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HYPERBOLIC TYPE HARMONICALLY CONVEX FUNCTION AND INTEGRAL INEQUALITIES

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Abstract. In this paper, we define a new class of harmonic convexity i.e. Hyperbolic type harmonic convexity and explore its algebraic properties. Employing this new definition, some integral inequalities of Hermite-Hadamard type are presented. Furthermore, we have presented Hermite-Hadamard inequality involving Riemann Liouville fractional integral operator. We believe the ideas and techniques of this paper may inspire further research in various branches of pure and applied sciences.

Keywords: Hyperbolic type Convex function,fractional calculus, Hölder integral inequality, Hermite-Hadamard inequality.

AMS Subject Classification: 26A51, 26D10, 26D15.

1. Introduction

In this section, we recall some basic concepts and results, which are useful in proving our results.

Definition 1.1. L*et* $K \subset \mathbb{R} \setminus \{0\}$ *be any interval and* $\varphi : K \to \mathbb{R}$ *be any mapping, then the function* φ *is said to be Harmonic convex function, if*

$$\varphi\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_1 + (1-\ell)\mathtt{d}_2}\right) \leq (1-\ell)\varphi(\mathtt{d}_1) + \ell\varphi(\mathtt{d}_2) \tag{1}$$

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for all $\mathtt{d}_1, \mathtt{d}_2 \in K, \ell \in [0,1]$.

Convexity theory provides powerful principles and techniques to study a wide class of problems in both pure and applied mathematics. Convex functions are significant in the hypothesis of numerical inequalities, some notable outcomes are immediate ramifications of these functions. The ideas of different sorts of new convex functions are developed from the basic definition of a convex function. During the last century, many researchers have contributed in the theory of convexity. The theory of convexity and their generalizations also play a magnificent role in the analysis of extremum problems. For the applications and interesting literature about convex analysis readers refer to [2, 3, 4, 5, 6].

Hermite-Hadamard inequality, which was proved by Hermite in 1883 and Hadamard in 1896 is extensively studied in the convex theory. The said inequality deals with a necessary and sufficient condition for a function to be convex in nature. The classical Hermite-Hadamard inequality is given as:

Consider $\varphi : K \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function with $\mathtt{d}_1 < \mathtt{d}_2$ and $\mathtt{d}_1, \mathtt{d}_2 \in K$. Then

$$\varphi\left(\frac{\mathtt{d}_1+\mathtt{d}_2}{2}\right) \leq \frac{1}{\mathtt{d}_2-\mathtt{d}_1}\int_{\mathtt{d}_1}^{\mathtt{d}_2}\varphi(x)dx \leq \frac{\varphi(\mathtt{d}_1)+\varphi(\mathtt{d}_2)}{2}. \tag{2}$$

İşcan[1] proved the following Hermite-Hadamard type inequality for the harmonically convex function.

Theorem 1.1. [1, Theorem 2.4, page 936] *Let* $K \subseteq (0,\infty)$ *be an interval and* $\varphi : K \to \mathbb{R}$ *be a harmonically convex function with* $\mathtt{d}_1 < \mathtt{d}_2$ *and* $\mathtt{d}_1, \mathtt{d}_2 \in K$. *Then the Hermite-Hadamard type inequality*

$$\varphi\left(\frac{2\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_1+\mathtt{d}_2}\right) \leq \frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\int_{\mathtt{d}_1}^{\mathtt{d}_2}\frac{\varphi(x)}{x^2}dx \leq \frac{\varphi(\mathtt{d}_1)+\varphi(\mathtt{d}_2)}{2}. \tag{3}$$

holds.

Toplu[10] introduced the concept of Hyperbolic type convexity as follows,

Definition 1.2. *A function* $\varphi : K \subseteq \mathbb{R} \to \mathbb{R}$ *is called hyperbolic type convex function if for every* $\mathtt{d}_1, \mathtt{d}_2 \in K$ *and* $\ell \in [0,1]$,

$$\varphi(\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2) \leq \frac{\sinh \ell}{\sinh 1}\varphi(\mathtt{d}_1) + \frac{\sinh 1 - \sinh \ell}{\sinh 1}\varphi(\mathtt{d}_2) \tag{4}$$

Theorem 1.2. [10, Theorem 3.1, page 305.]*Let* $\varphi : [\mathtt{d}_1, \mathtt{d}_2] \to \mathbb{R}$ *be a hyperbolic type convex function. If* $\mathtt{d}_1 < \mathtt{d}_2$ *and* $\varphi \in \mathcal{L}[\mathtt{d}_1, \mathtt{d}_2]$, *then the following Hermite-Hadamard type inequality holds.*

$$\varphi\left(\frac{\mathtt{d}_1+\mathtt{d}_2}{2}\right) \leq \frac{1}{\mathtt{d}_2-\mathtt{d}_1}\int_{\mathtt{d}_1}^{\mathtt{d}_2}\varphi(x)dx \leq \frac{\cosh 1 - 1}{\sinh 1}\varphi(\mathtt{d}_1) + \frac{e-1}{e\sinh 1}\varphi(\mathtt{d}_2). \tag{5}$$

Recently, many extensions/generalisations have been performed for the classical convex functions see[8], [9], [10], [11], [12], [13], [14] and the references therein. Some parts of these new ideas depend on expansion of the area of a convex function. Some newly developed theories in this aspect are GA-convex functions, Hyperbolic type convex functions, tgs convex functions, s-convex functions, invariant convex functions, $M_\varphi A$-convex functions, n-polynomial convex functions, preinvex functions, (η_1, η_2)-convex functions, h-convex functions etc. As there are huge applications of Hermite-Hadamard type inequalities, researchers put innovative ideas to study Hermite-Hadamard type inequalities involving integer integrals as well as fractional integrals. Very Recently, Hermite-Hadamard inequalities involving integrals have been obtained for different classes of convex functions; refer

[15], [16], [17], [18], [19] out of which Hyperbolic type convexity [10] provides the clue to develop the new concept of Hyperbolic type harmonic convexity.

Motivated by the results of hyperbolic type convex functions, the main purpose of the present paper is to establish new Hermite-Hadamard type integral inequalities for hyperbolic type harmonic convex functions. We have also used fractional calculus to prove Hermite-Hadamard type integral inequalities for Hyperbolic type harmonic convex functions.

Inspired by the advancement of theory of fractional calculus, ongoing research and literature about integral inequality and convexity, the present paper is structured in the following way, first in Section 1, we will give some necessary known definitions and literature. Second in Section 2, we will explore the concept of Hyperbolic type harmonically convex function. In addition, some algebraic properties for the newly introduced definition are elaborated. In Section 3, applying this we present Hermite–Hadamard type inequality and its refinements . Further, in Section 4, we investigate some novel refinements of the Hermite-Hadamard type inequality via Riemann-Liouville fractional integral operator.

2. Hyperbolic type Harmonically convexity

In this section, we introduce many new classes of hyperbolic type harmonic convex function and some basic properties of the function.

Definition 2.1. *Let $K \subseteq \mathbb{R}$ be an interval, then a real valued function $\varphi : K \to \mathbb{R}$ is called hyperbolic type harmonic convex function if for every $\mathtt{d}_1, \mathtt{d}_2 \in K$ and $\ell \in [0,1]$, the inequality*

$$\varphi\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2}\right) \leq \frac{\sinh\ell}{\sinh 1}\varphi(\mathtt{d}_2) + \frac{\sinh 1-\sinh\ell}{\sinh 1}\varphi(\mathtt{d}_1) \tag{6}$$

holds.

Definition 2.2. *Let $h : [0,1] \longrightarrow \mathbb{R}$ and $K \subseteq \mathbb{R}$ be an interval, then a real valued function $\varphi : K \to \mathbb{R}$ is called hyperbolic type harmonic h-convex function if for every $\mathtt{d}_1, \mathtt{d}_2 \in K$ and $\ell \in [0,1]$, the inequality*

$$\varphi\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2}\right) \leq h\left(\frac{\sinh\ell}{\sinh 1}\right)\varphi(\mathtt{d}_2) + h\left(\frac{\sinh 1-\sinh\ell}{\sinh 1}\right)\varphi(\mathtt{d}_1) \tag{7}$$

holds.

Remark 2.1. *If $h(\ell) = \ell$ then Definition 2.2 reduces to Definition 2.1.*

If $h(\ell) = \ell^s$ in Definition 2.2, then we have the following definition.

Definition 2.3. *Let $K \subseteq \mathbb{R}$ be an interval, then a real valued function $\varphi : K \to \mathbb{R}$ is called hyperbolic type harmonic s-convex function if for every $\mathtt{d}_1, \mathtt{d}_2 \in K$ and $\ell \in [0,1]$, the inequality*

$$\varphi\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2}\right) \leq \left(\frac{\sinh\ell}{\sinh 1}\right)^s\varphi(\mathtt{d}_2) + \left(\frac{\sinh 1-\sinh\ell}{\sinh 1}\right)^s\varphi(\mathtt{d}_1) \tag{8}$$

holds.

If $h(\ell) = \ell^{-s}$ in Definition (2.2), then we have the following definition.

Definition 2.4. *Let $K \subseteq \mathbb{R}$ be an interval, then a real valued function $\varphi : K \to \mathbb{R}$ is called hyperbolic type Godunova-Levin type harmonic s-convex function if for every $\mathtt{d}_1, \mathtt{d}_2 \in K$ and $\ell \in (0,1)$, the inequality*

$$\varphi\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2}\right) \leq \frac{1}{\left(\frac{\sinh\ell}{\sinh 1}\right)^s}\varphi(\mathtt{d}_2) + \frac{1}{\left(\frac{\sinh 1-\sinh\ell}{\sinh 1}\right)^s}\varphi(\mathtt{d}_1) \tag{9}$$

holds.

If $h(\ell)=\ell^{-1}$ in Definition (2.2) then we have the following definition,

Definition 2.5. *Let $K\subseteq\mathbb{R}$ be an interval, then a real valued function $\varphi:K\to\mathbb{R}$ is called hyperbolic type godunova-Levin type harmonic convex function if for every $\mathtt{d}_1,\mathtt{d}_2\in K$ and $\ell\in(0,1)$, the inequality*

$$\varphi\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2}\right)\leq\frac{1}{\frac{\sinh\ell}{\sinh 1}}\varphi(\mathtt{d}_2)+\frac{1}{\frac{\sinh 1-\sinh\ell}{\sinh 1}}\varphi(\mathtt{d}_1) \tag{10}$$

holds.

Proposition 2.1. *Consider φ and ψ be two real valued hyperbolic type harmonic convex functions and consider $\varphi,\psi:[\mathtt{d}_1,\mathtt{d}_2]\to\mathbb{R}$, then*
i. $\varphi+\psi$ is hyperbolic type harmonic convex function.
ii. For $c\in\mathbb{R}$ $(c\geq 0)$, the function $c\varphi$ is hyperbolic type harmonic convex function.

Proof. (i) Let φ and ψ be two hyperbolic type harmonic convex functions, then

$$\begin{aligned}
&(\varphi+\psi)\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2}\right)=\varphi\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2}\right)+\psi\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2}\right)\\
&\leq\frac{\sinh\ell}{\sinh 1}\varphi(\mathtt{d}_2)+\frac{\sinh 1-\sinh\ell}{\sinh 1}\varphi(\mathtt{d}_1)+\frac{\sinh\ell}{\sinh 1}\psi(\mathtt{d}_2)+\frac{\sinh 1-\sinh\ell}{\sinh 1}\psi(\mathtt{d}_1)\\
&=\frac{\sinh\ell}{\sinh 1}[\varphi(\mathtt{d}_2)+\psi(\mathtt{d}_2)]+\frac{\sinh 1-\sinh\ell}{\sinh 1}[\varphi(\mathtt{d}_1)+\psi(\mathtt{d}_1)]\\
&=\frac{\sinh\ell}{\sinh 1}(\varphi+\psi)(\mathtt{d}_2)+\frac{\sinh 1-\sinh\ell}{\sinh 1}(\varphi+\psi)(\mathtt{d}_1).
\end{aligned}$$

□

(ii) Let φ be hyperbolic type harmonic convex functions and $c\in\mathbb{R}$ $(c\geq 0)$, then

$$\begin{aligned}
(c\varphi)\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2}\right)&\leq c\left(\frac{\sinh\ell}{\sinh 1}\varphi(\mathtt{d}_2)+\frac{\sinh 1-\sinh\ell}{\sinh 1}\varphi(\mathtt{d}_1)\right)\\
&=\frac{\sinh\ell}{\sinh 1}c\varphi(\mathtt{d}_2)+\frac{\sinh 1-\sinh\ell}{\sinh 1}c\varphi(\mathtt{d}_1)\\
&=\frac{\sinh\ell}{\sinh 1}(c\varphi)(\mathtt{d}_2)+\frac{\sinh 1-\sinh\ell}{\sinh 1}(c\varphi)(\mathtt{d}_1).
\end{aligned}$$

Proposition 2.2. *If $\varphi:K\to K$ is a hyperbolic type harmonic convex and $\psi:K\to\mathbb{R}$ is a non-decreasing convex function , then $\psi\circ\varphi:K\to\mathbb{R}$ is a hyperbolic type harmonic convex function.*

Proof. For $\mathtt{d}_1,\mathtt{d}_2\in K$ and $k\in[0,1]$

$$\begin{aligned}
\psi\circ\varphi\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2}\right)&=\psi\left(\frac{\sinh\ell}{\sinh 1}\varphi(\mathtt{d}_2)+\frac{\sinh 1-\sinh k}{\sinh 1}\varphi(\mathtt{d}_1)\right)\\
&\leq\frac{\sinh\ell}{\sinh 1}\psi(\varphi(\mathtt{d}_2))+\frac{\sinh 1-\sinh k}{\sinh 1}\psi(\varphi(\mathtt{d}_1))\\
&\leq\frac{\sinh\ell}{\sinh 1}\psi\circ\varphi(\mathtt{d}_2)+\frac{\sinh 1-\sinh k}{\sinh 1}\psi\circ\varphi(\mathtt{d}_1).
\end{aligned}$$

□

Definition 2.6. *Two functions u and v are said to be of similar ordered if*

$$(u(\mathtt{d}_1)-u(\mathtt{d}_2))(v(\mathtt{d}_1)-v(\mathtt{d}_2))\geq 0,\ \forall\mathtt{d}_1,\mathtt{d}_2\in\mathbb{R}.$$

Proposition 2.3. *let φ and ψ be two similar ordered hyperbolic type harmonic convex function, then the product is also hyperbolic type harmonic convex function.*

Proof. Let φ and ψ be two hyperbolic type harmonic convex function, then

$$\begin{aligned}
&\varphi\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2}\right)\psi\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2}\right)\\
&\leq \left[\frac{\sinh\ell}{\sinh 1}\varphi(\mathtt{d}_2)+\frac{\sinh 1-\sinh\ell}{\sinh 1}\varphi(\mathtt{d}_1)\right]\left[\frac{\sinh\ell}{\sinh 1}\psi(\mathtt{d}_2)+\frac{\sinh 1-\sinh\ell}{\sinh 1}\psi(\mathtt{d}_1)\right]\\
&=\left(\frac{\sinh\ell}{\sinh 1}\right)^2\varphi(\mathtt{d}_2)\psi(\mathtt{d}_2)+\left(\frac{\sinh 1-\sinh\ell}{\sinh 1}\right)^2\varphi(\mathtt{d}_1)\psi(\mathtt{d}_1)\\
&+\left(\frac{\sinh\ell}{\sinh 1}\right)\left(\frac{\sinh 1-\sinh\ell}{\sinh 1}\right)[\psi(\mathtt{d}_2)\varphi(\mathtt{d}_1)+\varphi(\mathtt{d}_2)\psi(\mathtt{d}_1)]\\
&=\left[\frac{\sinh\ell}{\sinh 1}\varphi(\mathtt{d}_2)\psi(\mathtt{d}_2)+\frac{\sinh 1-\sinh\ell}{\sinh 1}\varphi(\mathtt{d}_1)\psi(\mathtt{d}_1)\right]\left[\frac{\sinh\ell}{\sinh 1}+\frac{\sinh 1-\sinh\ell}{\sinh 1}\right]\\
&=\frac{\sinh\ell}{\sinh 1}\varphi(\mathtt{d}_2)\psi(\mathtt{d}_2)+\frac{\sinh 1-\sinh\ell}{\sinh 1}\varphi(\mathtt{d}_1)\psi(\mathtt{d}_1).
\end{aligned}$$

Hence it is proved that the product of two similarly ordered hyperbolic type harmonic convex functions is also a hyperbolic type harmonic convex function. ☐

3. Hermite-Hadamard type Inequalities

Theorem 3.1. *Let $0<\mathtt{d}_1<\mathtt{d}_2$ and $\varphi: K\subseteq\mathbb{R}\to\mathbb{R}$ be a hyperbolic type harmonic convex function and $\mathtt{d}_1,\mathtt{d}_2\in K$. If $\varphi\in\mathcal{L}[\mathtt{d}_1,\mathtt{d}_2]$, then the following inequality holds.*

$$\varphi\left(\frac{2\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_1+\mathtt{d}_2}\right)\leq\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\int_{\mathtt{d}_1}^{\mathtt{d}_2}\frac{\varphi(x)}{x^2}dx\leq\frac{\cosh 1-1}{\sinh 1}\varphi(\mathtt{d}_2)+\frac{e-1}{e\sinh 1}\varphi(\mathtt{d}_1). \tag{11}$$

Proof. Since 'φ' is hyperbolic type harmonic convex function putting $k=\frac{1}{2}$ and choosing $x=\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2}$ and $y=\frac{\mathtt{d}_1\mathtt{d}_2}{(1-\ell)\mathtt{d}_1+k\mathtt{d}_2}$ in

$$\varphi\left(\frac{xy}{\ell x+(1-\ell)y}\right)\leq\frac{\sinh\ell}{\sinh 1}\varphi(y)+\frac{\sinh 1-\sinh\ell}{\sinh 1}\varphi(x),$$

we get

$$\varphi\left(\frac{2\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_1+\mathtt{d}_2}\right)\leq\frac{\sinh\frac{1}{2}}{\sinh 1}\varphi\left(\frac{\mathtt{d}_1\mathtt{d}_2}{(1-k)\mathtt{d}_1+k\mathtt{d}_2}\right)+\frac{\sinh 1-\sinh\frac{1}{2}}{\sinh 1}\varphi\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2}\right).$$

Integrating the above inequality with respect to k over $[0,1]$, we obtain

$$\begin{aligned}
\varphi\left(\frac{2\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_1+\mathtt{d}_2}\right)&\leq\frac{\sinh\frac{1}{2}}{\sinh 1}\int_0^1\varphi\left(\frac{\mathtt{d}_1\mathtt{d}_2}{(1-k)\mathtt{d}_1+k\mathtt{d}_2}\right)d\ell\\
&+\frac{\sinh 1-\sinh\frac{1}{2}}{\sinh 1}\int_0^1\varphi\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2}\right)d\ell\\
&=\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\int_{\mathtt{d}_1}^{\mathtt{d}_2}\frac{\varphi(x)}{x^2}dx.
\end{aligned}$$

Now, using the property of Hyperbolic type harmonic convex function and letting $x = \frac{\mathtt{d}_1\mathtt{d}_2}{t\mathtt{d}_1+(1-t)\mathtt{d}_2}$ then, we have

$$\begin{aligned}\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\int_{\mathtt{d}_1}^{\mathtt{d}_2}\frac{\varphi(x)}{x^2}dx &= \int_0^1 \varphi\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2}\right)d\ell\\ &\leq \int_0^1 \frac{\sinh\ell}{\sinh 1}\varphi(\mathtt{d}_2)+\frac{\sinh 1-\sinh\ell}{\sinh 1}\varphi(\mathtt{d}_1)d\ell\\ &= \frac{\cosh 1-1}{\sinh 1}\varphi(\mathtt{d}_2)+\frac{e-1}{e\sinh 1}\varphi(\mathtt{d}_1),\end{aligned}$$

where

$$\int_0^1 \sinh\ell d\ell = \cosh 1 - 1$$

and

$$\int_0^1 (\sinh 1-\sinh\ell)d\ell = \frac{e-1}{e}.$$

□

For our main results we need the following lemma.

Lemma 3.1. [1] *Let $\varphi : K \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable mapping on K° and $\mathtt{d}_1, \mathtt{d}_2 \in K$ with $\mathtt{d}_2 > \mathtt{d}_1$ If $\varphi' \in \mathcal{L}[\mathtt{d}_1, \mathtt{d}_2]$, then the following identity holds:*

$$\begin{aligned}&\frac{\varphi(\mathtt{d}_1)+\varphi(\mathtt{d}_2)}{2}-\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\int_{\mathtt{d}_1}^{\mathtt{d}_2}\frac{\varphi(x)}{x^2}dx\\ &= \frac{\mathtt{d}_1\mathtt{d}_2(\mathtt{d}_2-\mathtt{d}_1)}{2}\int_0^1\frac{(1-2\ell)}{(\ell\mathtt{d}_2+(1-\ell)\mathtt{d}_1)^2}\varphi'\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_2+(1-\ell)\mathtt{d}_1}\right)d\ell. \qquad (12)\end{aligned}$$

Theorem 3.2. *Let $\varphi : K \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on K° and $\varphi' \in L([\mathtt{d}_1, \mathtt{d}_2])$, where $[\mathtt{d}_1, \mathtt{d}_2] \subseteq K^\circ$. If $|\varphi'|$ is hyperbolic type harmonic convex function on $[\mathtt{d}_1, \mathtt{d}_2]$, then the inequality*

$$\left|\frac{\varphi(\mathtt{d}_1)+\varphi(\mathtt{d}_2)}{2}-\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\int_{\mathtt{d}_1}^{\mathtt{d}_2}\frac{\varphi(x)}{x^2}dx\right| \leq \frac{\mathtt{d}_1\mathtt{d}_2(\mathtt{d}_2-\mathtt{d}_1)}{2}\left[\frac{|\varphi'(\mathtt{d}_1)|}{\sinh 1}B+\frac{|\varphi'(\mathtt{d}_2)|}{\sinh 1}C\right]. \quad (13)$$

holds, where

$$\begin{aligned}A &= \int_0^1 \frac{|1-2\ell|}{(\ell\mathtt{d}_2+(1-\ell)\mathtt{d}_1)^2}d\ell\\ B &= \int_0^1 \frac{|1-2\ell|\sinh\ell}{\sinh 1(\ell\mathtt{d}_2+(1-\ell)\mathtt{d}_1)^2}d\ell\\ C &= \int_0^1 \frac{|1-2\ell|\,(\sinh 1-\sinh\ell)}{\sinh 1(\ell\mathtt{d}_2+(1-\ell)\mathtt{d}_1)^2}d\ell.\end{aligned}$$

Proof. From Lemma 3.1 , we get

$$\begin{aligned}&\frac{\varphi(\mathtt{d}_1)+\varphi(\mathtt{d}_2)}{2}-\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\int_{\mathtt{d}_1}^{\mathtt{d}_2}\frac{\varphi(x)}{x^2}dx\\ &\leq \frac{\mathtt{d}_1\mathtt{d}_2(\mathtt{d}_2-\mathtt{d}_1)}{2}\int_0^1\left[\frac{1-2\ell}{(\ell\mathtt{d}_2+(1-\ell)\mathtt{d}_1)^2}\varphi'\left(\frac{\mathtt{d}_1\mathtt{d}_2}{(\ell\mathtt{d}_2+(1-\ell)\mathtt{d}_1)}\right)\right]d\ell.\end{aligned}$$

Now, using the concept of Hyperbolic type harmonic convexity of φ',

$$\left|\frac{\varphi(\mathtt{d}_1)+\varphi(\mathtt{d}_2)}{2}-\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\int_{\mathtt{d}_1}^{\mathtt{d}_2}\frac{\varphi(x)}{x^2}dx\right|$$
$$\leq \frac{\mathtt{d}_1\mathtt{d}_2(\mathtt{d}_2-\mathtt{d}_1)}{2}\int_0^1 \frac{|1-2\ell|}{(\ell\mathtt{d}_2+(1-\ell)\mathtt{d}_1)^2}\left[\frac{\sinh\ell}{\sinh 1}\left|\varphi'(\mathtt{d}_1)\right|+\frac{\sinh 1-\sinh\ell}{\sinh 1}\left|\varphi'(\mathtt{d}_2)\right|\right]d\ell$$
$$\leq \frac{\mathtt{d}_1\mathtt{d}_2(\mathtt{d}_2-\mathtt{d}_1)}{2}\left[\frac{\varphi'(\mathtt{d}_1)}{\sinh 1}\int_0^1 \frac{|1-2\ell|\sinh\ell}{(\ell\mathtt{d}_2+(1-\ell)\mathtt{d}_1)^2}d\ell+\frac{\varphi'(\mathtt{d}_2)}{\sinh 1}\int_0^1 \frac{|1-2\ell|\left(\sinh 1-\sinh\ell\right)}{(\ell\mathtt{d}_2+(1-\ell)\mathtt{d}_1)^2}d\ell\right]$$
$$\leq \frac{\mathtt{d}_1\mathtt{d}_2(\mathtt{d}_2-\mathtt{d}_1)}{2}\left[\frac{|\varphi'(\mathtt{d}_1)|}{\sinh 1}B+\frac{|\varphi'(\mathtt{d}_2)|}{\sinh 1}C\right].$$

□

Theorem 3.3. *Let $\varphi : K \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on K° and $\varphi' \in L\left([\mathtt{d}_1,\mathtt{d}_2]\right)$, where $[\mathtt{d}_1,\mathtt{d}_2] \subseteq K^\circ$. If $|\varphi'|^q$ is hyperbolic type harmonic convex function on $[\mathtt{d}_1,\mathtt{d}_2]$ for $q \geq 1$, then the inequality*

$$\left|\frac{\varphi(\mathtt{d}_1)+\varphi(\mathtt{d}_2)}{2}-\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\int_{\mathtt{d}_1}^{\mathtt{d}_2}\frac{\varphi(x)}{x^2}dx\right| \leq \frac{\mathtt{d}_1\mathtt{d}_2(\mathtt{d}_2-\mathtt{d}_1)}{2}A^{1-\frac{1}{q}}\left[B|\varphi'(\mathtt{d}_1)|^q+C|\varphi'(\mathtt{d}_2)|^q\right]^{\frac{1}{q}}. \tag{14}$$

holds.

Proof. From Lemma 3.1 and using the Hölder's inequality, we get

$$\left|\frac{\varphi(\mathtt{d}_1)+\varphi(\mathtt{d}_2)}{2}-\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\int_{\mathtt{d}_1}^{\mathtt{d}_2}\frac{\varphi(x)}{x^2}dx\right|$$
$$\leq \frac{\mathtt{d}_1\mathtt{d}_2(\mathtt{d}_2-\mathtt{d}_1)}{2}\left(\int_0^1\left|\frac{|1-2\ell|}{(\ell\mathtt{d}_2+(1-\ell)\mathtt{d}_1)^2}d\ell\right|\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left|\frac{|1-2\ell|}{(\ell\mathtt{d}_2+(1-\ell)\mathtt{d}_1)^2}\right|\left|\varphi'\left(\frac{\mathtt{d}_1\mathtt{d}_2}{(\ell\mathtt{d}_2+(1-\ell)\mathtt{d}_1)}\right)\right|^q d\ell\right)^{\frac{1}{q}}.$$

Now using the concept of Hyperbolic type harmonic convexity of $|\varphi'|^q$, we get

$$\left|\frac{\varphi(\mathtt{d}_1)+\varphi(\mathtt{d}_2)}{2}-\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\int_{\mathtt{d}_1}^{\mathtt{d}_2}\frac{\varphi(x)}{x^2}dx\right|$$
$$\leq \frac{\mathtt{d}_1\mathtt{d}_2(\mathtt{d}_2-\mathtt{d}_1)}{2}\left(\int_0^1\frac{|1-2\ell|}{(\ell\mathtt{d}_2+(1-\ell)\mathtt{d}_1)^2}d\ell\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\frac{|1-2\ell|}{(\ell\mathtt{d}_2+(1-\ell)\mathtt{d}_1)^2}\left[\frac{\sinh\ell}{\sinh 1}\left|\varphi'(\mathtt{d}_1)\right|^q+\frac{\sinh 1-\sinh\ell}{\sinh 1}\left|\varphi'(\mathtt{d}_2)\right|^q\right]d\ell\right)^{\frac{1}{q}}$$
$$\leq \frac{\mathtt{d}_1\mathtt{d}_2(\mathtt{d}_2-\mathtt{d}_1)}{2}A^{1-\frac{1}{q}}\left[B|\varphi'(\mathtt{d}_1)|^q+C|\varphi'(\mathtt{d}_2)|^q\right]^{\frac{1}{q}}.$$

□

Theorem 3.4. *Let $\varphi : K \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on K° and $\varphi' \in L\left([\mathtt{d}_1,\mathtt{d}_2]\right)$, where $[\mathtt{d}_1,\mathtt{d}_2] \subseteq K^\circ$. If $|\varphi'|^q$ is hyperbolic type harmonic convex function on $[\mathtt{d}_1,\mathtt{d}_2]$ for $q \geq 1, \frac{1}{p}+\frac{1}{q}=1$ then the inequality.*

$$\left|\frac{\varphi(\mathtt{d}_1)+\varphi(\mathtt{d}_2)}{2}-\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\int_{\mathtt{d}_1}^{\mathtt{d}_2}\frac{\varphi(x)}{x^2}dx\right|$$
$$\leq \frac{\mathtt{d}_1\mathtt{d}_2(\mathtt{d}_2-\mathtt{d}_1)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(D\frac{|\varphi'(\mathtt{d}_1)|^q-|\varphi'(\mathtt{d}_2)|^q}{\sinh 1}+\left|\varphi'(\mathtt{d}_2)\right|^q L_{-2q}^{-2q}(\mathtt{d}_1,\mathtt{d}_2)\right)^{\frac{1}{q}}. \quad (15)$$

Where,

$$D=\int_0^1\frac{\sinh\ell}{(\ell\mathtt{d}_2+(1-\ell)\mathtt{d}_1)^{2q}}d\ell$$

Proof. From Lemma 3.1 and using the Hölder's Integral inequality, we get

$$\left|\frac{\varphi(\mathtt{d}_1)+\varphi(\mathtt{d}_2)}{2}-\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\int_{\mathtt{d}_1}^{\mathtt{d}_2}\frac{\varphi(x)}{x^2}dx\right|$$
$$\leq \frac{\mathtt{d}_1\mathtt{d}_2(\mathtt{d}_2-\mathtt{d}_1)}{2}\left(\int_0^1|1-2\ell|^p\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\frac{1}{(\ell\mathtt{d}_2+(1-\ell)\mathtt{d}_1)^{2q}}\left|\varphi'\left(\frac{\mathtt{d}_1\mathtt{d}_2}{(\ell\mathtt{d}_2+(1-\ell)\mathtt{d}_1)}\right)\right|^q d\ell\right)^{\frac{1}{q}}.$$

Now using the concept of Hyperbolic type harmonic convexity of $|\varphi'|^q$, we obtain

$$\left|\frac{\varphi(\mathtt{d}_1)+\varphi(\mathtt{d}_2)}{2}-\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\int_{\mathtt{d}_1}^{\mathtt{d}_2}\frac{\varphi(x)}{x^2}dx\right|$$
$$\leq \frac{\mathtt{d}_1\mathtt{d}_2(\mathtt{d}_2-\mathtt{d}_1)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|\varphi'(\mathtt{d}_1)|^q-|\varphi'(\mathtt{d}_2)|^q}{\sinh 1}\right.$$
$$\left.\times\int_0^1\frac{\sinh\ell}{(\ell\mathtt{d}_2+(1-\ell)\mathtt{d}_1)^{2q}}d\ell+\left|\varphi'(\mathtt{d}_2)\right|^q\int_0^1\frac{1}{(\ell\mathtt{d}_2+(1-\ell)\mathtt{d}_1)^{2q}}d\ell\right)^{\frac{1}{q}}$$
$$=\frac{\mathtt{d}_1\mathtt{d}_2(\mathtt{d}_2-\mathtt{d}_1)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(D\frac{|\varphi'(\mathtt{d}_1)|^q-|\varphi'(\mathtt{d}_2)|^q}{\sinh 1}+\left|\varphi'(\mathtt{d}_2)\right|^q L_{-2q}^{-2q}(\mathtt{d}_1,\mathtt{d}_2)\right)^{\frac{1}{q}}.$$

□

Theorem 3.5. *Let φ and ψ be two real valued hyperbolic type harmonic convex function, then*

$$\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\int_{\mathtt{d}_1}^{\mathtt{d}_2}\frac{\varphi(u)g(u)}{u^2}du \quad (16)$$
$$\leq\left(\frac{e^4-4e^2-1}{8e^2\sinh^2 1}\right)\varphi(\mathtt{d}_2)\psi(\mathtt{d}_2)+\left(\frac{-e^4+8e^3-8e^2-8e+5}{8e^2\sinh^2 1}\right)\varphi(\mathtt{d}_1)\psi(\mathtt{d}_1)$$
$$+\left(\frac{e^4-4e^3+4e^2+4e-1}{8e^2\sinh^2 1}\right)\left[\varphi(\mathtt{d}_1)\psi(\mathtt{d}_2)+\varphi(\mathtt{d}_2)\psi(\mathtt{d}_1)\right].$$

Proof. Considering φ and ψ be two hyperbolic type harmonic convex function, then

$$\begin{aligned}
&\varphi\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2}\right)\psi\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2}\right)\\
&\leq\left[\frac{\sinh\ell}{\sinh 1}\varphi(\mathtt{d}_2)+\frac{\sinh 1-\sinh\ell}{\sinh 1}\varphi(\mathtt{d}_1)\right]\left[\frac{\sinh\ell}{\sinh 1}\psi(\mathtt{d}_2)+\frac{\sinh 1-\sinh\ell}{\sinh 1}\psi(\mathtt{d}_1)\right]\\
&=\left(\frac{\sinh\ell}{\sinh 1}\right)^2\varphi(\mathtt{d}_2)\psi(\mathtt{d}_2)+\left(\frac{\sinh 1-\sinh\ell}{\sinh 1}\right)^2\varphi(\mathtt{d}_1)\psi(\mathtt{d}_1)\\
&+\frac{\sinh\ell}{\sinh 1}\frac{\sinh 1-\sinh\ell}{\sinh 1}\left[\psi(\mathtt{d}_2)\varphi(\mathtt{d}_1)+\varphi(\mathtt{d}_2)\psi(\mathtt{d}_1)\right].
\end{aligned}$$

Integrating both the sides with respect to ℓ over $[0,1]$, one has

$$\begin{aligned}
&\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\int_{\mathtt{d}_1}^{\mathtt{d}_2}\frac{\varphi(u)\psi(u)}{u^2}du\\
&\leq\frac{\varphi(\mathtt{d}_2)\psi(\mathtt{d}_2)}{\sinh^2 1}\int_0^1(\sinh\ell)^2d\ell+\frac{\varphi(\mathtt{d}_1)\psi(\mathtt{d}_1)}{\sinh^2 1}\int_0^1(\sinh 1-\sinh\ell)^2d\ell\\
&+\frac{\varphi(\mathtt{d}_1)\psi(\mathtt{d}_2)+\varphi(\mathtt{d}_2)\psi(\mathtt{d}_1)}{\sinh^2 1}\int_0^1\sinh\ell(\sinh 1-\sinh\ell)d\ell\\
&=\frac{\varphi(\mathtt{d}_2)\psi(\mathtt{d}_2)}{\sinh^2 1}\frac{(e^4-4e^2-1)}{8e^2}+\frac{\varphi(\mathtt{d}_1)\psi(\mathtt{d}_1)}{\sinh^2 1}\\
&\times\left[\frac{8e^2\sinh^2 1-16e^2\cosh 1\sinh 1+e^4-4e^2-1}{8e^2}+2\sinh 1\right]\\
&+\frac{[\varphi(\mathtt{d}_1)\psi(\mathtt{d}_2)+\varphi(\mathtt{d}_2)\psi(\mathtt{d}_1)]}{\sinh^2 1}\left[\frac{8e^2\cosh 1\sinh 1-e^4+4e^2+1}{8e^2}-\sinh 1\right]\\
&=\varphi(\mathtt{d}_2)\psi(\mathtt{d}_2)\left(\frac{e^4-4e^2-1}{8e^2\sinh^2 1}\right)+\varphi(\mathtt{d}_1)\psi(\mathtt{d}_1)\left(\frac{-e^4+8e^3-8e^2-8e+5}{8e^2\sinh^2 1}\right)\\
&+[\varphi(\mathtt{d}_1)\psi(\mathtt{d}_2)+\varphi(\mathtt{d}_2)\psi(\mathtt{d}_1)]\left(\frac{e^4-4e^3+4e^2+4e-1}{8e^2\sinh^2 1}\right).
\end{aligned}$$

□

4. Fractional Hermite-Hadamard type Inequalities

Definition 4.1. *Let $\varphi\in L[\mathtt{d}_1,\mathtt{d}_2]$ The Riemann-Liouville integrals $K^m_{\mathtt{d}_2-}\varphi$ and $K^m_{\mathtt{d}_1+}\varphi$ of order $m\geq 0$ are defined as*

$$K^m_{\mathtt{d}_1+}\varphi(x)=\frac{1}{\Gamma(m)}\int_{\mathtt{d}_1}^{x}(x-\ell)^{m-1}\varphi(\ell)d\ell,x\geq\mathtt{d}_1$$

and

$$K^m_{\mathtt{d}_2-}\varphi(x)=\frac{1}{\Gamma(m)}\int_{x}^{\mathtt{d}_2}(\ell-x)^{m-1}\varphi(\ell)d\ell,x\leq\mathtt{d}_2$$

For our results on fractional integrals we need the following lemma.

Lemma 4.1. [7] *Let $\varphi:I\subseteq(0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function on I° such that $\varphi'\in\mathcal{L}[\mathtt{d}_1,\mathtt{d}_2]$, where $\mathtt{d}_1,\mathtt{d}_2\in I$ with $\mathtt{d}_1<\mathtt{d}_2$. Then the following equality for fractional* *integral holds:*

$$\frac{\varphi(\mathtt{d}_1)+\varphi(\mathtt{d}_2)}{2}-\frac{\Gamma(m+1)}{2}\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\right)^m\left\{J^m_{\frac{1}{\mathtt{d}_1}-}\varphi\circ\sigma\left(\frac{1}{\mathtt{d}_2}\right)+J^m_{\frac{1}{\mathtt{d}_2}+}\varphi\circ\sigma\left(\frac{1}{\mathtt{d}_1}\right)\right\}$$
$$=\frac{\mathtt{d}_1\mathtt{d}_2(\mathtt{d}_2-\mathtt{d}_1)}{2}\int_0^1\frac{\ell^m-(1-\ell)^m}{(\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2)^2}\varphi'\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2}\right)d\ell. \tag{17}$$

Theorem 4.1. *Let $\varphi: I\longrightarrow\mathbb{R}$ be a differentiable mapping on I°, where $\mathtt{d}_1,\mathtt{d}_2\in I^\circ$ with $0\leq\mathtt{d}_1\leq\mathtt{d}_2$ and $\varphi'\in\mathcal{L}[\mathtt{d}_1,\mathtt{d}_2]$. If $|\varphi'|^q$ is hyperbolic type harmonic convex function, then the following inequality for fractional integral holds*

$$\left|\frac{\varphi(\mathtt{d}_1)+\varphi(\mathtt{d}_2)}{2}-\frac{\Gamma(\mathtt{d}_1+1)}{2}\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\right)^{\mathtt{d}_1}\left\{J^m_{\frac{1}{\mathtt{d}_1}-}\varphi\circ\sigma\left(\frac{1}{\mathtt{d}_2}\right)+J^m_{\frac{1}{\mathtt{d}_2}+}\varphi\circ\sigma\left(\frac{1}{\mathtt{d}_1}\right)\right\}\right|$$
$$\leq\frac{\mathtt{d}_1\mathtt{d}_2(\mathtt{d}_2-\mathtt{d}_1)}{2}\xi_1^{1-1/q}(\mathtt{d}_1,\mathtt{d}_2)\left(\xi_2(\mathtt{d}_1,\mathtt{d}_2)|\varphi'(\mathtt{d}_2)|^q+\xi_3(\mathtt{d}_1,\mathtt{d}_2)|\varphi'(\mathtt{d}_2)|^q\right)^{1/q} \tag{18}$$

Where

$$\xi_1(\mathtt{d}_1,\mathtt{d}_2)=\int_0^1\frac{\ell^m+(1-\ell)^m}{(\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2)^2}dt$$
$$\xi_2(\mathtt{d}_1,\mathtt{d}_2)=\int_0^1\frac{\ell^m+(1-\ell)^m}{(\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2)^2}\left[\frac{sinh\ell}{sinh1}|\varphi'(\mathtt{d}_2)|^q\right]d\ell$$
$$\xi_3(\mathtt{d}_1,\mathtt{d}_2)=\int_0^1\frac{\ell^m+(1-\ell)^m}{(\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2)^2}\left[\frac{sinh1-sinh\ell}{sinh1}|\varphi'(\mathtt{d}_1)|^q\right]d\ell$$

Proof. Using lemma 4.1 , Well known Power mean Inequality and $|\varphi'|^q$ as hyperbolic type harmonic convexity

$$\left|\frac{\varphi(\mathtt{d}_1)+\varphi(\mathtt{d}_2)}{2}-\frac{\Gamma(m+1)}{2}\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\right)^{\mathtt{d}_1}\left\{J^m_{1/\mathtt{d}_1-}\varphi\circ\sigma\left(\frac{1}{\mathtt{d}_1}\right)+J^m_{1/\mathtt{d}_2+}\varphi\circ\sigma\left(\frac{1}{\mathtt{d}_1}\right)\right\}\right|$$
$$\leq\frac{\mathtt{d}_1\mathtt{d}_2(\mathtt{d}_2-\mathtt{d}_1)}{2}\int_0^1\left|\frac{\ell^m-(1-\ell)^m}{(\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2)^2}\right|\left|\varphi'\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2}\right)\right|d\ell$$
$$\leq\frac{\mathtt{d}_1\mathtt{d}_2(\mathtt{d}_2-\mathtt{d}_1)}{2}\left(\int_0^1\left|\frac{\ell^{\mathtt{d}_1}-(1-\ell)^{\mathtt{d}_1}}{(\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2)^2}\right|d\ell\right)^{1-1/q}$$
$$\times\left(\left|\frac{\ell^{\mathtt{d}_1}-(1-\ell)^{\mathtt{d}_1}}{(\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2)^2}\right|\left|\varphi'\left(\frac{\mathtt{d}_1\mathtt{d}_2}{(\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2)}\right)\right|^q d\ell\right)^{1/q}$$
$$\leq\frac{\mathtt{d}_1\mathtt{d}_2(\mathtt{d}_2-\mathtt{d}_1)}{2}\left(\int_0^1\frac{[\ell^m+(1-\ell)^m]}{(\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2)^2}dt\right)^{1-1/q}$$
$$\times\left(\int_0^1\frac{[\ell^m+(1-\ell)^m]}{(\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2)^2}\left[\frac{sinh\ell}{sinh1}|\varphi'(\mathtt{d}_2)|^q+\frac{sinh1-sinh\ell}{sinh1}|\varphi'(\mathtt{d}_1)|^q\right]d\ell\right)^{1/q}$$

□

Theorem 4.2. *Let $\varphi:K\subseteq\mathbb{R}\to\mathbb{R}$ be a function such that $\varphi\in\mathcal{L}[\mathtt{d}_1,\mathtt{d}_2]$, where $\mathtt{d}_1,\mathtt{d}_2\in K$ with $\mathtt{d}_1<\mathtt{d}_2$. If φ is a hyperbolic type harmonic convex function on $[\mathtt{d}_1,\mathtt{d}_2]$, then the* *following inequalities for fractional integrals hold.*

$$\begin{aligned}
&\frac{\sinh 1}{\sinh\frac{1}{2}}\left\{\varphi\left(\frac{2\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_1+\mathtt{d}_2}\right)-\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\right)^m\Gamma(m+1)J^m_{\frac{1}{\mathtt{d}_1}-}(\varphi\circ\sigma)\left(\frac{1}{\mathtt{d}_2}\right)\right\}\\
&\quad\le\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\right)^m\Gamma(m+1)\left\{J^m_{\frac{1}{\mathtt{d}_2}+}(\varphi\circ\sigma)\left(\frac{1}{\mathtt{d}_1}\right)-J^m_{\frac{1}{\mathtt{d}_1}-}(\varphi\circ\sigma)\left(\frac{1}{\mathtt{d}_2}\right)\right\}\\
&\qquad\le\varphi(\mathtt{d}_1)+\varphi(\mathtt{d}_2)-2\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\right)^m\Gamma(m+1)J^m_{\frac{1}{\mathtt{d}_1}-}(\varphi\circ\sigma)\left(\frac{1}{\mathtt{d}_2}\right),
\end{aligned}\tag{19}$$

where $\sigma(x)=\frac{1}{x}$.

Proof. Since φ is hyperbolic type harmonic convex function putting $\ell=\frac{1}{2}$ and choosing $x=\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2}$and $y=\frac{\mathtt{d}_1\mathtt{d}_2}{(1-\ell)\mathtt{d}_1+\ell\mathtt{d}_2}$ in

$$\varphi\left(\frac{xy}{\ell x+(1-\ell)y}\right)\le\frac{\sinh\ell}{\sinh1}\varphi(y)+\frac{\sinh1-\sinh\ell}{\sinh1}\varphi(x),$$

we get

$$\varphi\left(\frac{2\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_1+\mathtt{d}_2}\right)\le\frac{\sinh\frac{1}{2}}{\sinh1}\varphi\left(\frac{\mathtt{d}_1\mathtt{d}_2}{(1-\ell)\mathtt{d}_1+\ell\mathtt{d}_2}\right)+\frac{\sinh1-\sinh\frac{1}{2}}{\sinh1}\varphi\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2}\right).$$

Multiplying both the sides by ℓ^{m-1} and Integrating with respect to k over $[0,1]$,

$$\begin{aligned}
\varphi\left(\frac{2\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_1+\mathtt{d}_2}\right)\int_0^1\ell^{m-1}d\ell\le{}&\frac{\sinh\frac{1}{2}}{\sinh1}\int_0^1\varphi\left(\frac{\mathtt{d}_1\mathtt{d}_2}{(1-\ell)\mathtt{d}_1+\ell\mathtt{d}_2}\right)\ell^{m-1}d\ell\\
&+\frac{\sinh1-\sinh\frac{1}{2}}{\sinh1}\int_0^1\varphi\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2}\right)\ell^{m-1}d\ell
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\varphi\left(\frac{2\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_1+\mathtt{d}_2}\right)\le{}&\frac{\sinh\frac{1}{2}}{\sinh1}\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\right)^m\Gamma(m+1)J^m_{\frac{1}{\mathtt{d}_2}+}(\varphi\circ\sigma)\left(\frac{1}{\mathtt{d}_1}\right)\\
&+\frac{\sinh1-\sinh\frac{1}{2}}{\sinh1}\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\right)^m\Gamma(m+1)J^m_{\frac{1}{\mathtt{d}_1}-}(\varphi\circ\sigma)\left(\frac{1}{\mathtt{d}_2}\right)
\end{aligned}$$

Consequently,

$$\begin{aligned}
\frac{\sinh1}{\sinh\frac{1}{2}}\varphi\left(\frac{2\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_1+\mathtt{d}_2}\right)\le{}&\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\right)^m\Gamma(m+1)J^m_{\frac{1}{\mathtt{d}_2}+}(\varphi\circ\sigma)\left(\frac{1}{\mathtt{d}_1}\right)\\
&+\frac{\sinh1-\sinh\frac{1}{2}}{\sinh\frac{1}{2}}\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\right)^m\Gamma(m+1)J^m_{\frac{1}{\mathtt{d}_1}-}(\varphi\circ\sigma)\left(\frac{1}{\mathtt{d}_2}\right)
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow\ &\frac{\sinh1}{\sinh\frac{1}{2}}\left\{\varphi\left(\frac{2\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_1+\mathtt{d}_2}\right)-\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\right)^m\Gamma(m+1)J^m_{\frac{1}{\mathtt{d}_1}-}(\varphi\circ\sigma)\left(\frac{1}{\mathtt{d}_2}\right)\right\}\\
&\quad\le\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\right)^m\Gamma(m+1)\left\{J^m_{\frac{1}{\mathtt{d}_2}+}(\varphi\circ\sigma)\left(\frac{1}{\mathtt{d}_1}\right)-J^m_{\frac{1}{\mathtt{d}_1}-}(\varphi\circ\sigma)\left(\frac{1}{\mathtt{d}_2}\right)\right\}.
\end{aligned}\tag{20}$$

For the second part of the proof, let φ be a hyperbolic type harmonic convex function. Then

$$\varphi\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2}\right)\le\frac{\sinh\ell}{\sinh1}\varphi(\mathtt{d}_2)+\frac{\sinh1-\sinh\ell}{\sinh1}\varphi(\mathtt{d}_1)$$

and

$$\varphi\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_2+(1-\ell)\mathtt{d}_1}\right) \leq \frac{\sinh\ell}{\sinh 1}\varphi(\mathtt{d}_1)+\frac{\sinh 1-\sinh\ell}{\sinh 1}\varphi(\mathtt{d}_2).$$

Adding both the above inequalities,

$$\varphi\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_2+(1-\ell)\mathtt{d}_1}\right)+\varphi\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\ell\mathtt{d}_1+(1-\ell)\mathtt{d}_2}\right) \leq \varphi(\mathtt{d}_1)+\varphi(\mathtt{d}_2).$$

Multiplying both the sides by ℓ^{m-1} and Integrating with respect to ℓ over $[0,1]$, we get

$$\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\right)^m \Gamma(m+1)\left\{J^m_{\frac{1}{\mathtt{d}_2}+}(\varphi\circ\sigma)\left(\frac{1}{\mathtt{d}_1}\right)-J^m_{\frac{1}{\mathtt{d}_1}-}(\varphi\circ\sigma)\left(\frac{1}{\mathtt{d}_2}\right)\right\}$$
$$\leq \varphi(\mathtt{d}_1)+\varphi(\mathtt{d}_2)-2\left(\frac{\mathtt{d}_1\mathtt{d}_2}{\mathtt{d}_2-\mathtt{d}_1}\right)^m \Gamma(m+1)J^m_{\frac{1}{\mathtt{d}_1}-}(\varphi\circ\sigma)\left(\frac{1}{\mathtt{d}_2}\right). \quad (21)$$

Combining (20) and (21) we get (19). □

5. Conclusions

In this paper, a new definition of generalized harmonic convexity is introduced and some refinements of the Hermite-Hadamard inequality for hyperbolic type functions which are harmonic convex in nature are presented as well. We have also proved two result using Riemann -Liouville fractional integral operator. Similar method can be applied to different type of convex functions to obtain many refinements of Hermite-Hadamard, Ostrowski, Fejér type inequalities. We believe the results of this article will attract future researchers in this field.

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