r+s+1-y})+ \phi(f_{3(r+s+1-y)})= 82(r+s)+4-y$; $1 \leq y \leq r+s$ and $w(f_4)= \phi(s'_{r+s+1-y})+ \phi(t_{r+s+1-y})+ \phi(k_{r+s+1-y})+ \phi(s'_{r+s+1-y}\ t_{r+s+1-y})+ \phi(t_{r+s+1-y}\ k_{r+s+1-y})+ \phi(s'_{r+s+1-y}\ k_{r+s+1-y})+ \phi(f_{4(r+s+1-y)})= 82(r+s)+3+y$; $1 \leq y \leq r+s$, which forms an arithmetic progression $\{81(r+s)+4, 81(r+s)+4+1\times 1, \ldots, 81(r+s)+4+(4(r+s)-1)1\}$ and the face weight of $r-$ sided of $T_{(r,s)} = \frac{44r^2+42rs+54r+52s+2}{2}$.

□

**Theorem 2.3.** *The graph $DD_{VV}(mP_n)$; $m \geq 1$, $n \geq 2$ of types $(1,0,0)$, $(1,0,1)$, $(0,1,0)$, $(0,1,1)$, $(1,1,0)$ and $(1,1,1)$ is (a,d)-face antimagic.*

*Proof.* Let $G(V,E)$ be a graph obtained by $m$ copies of path graph $P_n$; $n \geq 2$ which is denoted by $mP_n; m \geq 1, n \geq 2$ with vertex set $V = \{s_y; 1 \leq y \leq mn\}$ and edge set $E = \{s_y s_{y+1}; 1 \leq y \leq mn-m$, where $y \not\equiv 0 \pmod n\}$. Let $G'(V',E',F')$ be a duplication of all vertices by edges of $G$ with $V'= \{b_y, a_y; 1 \leq y \leq mn\} \cup V$, $E' = \{s_y b_y, s_y a_y, b_y a_y; 1 \leq y \leq mn\} \cup E$ and $F'= \{f_1 : s_y b_y a_y; 1 \leq y \leq mn\}$. Let $G''(V'',E'',F'')$ be a duplication of all vertices by edges of $G'$ with $V''= \{t_y, k_y, v_y, r_y, p_y, q_y; 1 \leq y \leq mn\} \cup V'$, $E''= \{s_y t_y, s_y k_y, t_y k_y, b_y v_y, b_y r_y, v_y r_y, a_y p_y, a_y q_y, p_y q_y; 1 \leq y \leq mn\} \cup E'$ and $F''= \{f_2 : b_y v_y r_y$, $f_3 : a_y p_y q_y$, $f_4 : s_y t_y k_y; 1 \leq y \leq mn\} \cup F'$. The following provides the $(a,d)-$ face antimagic labeling for various types.

**i.** Type $(1,0,0)$

Consider $\phi : V'' \to \{1,2,\ldots,9mn\}$. The vertex labeling pattern are as follows: $\phi(s_y)= 9mn+1-y$; $1 \leq y \leq mn$, $\phi(b_y)=2mn+y$; $1 \leq y \leq mn$, $\phi(a_y)=5mn+1-y$; $1 \leq y \leq mn$, $\phi(v_y)=5mn+y$; $1 \leq y \leq mn$, $\phi(r_y)=7mn+1-y$; $1 \leq y \leq mn$, $\phi(p_y)=y$; $1 \leq y \leq mn$, $\phi(q_y)=8mn+1-y$; $1 \leq y \leq mn$, $\phi(t_y)=3mn+y$; $1 \leq y \leq mn$, $\phi(k_y)=mn+y$; $1 \leq y \leq mn$ with the following $3-$ sided face weights $w(f_1)= \phi(s_y)+ \phi(b_y)+ \phi(a_y)=16mn+2-y$; $1 \leq y \leq mn$, $w(f_2)=\phi(b_y)+ \phi(v_y)+ \phi(r_y)=14mn+1+y$; $1 \leq y \leq mn$, $w(f_3)=\phi(a_y)+\phi(p_y)+ \phi(q_y)=13mn+2-y$; $1 \leq y \leq mn$ and $w(f_4)=\phi(s_y)+ \phi(t_y)+ \phi(k_y)=13mn+1+y$; $1 \leq y \leq mn$, which forms an arithmetic progression $\{12mn+2, 12mn+2+1\times 1, \ldots, 12mn+2+(4mn-1)1\}$.

**ii.** Type (1, 0, 1)

Consider $\phi : V'' \cup F'' \to \{1,2,\ldots,13mn\}$. The vertex labeling pattern is same as type $(i)$. The face labeling pattern is defined by $\phi(f_{1y})=13mn+1-y$; $1 \leq y \leq mn$, $\phi(f_{2y})=11mn+y$; $1 \leq y \leq mn$, $\phi(f_{3y})=10mn+1-y$; $1 \leq y \leq mn$, $\phi(f_{4y})=10mn+y$; $1 \leq y \leq mn$ with the following $3-$ sided face weights $w(f_1)=\phi(s_y)+ \phi(b_y)+ \phi(a_y)+ \phi(f_{1y})=29mn+3-2y$; $1 \leq y \leq mn$, $w(f_2)=\phi(b_y)+ \phi(v_y)+ \phi(r_y)+ \phi(f_{2y})=25mn+1+2y$; $1 \leq y \leq mn$, $w(f_3)=\phi(a_y)+ \phi(p_y)+ \phi(q_y)+ \phi(f_{3y})=23mn+3-2y$; $1 \leq y \leq mn$ and $w(f_4)=\phi(s_y)+ \phi(t_y)+ \phi(k_y)+ \phi(f_{4y})=23mn+1+2y$; $1 \leq y \leq mn$, which forms an arithmetic progression $\{21mn+3, 21mn+3+1\times 2, \ldots, 21mn+3+(4mn-1)2\}$.

**iii.** Type (0, 1, 0)

Consider $\phi : E'' \to \{1,2,\ldots,13mn-m\}$. The edge labeling pattern is given by $\phi(s_y b_y) = 9mn+y; 1 \leq y \leq mn$, $\phi(s_y a_y) = 8mn+1-y; 1 \leq y \leq mn$, $\phi(b_y a_y) = y; 1 \leq y \leq mn$, $\phi(b_y v_y) = 6mn+y; 1 \leq y \leq mn$, $\phi(b_y r_y) = 11mn+1-y; 1 \leq y \leq mn$, $\phi(v_y r_y) = mn+y; 1 \leq y \leq mn$, $\phi(a_y p_y) = 12mn+1-y; 1 \leq y \leq mn$, $\phi(a_y q_y) = 6mn+1-y; 1 \leq y \leq mn$, $\phi(p_y q_y) = 2mn+y; 1 \leq y \leq mn$, $\phi(s_y t_y) = 8mn+y; 1 \leq y \leq mn$, $\phi(s_y k_y) = 5mn+1-y; 1 \leq y \leq mn$, $\phi(t_y k_y) = 3mn+y; 1 \leq y \leq mn$,

$\phi(s_y s_{y+1}) = 12mn + y; 1 \le y \le mn$, where $y \not\equiv 0 \pmod{n}$ with the following 3-sided face weights $w(f_1) = \phi(s_y b_y) + \phi(s_y a_y) + \phi(b_y a_y) = 17mn + 1 + y; 1 \le y \le mn$, $w(f_2) = \phi(b_y v_y) + \phi(b_y r_\phi) + \phi(v_y r_y) = 18mn + 1 + y; 1 \le y \le mn$, $w(f_3) = \phi(a_y p_y) + \phi(a_y q_y) + \phi(p_y q_y) = 20mn + 2 - y; 1 \le y \le mn$ and $w(f_4) = \phi(s_y t_y) + \phi(s_y k_y) + \phi(t_y k_y) = 16mn + 1 + y; 1 \le y \le mn$, which forms an arithmetic progression $\{16mn + 2, 16mn + 2 + 1 \times 1, \ldots, 16mn + 2 + (4mn - 1)1\}$.

**iv.** Type (0, 1, 1)

Consider $\phi : E^{''} \cup F^{''} \to \{1, 2, \ldots, 17mn - m\}$. The edge labeling pattern is same as type (*iii*). The face labeling pattern is defined as $\phi(f_{1y})$=$14mn - m + y$; $1 \le y \le mn$, $\phi(f_{2y})$=$15mn-m+y$; $1 \le y \le mn$, $\phi(f_{3y})$=$17mn-m-y+1$; $1 \le y \le mn$, $\phi(f_{4y})$=$13mn-m + y$; $1 \le y \le mn$ with the following 3− sided face weights $w(f_1)$=$\phi(s_y b_y)$+ $\phi(s_y a_y)$+ $\phi(b_y a_y)$+ $\phi(f_{1y})$=$31mn - m + 1 + 2y$; $1 \le y \le mn$, $w(f_2)$=$\phi(b_y v_y)$+ $\phi(b_y r_y)$+ $\phi(v_y r_y)$+ $\phi(f_{2y})$=$33mn-m+1+2y$; $1 \le y \le mn$, $w(f_3)$=$\phi(a_y p_y)$+ $\phi(a_y q_y)$+ $\phi(p_y q_y)$+ $\phi(f_{3y})$=$37mn-m+3-2y$; $1 \le y \le mn$ and $w(f_4)$=$\phi(s_y t_y)$+ $\phi(s_y k_y)$+ $\phi(t_y k_y)$+ $\phi(f_{4y})$=$29mn-m+1+2y$; $1 \le y \le mn$, which forms an arithmetic progression $\{29mn - m + 3, 29mn - m + 3 + 1 \times 2, \ldots, 29mn - m + 3 + (4mn - 1)2\}$.

**v.** Type (1, 1, 0)

Consider $\phi : V^{''} \cup E^{''} \to \{1, 2, \ldots, 22mn - m\}$. The vertex labeling pattern is same as type (*i*). The edge labeling pattern is defined by $\phi(s_y b_y)$=$13mn + 1 - y$; $1 \le y \le mn$, $\phi(s_y a_y)$=$17mn + 1 - y$; $1 \le y \le mn$, $\phi(b_y a_y)$=$17mn + y$; $1 \le y \le mn$, $\phi(b_y v_y)$=$11mn + y$; $1 \le y \le mn$, $\phi(b_y r_y)$=$15mn + y$; $1 \le y \le mn$, $\phi(v_y r_y)$=$19mn + 1 - y$; $1 \le y \le mn$, $\phi(a_y p_y)$=$10mn+1-y$; $1 \le y \le mn$, $\phi(a_y q_y)$=$14mn+1-y$; $1 \le y \le mn$, $\phi(p_y q_y)$=$20mn+y$; $1 \le y \le mn$, $\phi(s_y t_y)$=$10mn + y$; $1 \le y \le mn$, $\phi(s_y k_y)$=$14mn + y$; $1 \le y \le mn$, $\phi(t_y k_y)$= $20mn + 1 - y$; $1 \le y \le mn$, $\phi(s_y s_{y+1})$=$21mn + y$; $1 \le y \le mn$, where $y \not\equiv 0 \pmod{n}$ with the following 3-sided face weights $w(f_1)$=$\phi(s_y)$+ $\phi(b_y)$+ $\phi(a_y)$+ $\phi(s_y b_y)$+ $\phi(s_y a_y)$+ $\phi(b_y a_y)$=$63mn + 4 - 2y$; $1 \le y \le mn$, $w(f_2)$=$\phi(b_y)$+ $\phi(v_y)$+ $\phi(r_y)$+ $\phi(b_y v_y)$+ $\phi(b_y r_y)$+ $\phi(v_y r_y)$=$59mn + 2 + 2y$; $1 \le y \le mn$, $w(f_3)$=$\phi(a_y)$+ $\phi(p_y)$+ $\phi(q_y)$+ $\phi(a_y p_y)$+ $\phi(a_y q_y)$+ $\phi(p_y q_y)$=$57mn+4-2y$; $1 \le y \le mn$ and $w(f_4)$=$\phi(s_y)$+ $\phi(t_y)$+ $\phi(k_y)$+ $\phi(s_y t_y)$+ $\phi(s_y k_y)$+ $\phi(t_y k_y)$=$57mn + 2 + 2y$; $1 \le y \le mn$, which forms an arithmetic progression $\{55mn + 4, 55mn + 4 + 1 \times 2, \ldots, 55mn + 4 + (4mn - 1)2\}$.

**(vi)** Type $(1, 1, 1)$

Consider $\phi : V^{''} \cup E^{''} \cup F^{''} \to \{1, 2, \ldots, 26mn-m\}$. The vertex and edge labeling pattern is same as type (*v*). The face labeling pattern is defined by $\phi(f_{1y})$= $22mn-m+y$; $1 \le y \le mn$, $\phi(f_{2y})$= $24mn-m+1-y$; $1 \le y \le mn$, $\phi(f_{3y})$ =$25mn-m+y$; $1 \le y \le mn$, $\phi(f_{4y})$= $25mn - m + 1 - y$; $1 \le y \le mn$ with the following 3-sided face weights $w(f_1)$= $\phi(s_y)$+ $\phi(b_y)$+ $\phi(a_y)$+ $\phi(s_y b_y)$+ $\phi(s_y a_y)$+ $\phi(b_y a_y)$+ $\phi(f_{1y})$= $85mn - m + 4 - y$; $1 \le y \le mn$, $w(f_2)$= $\phi(b_y)$ + $\phi(v_y)$+ $\phi(r_y)$+ $\phi(b_y v_y)$+ $\phi(b_y r_y)$+ $\phi(v_y r_y)$+ $\phi(f_{2y})$= $83mn - m + 3 + y$; $1 \le y \le mn$, $w(f_3)$=$\phi(a_y)$+ $\phi(p_y)$+ $\phi(q_y)$+ $\phi(a_{uy} p_y)$+ $\phi(a_y q_y)$+ $\phi(p_y q_y)$+ $\phi(f_{3y})$= $82mn - m + 4 - y$; $1 \le y \le mn$ and $w(f_4)$= $\phi(s_y)$+ $\phi(t_y)$+ $v(k_y)$+ $\phi(s_y t_y)$+ $\phi(s_y k_y)$+ $\phi(t_y k_y)$+ $\phi(f_{4y})$= $82mn - m + 3 + y$; $1 \le y \le mn$, which forms an arithmetic progression $\{81mn - m + 4, 81mn - m + 4 + 1 \times 1, \ldots, 81mn - m + 4 + (4mn - 1)1\}$.

□

**Theorem 2.4.** *If a graph $G$ is $(a, d)-$ face antimagic except for $3-$ sided faces, then $DD_{VV}(G)$ is also $(a, d)-$ face antimagic.*

*Proof.* Assume $G$ to be $(a, d)-$ face antimagic with vertex set $V = \{v_i : 1 \le i \le p\}$, edge set $E = \{e_i : 1 \le i \le q\}$ and face set $F = \{f_i : 1 \le i \le f\}$, where $p$, $q$ and $f$ denotes the number of vertices, edges and faces of $G$ respectively. Let $G^{'} = DD_{VV}(G)$ denotes the double

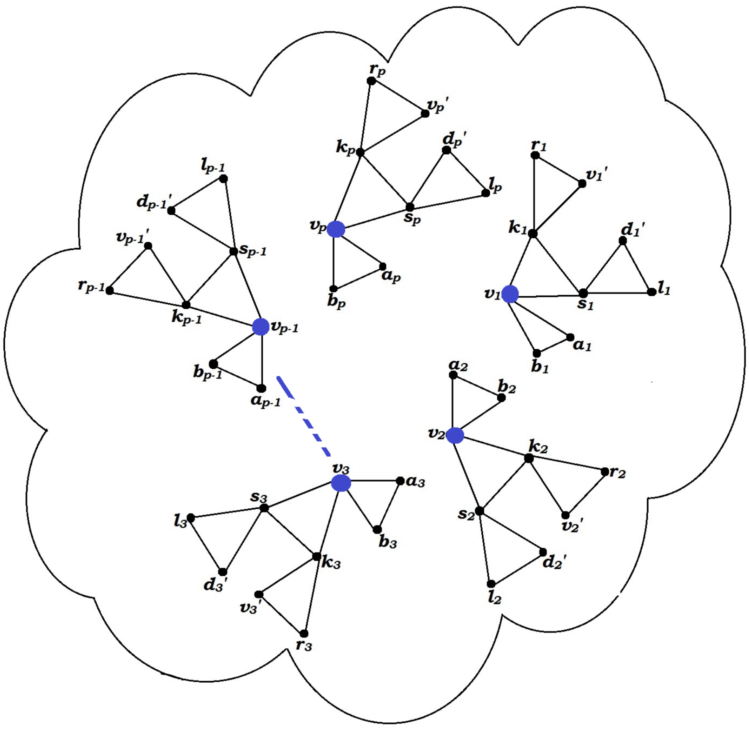

FIGURE 4. $DD_{VV}(G)$

duplications of all vertices by edges of $G$ with vertex set $V' = \{k_i, s_i, r_i, v_i', d_i', l_i, b_i, a_i : 1 \leq i \leq p\} \cup V$, edge set $E' = \{v_ik_i, v_is_i, k_is_i, k_ir_i, k_iv_i', r_iv_I', s_id_i', s_il_i, d_i'l_i, v_ia_i, v_ib_i, b_ia_i : 1 \leq i \leq p\} \cup E$ and face set $F' = \{f_1 : k_ir_iv_i'; f_2 : v_ia_ib_i; f_3 : s_id_i'l_i; f_4 : v_ik_is_i : 1 \leq i \leq p\} \cup F$. The newly added vertices of $G'$ are assigned the following labeling pattern.

Define a mapping $\phi(G') : V'(G') \to \{1, 2, \ldots, 9p\}$ such that $\phi(v_i) = i : 1 \leq i \leq p$, $\phi(r_i) = p + i : 1 \leq i \leq p$, $\phi(d_i') = 2p + i : 1 \leq i \leq p$, $\phi(l_i) = 4p - i + 1 : 1 \leq i \leq p$, $\phi(v_i') = 5p - i + 1 : 1 \leq i \leq p$, $\phi(b_i) = 6p - i + 1 : 1 \leq i \leq p$, $\phi(k_i) = 6p + i : 1 \leq i \leq p$, $\phi(a_i) = 7p + i : 1 \leq i \leq p$ and $\phi(s_i) = 9p - i + 1 : 1 \leq i \leq p$.

To prove $G'$ is $(a, d)-$ face antimagic it is enough to prove $(a, d)-$ face antimagic for newly added triangles. The $3-$ sided face weights of $G'$ are as follows:

$$\begin{aligned} w(f_1) &= \phi(k_i) + \phi(r_i) + \phi(v_i'); 1 \leq i \leq p \\ &= 12p + i + 1; 1 \leq i \leq p \\ w(f_2) &= \phi(v_i) + \phi(a_i) + \phi(b_i); 1 \leq i \leq p \\ &= 13p + i + 1; 1 \leq i \leq p \\ w(f_3) &= \phi(s_i) + \phi(d_i') + \phi(l_i); 1 \leq i \leq p \\ &= 15p - i + 2; 1 \leq i \leq p \\ w(f_4) &= \phi(v_i) + \phi(k_i) + \phi(s_i); 1 \leq i \leq p \\ &= 15p + i + 1; 1 \leq i \leq p \end{aligned}$$

which forms an arithmetic progression $\{12p+2, 12p+2+1\times 1, 12p+2+2\times 1, \ldots, 12p+2+(4p-1)1\}$.

□

## 3. Conclusions

In this paper, $(a, d)-$ face antimagic labeling for various special graphs such as double duplication of all vertices by edges of ladder graph, tadpole graph and $m-$copies of path graph are proved. Also, if G is $(a, d)-$ face antimagic except for $3-$ sided faces, then double duplication of all vertices by edges of G is also $(a, d)-$ face antimagic. In future, the constants $a$ and $d$ from $(a, d)-$ face antimagic labeling can be used to encode and decode the message using hill cipher.

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