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ALTERNATING METHOD FOR SOLVING A BIHARMONIC INVERSE PROBLEM IN DETECTION OF ROBIN COEFFICIENTS

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Abstract. In this paper, we are interested in an inverse problem of detection of corrosion for a biharmonic equation which consists to determine the Robin coefficients in the inaccessible part of the boundary from the Riquiert-Neumann data on the accessible one. For this end, we consider the factorisation of the biharmonic problem which gives rise to two Cauchy problems for Laplace and Poisson equations. An algorithm based on an alternative iterative method is proposed allowing to complete the missing Cauchy data and then recover the Robin coefficients. We show the feasibility of this approach by numerical reconstructions.

Keywords: ill-posed problem, inverse problem, biharmonic equation, finite element method.

AMS Subject Classification: 65N21, 35J05, 31A30.

1. Introduction

The biharmonic equation in two dimensions arises naturally in many applications. It is a well-known example of a mathematical model governing the interior two-dimensional flow of viscous fluids at small Reynolds numbers, i.e., the Stokes flow or the Kirchhoff theory of plates in elasticity, elastic bending beam and determining an unknown boundary [1, 2, 3].

The knowledge of the appropriate boundary conditions across the boundary of the domain results in direct problems of the biharmonic equation which have been widely studied in the literature [4, 5].

The biharmonic problem is defined by their boundary conditions [6]. For the biharmonic equation, the Dirichlet problem is well-known and polynomial solutions were constructed in [7]. Recently, other boundary-value problems were also studied actively such as the

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Riquier problem [8], the Neumann problem [9], the Robin problem [10] and the biharmonic operator under Steklov boundary conditions [11].
Unfortunately, many problems are not part of this category. Indeed, the boundary conditions are often incomplete or in the form of the boundary conditions, under-specified and over-specified on different parts of the boundary, or the solution is prescribed at some internal points in the domain, or some coefficients are not identified [12, 13]. It is the inverse problem known to be generally ill-posed, namely, the existence, uniqueness and stability of their solutions are not always guaranteed [14].
The Laplace inverse problem in detection of Robin coefficient [15] has been subject of several works. Some of them are focused on the study of its ill-posed aspect and results of identifiability are shown in [16, 17]. The stability, i.e. the continuous dependence of the unknown parameter on the measured data are investigated in [18, 19]. In addition, identification algorithms are developed by different authors with different geometry (smooth and non-smooth) with different choice of conditions [20, 21, 22].
Several situations lead to different inverse problems. In particular, we will find the problem in which the conditions on an inaccessible part of the boundary are not known, it is the famous problem of completion of data subject to several studies with different operators. For the data completion problem for the biharmonic equation, five formulations considering different conditions on the boundary are proposed in [23]. In addition, different methods of resolution are proposed in [24, 25, 26]. Another situation arises in the study of the static deflection of a beam in elastic bending which will give rise to an important inverse problem which consists on determining the unknown Robin coefficients from data measured Riquier-Neumann on the accessible part of the boundary [27].
In this paper, we consider the biharmonic inverse problem in detection of Robin coefficients which consist of identifying the two unknown Robin coefficients in the inaccessible part of the boundary from the overspecified conditions in the accessible part of the boundary, namely, the Navier-boundary condition and the measured Riquier-Neumann data. It is an ill-posed problem which requires a regularisation method to be solved. The proposed approach is based on the iterative algorithm developed by Kozlov and al. [28], subject of several studies and used in the resolution of various problems [25, 29, 30, 31], which will initially allow to complete the missing data on the inaccessible part and subsequently the coefficients to be determined will be constructed as being the quotient of the data found. The implementation of the proposed algorithm is carried out by the finite element method.
This paper is organized as follows. Section 2 deals with presenting the mathematical formulation of the inverse problem for the biharmonic equation with the adapted boundary conditions and its corresponding factorized problem. The iterative algorithm is presented in section 3. Section 4 is devoted to numerical results for typical examples of a square domain.

2. Mathematical formulation

2.1. **Inverse problem of detection of Robin coefficients.** Let Ω be a simply connected bounded domain in the plane, with piece-wise smooth boundary $\partial\Omega = \bar{\Gamma}_0 \cup \bar{\Gamma}_1$, where Γ_0 and Γ_1 are two open disjoint portions of the boundary, and u is a solution of the boundary value problem:

$$\Delta^2 u \quad = 0 \quad \text{in} \quad \Omega \tag{1}$$

with Navier-boundary condition on Γ_0, which is in the following form:

$$\left\{\begin{array}{lll} u & = u_0 & \text{on} \quad \Gamma_0 \\ \Delta u & = u_2 & \text{on} \quad \Gamma_0 \end{array}\right. \tag{2}$$

and homogeneous Robin-conditions on Γ_1, which are in the following form:

$$\left\{\begin{array}{lll} \frac{\partial u}{\partial n} + \mu u & = 0 & \text{on} \quad \Gamma_1 \\ \frac{\partial (\Delta u)}{\partial n} + \lambda \Delta u & = 0 & \text{on} \quad \Gamma_1 \end{array}\right. \tag{3}$$

where $(\mu, \lambda) \in (L^\infty(\Gamma_1) \times L^\infty(\Gamma_1))$ and $\mu \geq 0, \lambda \geq 0$, which are two specified functions [32].

When the two coefficients are given, we will have the direct problem where the problem (1) - (3) comes from the study of the static deflection of a beam in elastic bending, where u denotes the transverse deflection of the beam [27].

It is well known [28, 31], that for $(u_0, u_2) \in H^{\frac{3}{2}}(\Gamma_0) \times H^{-\frac{1}{2}}(\Gamma_0)$, there exists a unique solution $u \in H^2(\Omega)$ to the direct problem (1)-(3). Some results of solvability and uniqueness for the biharmonic problem with a class of boundary conditions which generalize in particular the Robin problem, the Dirichlet problem and the require problem can be found in [33].

The biharmonic inverse problem we are concerned with is to determine the functions λ and μ from a given Navier data (u_0, u_2) and the measured Riquier-Neumann data [8]:

$$\left\{\begin{array}{lll} \frac{\partial u}{\partial n} & = u_1 & \text{on} \quad \Gamma_0 \\ \frac{\partial (\Delta u)}{\partial n} & = u_3 & \text{on} \quad \Gamma_0 \end{array}\right. \tag{4}$$

In other words, given Γ_1 and $u_0 \in H^{\frac{3}{2}}(\Gamma_0)$, $u_1 \in H^{\frac{1}{2}}(\Gamma_0), u_2 \in H^{-\frac{1}{2}}(\Gamma_0), u_3 \in H^{-\frac{3}{2}}(\Gamma_0)$, we determine λ and μ such that the unique solution $u \in H^2(\Omega)$ of $(1)-(2)-(3)$ which satisfies $\frac{\partial u}{\partial n}_{/\Gamma_0} = u_1$ and $\frac{\partial \Delta u}{\partial n}_{/\Gamma_0} = u_3$.

Details regarding sobolev spaces can be fund in [27, 34, 35]. In particular :

Let $\Gamma \subset \partial\Omega$ be an open subset of the boundary. We define the traces $u_{/\Gamma}$ and $\frac{\partial u}{\partial n}_{/\Gamma}$ for $u \in H^2(\Omega)$, let $H^{\frac{3}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)$, the trace spaces of $H^2(\Omega)$, and $H^{-\frac{3}{2}}(\partial\Omega), H^{-\frac{1}{2}}(\partial\Omega)$ the dual spaces of $H^{\frac{3}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)$, respectively.

We define the spaces as the following:

$$\begin{aligned} H^{\frac{3}{2}}(\Gamma) &= \left\{ u_{/\Gamma}, u \in H^{\frac{3}{2}}(\partial\Omega) \right\} \\ H^{\frac{1}{2}}(\Gamma) &= \left\{ u_{/\Gamma}, u \in H^{\frac{1}{2}}(\partial\Omega) \right\} \\ \tilde{H}^{\frac{3}{2}}(\Gamma) &= \left\{ u \in H^{\frac{3}{2}}(\Gamma) : \operatorname{Supp} u \subset \bar{\Gamma} \right\} \\ \tilde{H}^{\frac{1}{2}}(\Gamma) &= \left\{ w \in H^{\frac{1}{2}}(\Gamma) : \operatorname{Supp} w \subset \bar{\Gamma} \right\} \end{aligned}$$

Then, we define by $H^{-\frac{3}{2}}(\Gamma)$ the dual space of $\tilde{H}^{\frac{3}{2}}(\Gamma)$. In addition, we can see that $H^{-\frac{1}{2}}(\Gamma)$ is the dual space of $\tilde{H}^{\frac{1}{2}}(\Gamma)$.

2.2. **The factorised problem.** The inverse problem is to find μ and λ from the knowledge of the data u_0, u_1, u_2 and u_3 in the accessible part of the boundary considering two steps:

The first step is to complete the missing data on the inaccessible part Γ_1, which amount to solving data completion problem given by:

$$\begin{cases} \Delta^2 u & = 0 & \text{in} & \Omega \\ u & = u_0 & \text{on} & \Gamma_0 \\ \Delta u & = u_2 & \text{on} & \Gamma_0 \\ \frac{\partial u}{\partial n} & = u_1 & \text{on} & \Gamma_0 \\ \frac{\partial(\Delta u)}{\partial n} & = u_3 & \text{on} & \Gamma_0 \end{cases} \tag{5}$$

where the goal is to get the missing data u, $\frac{\partial u}{\partial n}$, Δu and $\frac{\partial \Delta u}{\partial n}$ on Γ_1.
The second step consists to get the two Robin coefficients that can be obtained by:

$$\begin{array}{lll} \mu(x) = -\frac{\partial u(x)/\partial n}{u(x)}_{/\Gamma_1} & \text{if} & u(x) \neq 0 \\ \lambda(x) = -\frac{\partial \Delta u(x)/\partial n}{\Delta u(x)}_{/\Gamma_1} & \text{if} & \Delta u(x) \neq 0 \end{array}$$

by making the change in the form:

$$\Delta u = \omega \tag{6}$$

Then, the problem becomes:

$$\begin{cases} \Delta u & = \omega & \text{in} & \Omega \\ \Delta \omega & = 0 & \text{in} & \Omega \\ u & = u_0 & \text{on} & \Gamma_0 \\ w & = u_2 & \text{on} & \Gamma_0 \\ \frac{\partial u}{\partial n} & = u_1 & \text{on} & \Gamma_0 \\ \frac{\partial \omega}{\partial n} & = u_3 & \text{on} & \Gamma_0 \end{cases} \tag{7}$$

then,

$$\begin{array}{lll} \mu(x) = -\frac{\partial u(x)/\partial n}{u(x)}_{/\Gamma_1} & \text{if} & u(x) \neq 0 \\ \lambda(x) = -\frac{\partial \omega(x)/\partial n}{\omega(x)}_{/\Gamma_1} & \text{if} & \omega(x) \neq 0 \end{array}$$

.
This change of variable will make it possible to compose the problem in two inverse problems:

$$(P_1) \begin{cases} \Delta \omega = 0 & \text{in} & \Omega \\ \omega = u_2 & \text{on} & \Gamma_0 \\ \frac{\partial \omega}{\partial n} = u_3 & \text{on} & \Gamma_0 \end{cases} \quad \text{and} \quad (P_2) \begin{cases} \Delta u = \omega & \text{in} & \Omega \\ u = u_0 & \text{on} & \Gamma_0 \\ \frac{\partial u}{\partial n} = u_1 & \text{on} & \Gamma_0 \end{cases} \tag{8}$$

with,

$$\lambda(x) = -\frac{\partial \omega(x)/\partial n}{\omega(x)}_{/\Gamma_1} \quad \text{and} \quad \mu(x) = -\frac{\partial u(x)/\partial n}{u(x)}_{/\Gamma_1} \tag{9}$$

with $(P1)$ is an inverse problem for the Laplace equation and $(P2)$ is the inverse problem associated with the Poisson equation which can be solved by the iterative algorithm described in the next section.

3. Description of the iterative procedure

The investigated iterative algorithm consists in constructing iteratively the traces which approximate the data on the part of the boundary Γ_1 of a function which verifies the biharmonic equation in the domain. This algorithm, called also KMF algorithm, is a convergent algorithm for solving elliptic Cauchy problems and is widely used in the case of Cauchy

problems with Laplace's and Poisson's equation by different researchers [29, 30]. In this work, we propose the adaptation of this algorithm to identify the Robin coefficients for an inverse problem for biharmonic equation. This algorithm is based on reducing the ill-posed problem (P1) and (P2) to a sequence of mixed well-posed boundary value problems. Then, we can identify the desired coefficients from the obtained Dirichlet and Neumann conditions in the inaccessible part of the boundary. The proposed algorithm consists of the following steps:
Step 1: Specify an initial guess $w_{0,0}$ and $u_{0,0}$ on Γ_1.
Step 2: Solve the following mixed well-posed boundary value problems:

$$\begin{cases} \Delta\omega^0 = 0 & 1 \text{ in } & \Omega \\ \omega^0 = w_{0,0} & \text{on} & \Gamma_1 \\ \frac{\partial \omega^0}{\partial n} = u_3 & \text{on} & \Gamma_0 \end{cases} \quad \text{and} \quad \begin{cases} \Delta u^0 = \omega^0 & \text{in} & \Omega \\ u^0 = u_{0,0} & \text{on} & \Gamma_1 \\ \frac{\partial u^0}{\partial n} = u_1 & \text{on} & \Gamma_0 \end{cases} \tag{10}$$

$$\text{to obtain } \quad \frac{\partial \omega^0}{\partial n}_{/\Gamma_1} = v_0 \qquad \qquad \text{to obtain } \quad \frac{\partial u^0}{\partial n}_{/\Gamma_1} = h_0$$

$$\text{then} \qquad \lambda_0 = -\frac{v_0}{\omega_{0,0}}_{/\Gamma_1} \qquad \qquad \text{and} \qquad \qquad \mu_0 = -\frac{h_0}{u_{0,0}}_{/\Gamma_1} \tag{11}$$

Step 3: for $k \geq 0$, solve alternatively the following well-posed problems:
i)If the approximation $(u^{(2k)}; \omega^{(2k)})$ is constructed, solve the two mixed well-posed boundary value problems:

$$\begin{cases} \Delta\omega^{(2k+1)} = 0 & \text{in} & \Omega \\ \omega^{(2k+1)} = u_2 & \text{on} & \Gamma_0 \\ \frac{\partial \omega}{\partial n}^{(2k+1)} = v_k & \text{on} & \Gamma_1 \end{cases} \quad \text{and} \quad \begin{cases} \Delta u^{(2k+1)} = \omega^{(2k+1)} & \text{in} & \Omega \\ u^{(2k+1)} = u_0 & \text{on} & \Gamma_0 \\ \frac{\partial u}{\partial n}^{(2k+1)} = h_k & \text{on} & \Gamma_1 \end{cases} \tag{12}$$

$$\text{to obtain } \quad \omega^{(2k+1)}_{/\Gamma_1} = \omega_{0,k+1} \qquad \qquad \text{to obtain } \quad u^{(2k+1)}_{/\Gamma_1} = u_{0,k+1}$$

ii) If the approximation $(u^{(2k+1)}; \omega^{(2k+1)})$ is constructed, solve alternatively the two mixed well-posed boundary value problems:

$$\begin{cases} \Delta\omega^{(2k+2)} = 0 & \text{in} & \Omega \\ \omega^{(2k+2)} = \omega_{0,k+1} & \text{on} & \Gamma_1 \\ \frac{\partial \omega}{\partial n}^{(2k+2)} = u_3 & \text{on} & \Gamma_0 \end{cases} \quad \text{and} \quad \begin{cases} \Delta u^{(2k+2)} = \omega^{(2k+2)} & \text{in} & \Omega \\ u^{(2k+2)} = u_{0,k+1} & \text{on} & \Gamma_1 \\ \frac{\partial u}{\partial n}^{(2k+2)} = u_1 & \text{on} & \Gamma_0 \end{cases} \tag{13}$$

$$\text{to obtain } \quad \frac{\partial \omega^{(2k+2)}}{\partial n}_{/\Gamma_1} = v_{k+1} \qquad \qquad \text{to obtain } \quad \frac{\partial u^{(2k+2)}}{\partial n}_{/\Gamma_1} = h_{k+1}$$

$$\text{then} \qquad \lambda_{k+1} = -\frac{v_{k+1}}{\omega_{0,k+1}}_{/\Gamma_1} \qquad \qquad \text{and} \qquad \qquad \mu_{k+1} = -\frac{h_{k+1}}{u_{0,k+1}}_{/\Gamma_1} \tag{14}$$

Step 4: Repeat step 3 for $k \geq 0$ until a specified stopping criterion is satisfied.

4. Numerical results

The solution domain considered is an example with non-smooth boundary, which is a square $\Omega = (0,1) \times (0,1)$ with a boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$ such that $\Gamma_0 \cap \Gamma_1 = \emptyset$ where; Γ_1 is the under-specified boundary and Γ_0 is the over-specified boundary.
$\Gamma_0 = \{0\} \times (0,1) \cup (0,1) \times \{0\} \cup (0,1) \times \{1\}$
$\Gamma_1 = \{1\} \times (0,1)$

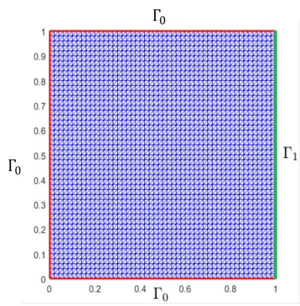


Figure 1. The considered domain

The experiments are done on a intel(R) Core(TM) i5-431OU CPU @ 2.6 GHz machine with 4.00 Go RAM.

4.1. **First example:** The analytical biharmonic function u and the harmonic function ω which verified $\Delta u = \omega$ to be retrieved are given by:

$$u(x,y) = x^3 + y^3 \tag{15}$$

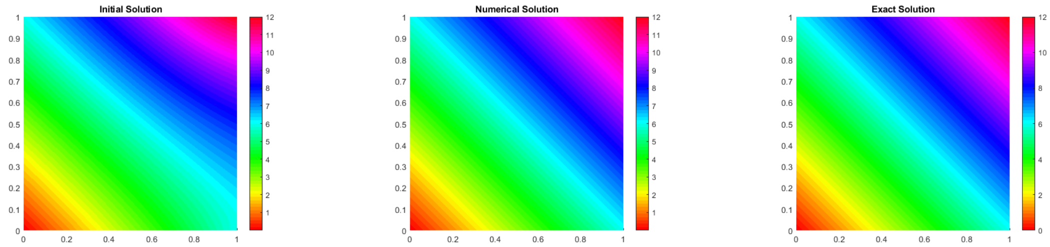
and

$$w(x,y) = 6(x+y) \tag{16}$$

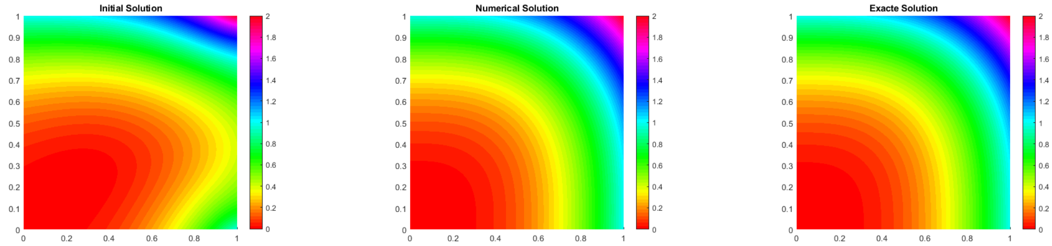
The Robin coefficients to be retrieved are given by:

$$\lambda_{ex} = \frac{-1}{1+y} \quad \text{and} \quad \mu_{ex} = \frac{-3}{1+y^3} \tag{17}$$

We note that u_0, u_1, u_2 and u_3 are known functions and can be calculated easily for the considered typical test example.
It should be noted that several choices of these conditions were considered which led to the same results. In our case, we have taken $u_{0,0} = 3x + 1$ and $w_{0,0} = x + y$.
Figure 2 presents a comparison between the initial solutions, the exact solutions and numerical solutions of u and w showing that we obtain a good numerical approximation for the two functions u and w.
Figure 3 shows the evolution of the two Robin coefficients during the iterative process in comparison with the exact solution. The figure shows that we can obtain a very good approximation even with initial data quite far from the exact solution. In particular, we obtain an approximate solution with an error equal to 8.64 10^{-8} after 100 iterations for the coefficient λ and an approached solution with an error equal to 4.3 10^{-2} after 44 iterations for μ.
The evolution of the error as a function of the number of iterations is presented in figure 4 showing that the results obtained for the coefficient λ are more precise compared to those obtained for the coefficient μ which requires more iterations to have better precision.



(a)



(b)

FIGURE 2. Comparison between the initial solution, numerical solution and exact solution for u (a) and w (b)

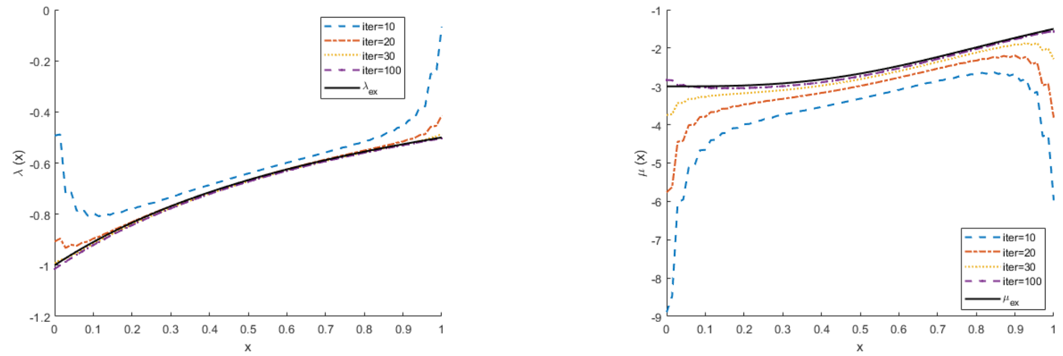


FIGURE 3. Numerical results during iterative process

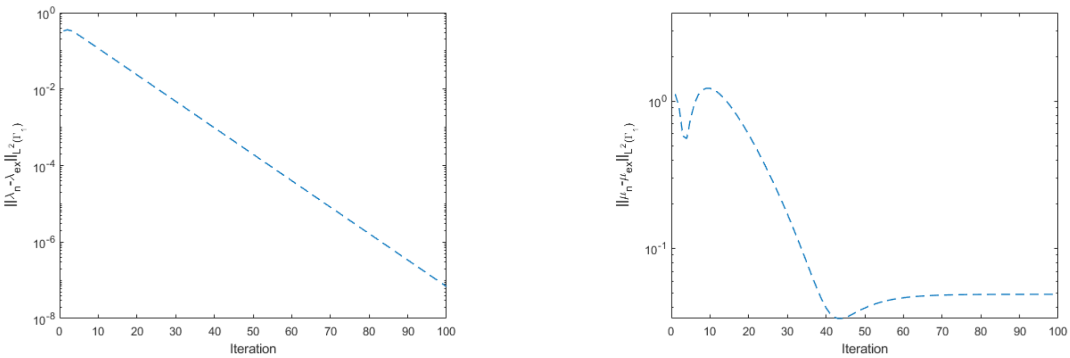


FIGURE 4. Error calculated in function of iterations

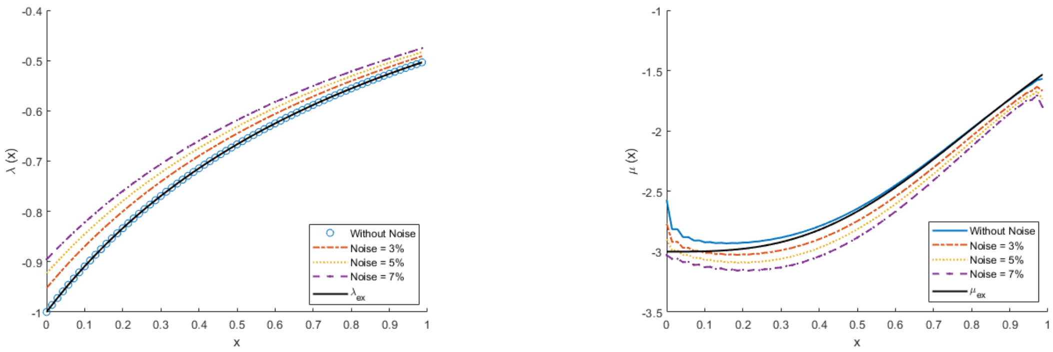


FIGURE 5. The numerical solution for λ and μ on the underspecified boundary for various levels of noise, namely 3%, 5% and 7% in comparison with the exact solution

The boundary data was perturbed with $s = 3\%$ noise in order to simulate the measurement errors. It can be seen that there is a good agreement between the two numerical solutions, and they are both good approximations to the exact solution bearing in mind that we have solved a highly ill-posed problem. Similar results are obtained for various levels of noise added into the input data. Figure 5 presents the numerical soulution obtained for various levels of noise and it can be seen that as the level of noise decreases then the numerical solution approaches the exact solution.

4.2. **Second example:** In this second example, the analytical biharmonic function u and the harmonic function ω which verified $\Delta u = \omega$ to be retrieved are given by:

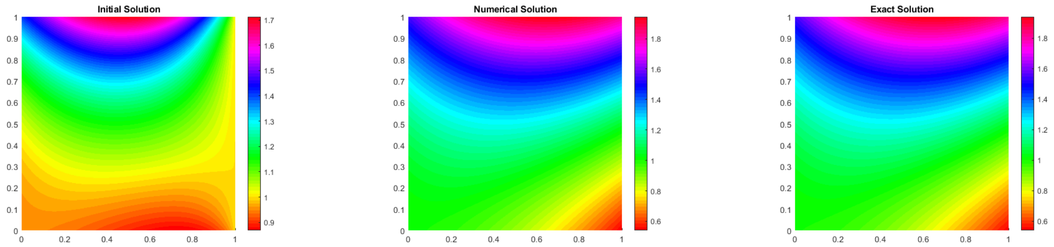
$$u(x, y) = \frac{x \sin(x) \cosh(y) - x \cos(x) \sinh(y)}{2} \tag{18}$$

and

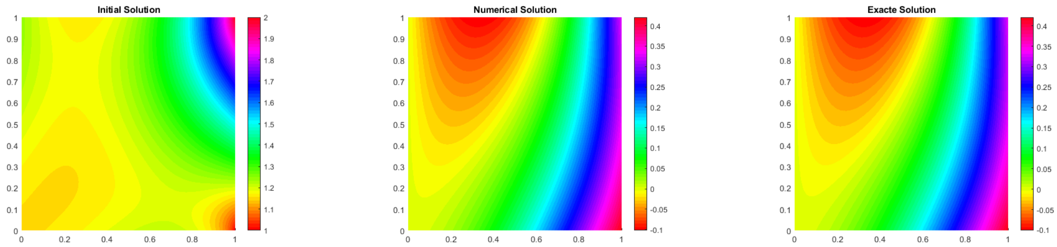
$$w(x, y) = \cos(x) \cosh(y) + \sin(x) \sinh(y) \tag{19}$$

The considered initial guess in this case is given by: $u_{0,0} = x$ and $w_{0,0} = x + y$.

Figure 6 presents a comparison between the initial solutions of u and w in comparison with the exact and numerical solutions showing that we obtain similar numerical approximations for the two solutions u and w.
The results presented in figure 7 shows that starting from initial conditions not necessarily close to the exact solution, we manage to obtain the desired Robin coefficients closer and closer as the process proceeds iteratively. However, figure 8 shows the evolution of the error evaluated at each iteration. For example, we obtain an approximate solution of coefficient λ with an error of 2.51 10^{-2} after 40 iterations and an approximation of coefficient μ with an error of 1.32 10^{-2} after 81 iterations.
The boundary data was perturbed with $s = 3\%$ noise in order to simulate the measurement errors. It can be seen that there is a good agreement between the two numerical solutions, and they are both good approximations to the exact solution bearing in mind that we have solved a highly ill-posed problem.
In figure 9, we present the two numerical solutions with various levels of noise added into the input data. It can be seen that as the level of noise decreases then the numerical solution approaches the exact solution.



(a)



(b)

FIGURE 6. Comparison between the initial solution, numerical solution and exact solution for u (a) and w (b)

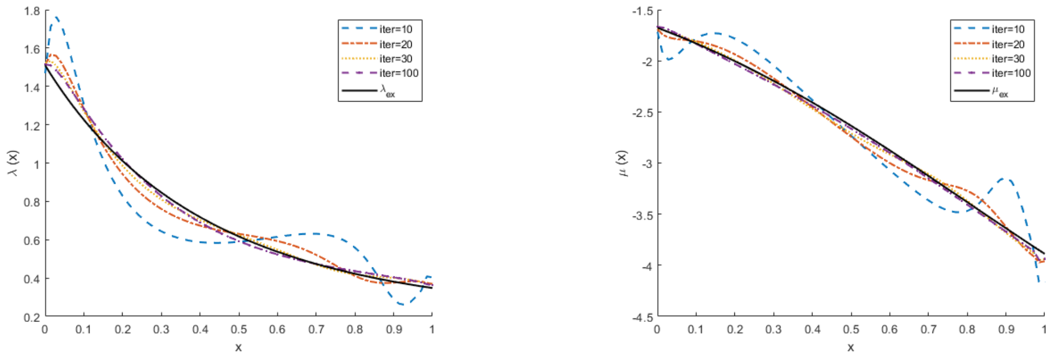


FIGURE 7. Numerical results during iterative process for λ and μ

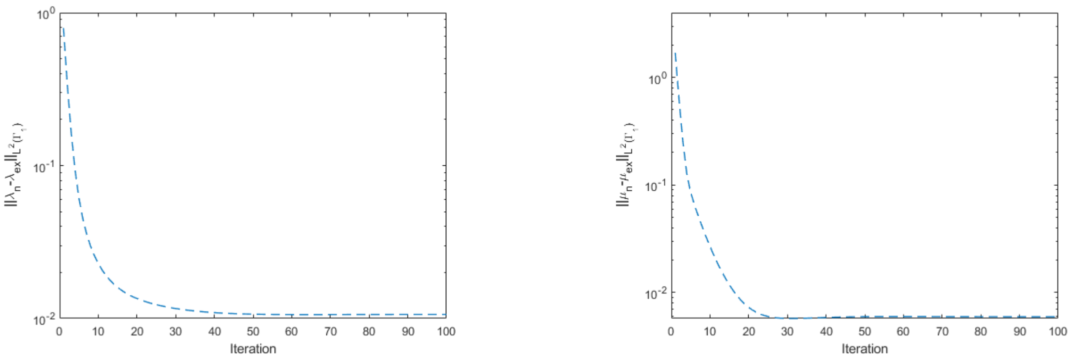


FIGURE 8. Error calculated in function of iterations

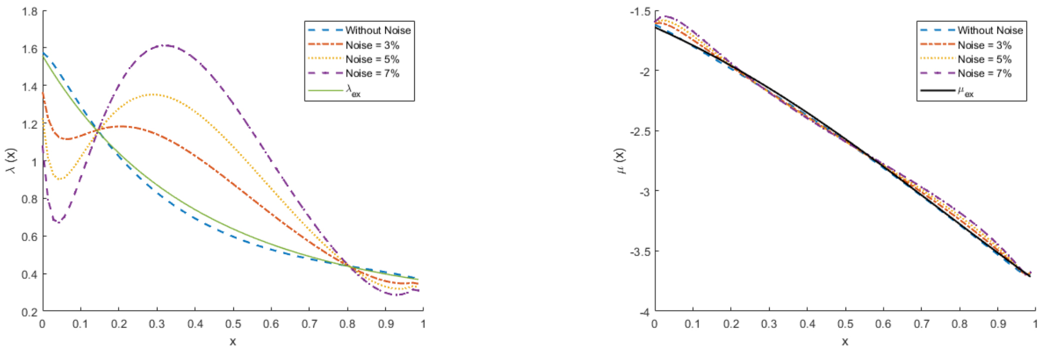


FIGURE 9. The numerical solution for λ and μ on the underspecified boundary for various levels of noise, namely 3%, 5% and 7% in comparison with the exact solution

5. Conclusion

In this paper, an inverse problem of identifying Robin coefficients for the biharmonic equation is considered. The ill-posed aspect of this type of problem led us to consider the regularizing method developed by Kozlov and al. [28] called alternative iterative method to solve it. The method is implemented by the finite element method. The numerical results obtained for various examples in the case of an irregular (square) domain show the efficiency of the algorithm in approaching the coefficients sought in the inaccessible part of the boundary even for disturbed data.
To improve the numerical results obtained, knowing that the KMF algorithm requires a large number of iterations to converge, we can introduce a relaxation procedure and some developed error estimators to adapt the mesh and refine the parts of the domain where the solution can be improved, especially since we are working on an irregular domain.

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