TWMS J. App. and Eng. Math. V.14, N.1, 2024, pp. 42-51

APPROXIMATION BY NONLINEAR q-BERNSTEIN-CHLODOWSKY OPERATORS

ECEM ACAR[1*], SEVILAY KIRCI SERENBAY[2], §

ABSTRACT. Max-Product algebra is new direction in constructive approximation of functions by operators. In this study, we introduce the q-analog of Bernstein-Chlodowsky operators using max-product algebra and investigate approximation properties of a sequence of these operators. Also, an upper estimate of the approximation error of the form $C\omega_1(f;1/\sqrt{n+1})$ with $C>0$ obvious constant is obtained.

Keywords: q-integers, nonlinear operators, Bernstein-Chlodowsky operators.

AMS Subject Classification: 41A30, 41A46, 41A25.

1. Introduction

In recent years, many articles have focused on the problem of approximating continuous functions using q-Calculus (see [2]-[4],[8]-[11]) and (p,q)-calculus (see [19]-[22]). Initially, Lupas [10] and Philips [11] introduced the generalization of q-Bernstein operators and investigated approximation of these operators. Then, Derriennic introduced many properties of the q-analogue of the Durrmeyer operators in [8]. Later, generalized q-Durrmeyer operators were studied in [9], [12].

In addition to these studies, the nonlinear positive operators by means of discrete linear approximating operators were introduced by Bede et al., in [6]. In [13]-[15]-[18] "max-product kind operators" were introduced by using maximum in the name of sum in usual linear operators and gave Jackson-type error estimate in terms of modulus of continuity. Since max-product kind of approximation theory is a very rich and useful phenomena of approximating continuous functions, researchers have turned to this new field in recent years. Especially, Bernstein-Chlodowsky polynomials have not been studied so extensively. The nonlinear Bernstein-Chlodowsky operators of max-product type are defined

[1] Harran University, Department of Mathematics, Şanlıurfa, Turkey.
e-mail: karakusecem@harran.edu.tr; ORCID: https://orcid.org/0000-0002-2517-5849.
[*] Corresponding author.
e-mail: sevilaykirci@gmail.com; ORCID: https://orcid.org/0000-0001-5819-9997.
§ Manuscript received: November 13, 2021; accepted: March 16, 2022.
TWMS Journal of Applied and Engineering Mathematics, Vol.14, No.1

by Güngör et al., in [13], as below

$$C_n^{(M)}\left(f\right)\left(x\right)=\frac{\bigvee_{k=0}^{n}h_{n,k}(x)f\left(\frac{b_nk}{n}\right)}{\bigvee_{k=0}^{n}h_{n,k}(x)}, \tag{1}$$

with

$$h_{n,k}(x)=\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

which $0\leq x\leq b_n$ and n is a sequence of positive real numbers such that $\lim_{n\to\infty}b_n=\infty$.

In this study, we define nonlinear q-Bernstein-Chlodowsky operators of max-product kind and give the approximation properties of these operators. Firstly, we indicate some basic definition and general notations which will be used in this paper. We consider the operations "$\bigvee$" (maximum) and "." (product) over the max-product algebra $(\mathbb{R}_+,\vee,\cdot)$. Let $I\subset\mathbb{R}$ be a finite or infinite interval, and set

$$CB_+(I)=\left\{f:I\longrightarrow\mathbb{R}_+;f\quad continous\quad and\quad bounded\quad on\quad I\right\}.$$

The general form of discrete max-product-type approximation operators

$$L_n(f)(x)=\bigvee_{i=0}^{n}K_n(x,x_i)f(x_i),\quad L_n(f)(x)=\bigvee_{i=0}^{\infty}K_n(x,x_i)f(x_i),$$

where $n\in\mathbb{N}$, $f\in CB_+(I)$, $K_n(.,x_i)\in CB_+(I)$ and $x_i\in I$, for all i. These operators are nonlinear positive operators satisfying pseudo-linearity property

$$L_n(\alpha.f\vee\beta.g)(x)=\alpha.L_n(f)(x)\vee\beta.L_n(g)(x),$$

where $\forall\alpha,\ \beta\in\mathbb{R}_+,\quad f,g:I\to\mathbb{R}_+$. Additionally, the max-product operators are positive homogenous, in other words $\forall\lambda\geq 0$, $L_n(\lambda f)=\lambda L_n(f)$ (for the other details one can see [5]).

Now, let give some basic definition of the q-calculus. For the parameter $q>0$ and $n\in\mathbb{N}$, we define the q-integer $[n]_q$ as follow

$$[n]_q=\begin{cases}\frac{1-q^n}{1-q} & \text{if}\quad q\neq 1\\ n & \text{if}\quad q=1\end{cases},\quad [0]_q=0 \tag{2}$$

and q-factorial $[n]_q!$ as

$$[n]_q!=[1]_q[2]_q...[n]_q\quad\text{for}\quad n\in\mathbb{N}\quad\text{and}\quad [0]_q!=1. \tag{3}$$

For integers $0\leq k\leq n$ q-binomial is defined as

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\frac{[n]_q!}{[k]_q![n-k]_q!}. \tag{4}$$

2. Construction of The Operators

In this section, we define nonlinear q-Bernstein-Chlodowsky operators of max-product kind as below:

$$C_{n,q}^{M}\left(f\right)\left(x\right)=\frac{\bigvee_{k=0}^{n}s_{n,k}(x,q)f\left(\frac{\alpha_n[k]_q}{[n]_q}\right)}{\bigvee_{k=0}^{n}s_{n,k}(x,q)}, \tag{5}$$

with

$$s_{n,k}(x,q)=\begin{bmatrix}n\\k\end{bmatrix}_q\left(\frac{x}{\alpha_n}\right)^k\left(1-\frac{x}{\alpha_n}\right)_q^{n-k},\quad\left(1-\frac{x}{\alpha_n}\right)_q^{n-k}=\prod_{s=1}^{n-k}\left(1-q^s\frac{x}{\alpha_n}\right)$$

where $0 \leq x \leq \alpha_n$, α_n is a sequence of positive numbers such that $\lim_{n\to\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \frac{\alpha_n}{\sqrt{[n]_q}} = 0$, $n \in \mathbb{N}$, $q \in (0,1)$, and the function $f : [0, \alpha_n] \to \mathbb{R}^+$ is a contiuous.

The operators $C_{n,q}^{M}(f)(x)$ are positive and continuous on the interval $[0, \alpha_n]$ for a continuous function $f : [0, \alpha_n] \to \mathbb{R}^+$. Also, these operators satisfy the pseudo-linearity property and these operators also are positive homogenous. Since it is esay to show that $C_{n,q}^{M}(f)(0) - f(0) = 0$ for all n, we may assume that $0 \leq x \leq \alpha_n$.

Additionally, we provide an error estimate for the operators $C_{n,q}^{M}(f)(x)$ defined by (5) in terms of the modulus of continuity. Therefore, we need some notations an lemmas for the proof of the main results.

For each $k, j \in \{0, 1, 2, \cdots, n\}$ and $x \in \left[\frac{\alpha_n [j]_q}{[n+1]_q}, \frac{\alpha_n [j+1]_q}{[n+1]_q}\right]$, we obtained in the following structure

$$M_{k,n,j}(x,q) = \frac{s_{n,k}(x,q)\left|\frac{\alpha_n [k]_q}{[n]_q} - x\right|}{s_{n,j}(x,q)}, \tag{6}$$

$$m_{k,n,j}(x,q) = \frac{s_{n,k}(x,q)}{s_{n,j}(x,q)}. \tag{7}$$

It can easily see that if $k \geq j+1$, then

$$M_{k,n,j}(x,q) = \frac{s_{n,k}(x,q)\left(\frac{\alpha_n [k]_q}{[n]_q} - x\right)}{s_{n,j}(x,q)} \tag{8}$$

and if $k \leq j-1$, then

$$M_{k,n,j}(x,q) = \frac{s_{n,k}(x,q)\left(x - \frac{\alpha_n [k]_q}{[n]_q}\right)}{s_{n,j}(x)}. \tag{9}$$

Additionally, for each $k, j \in \{0, 1, 2, \cdots, n\}$, $k \geq j+2$ and $x \in \left[\frac{\alpha_n [j]_q}{[n+1]_q}, \frac{\alpha_n [j+1]_q}{[n+1]_q}\right]$, we will obtain the following

$$\overline{M}_{k,n,j}(x,q) = \frac{s_{n,k}(x,q)\left(\frac{\alpha_n [k]_q}{[n+1]_q} - x\right)}{s_{n,j}(x,q)} \tag{10}$$

and for each $k, j \in \{0, 1, 2, \cdots, n\}$, $k \leq j-2$ and $x \in \left[\frac{\alpha_n [j]_q}{[n+1]_q}, \frac{\alpha_n [j+1]_q}{[n+1]_q}\right]$, we will get the following

$$\widehat{M}_{k,n,j}(x,q) = \frac{s_{n,k}(x,q)\left(x - \frac{\alpha_n [k]_q}{[n+1]_q}\right)}{s_{n,j}(x,q)}. \tag{11}$$

Lemma 2.1. *Let $q \in (0,1)$, $j \in \{0, 1, \cdots, n\}$ and $x \in \left[\frac{\alpha_n [j]_q}{[n+1]_q}, \frac{\alpha_n [j+1]_q}{[n+1]_q}\right]$. Then, we have*

(1) *for all $k \in \{0, 1, \cdots, n\}$ and $k \geq j+2$*

$$\overline{M}_{k,n,j}(x,q) \leq M_{k,n,j}(x,q) \leq \left(1 + \frac{2}{q^{n+1}}\right) \overline{M}_{k,n,j}(x,q).$$

(2) *for all $k \in \{0, 1, \cdots, n\}$ and $k \leq j-2$*

$$M_{k,n,j}(x,q) \leq \widehat{M}_{k,n,j}(x,q) \leq \left(1 + \frac{2}{q^{n}}\right) M_{k,n,j}(x,q).$$

The proof process is similar to the book [7].

Lemma 2.2. *For all $k,j\in\{0,1,2,\cdots,n\}$ and $x\in\left[\frac{\alpha_n[j]_q}{[n+1]_q},\frac{\alpha_n[j+1]_q}{[n+1]_q}\right]$ we obtain the following inequalities:*

$$m_{k,n,j}\left(x,q\right)\leq 1. \tag{12}$$

Proof. We have two cases for the proof of the above lemma: 1) $k\geq j$, 2) $k\leq j$. *Case 1:* Let $k\geq j$. From the definition $m_{k,n,j}\left(x,q\right)$ given (7) and since the function $\frac{\alpha_n-q^{n-k}x}{x}$ is nonincreasing on $\left[\frac{\alpha_n[j]_q}{[n+1]_q},\frac{\alpha_n[j+1]_q}{[n+1]_q}\right]$, we get

$$\begin{aligned}\frac{m_{k,n,j}(x)}{m_{k+1,n,j}(x)}=&\frac{[k+1]_q}{[n-k]_q}\cdot\frac{\alpha_n-q^{n-k}x}{x}\geq\frac{[k+1]_q}{[n-k]_q}\cdot\frac{\alpha_n-q^{n-k}\frac{\alpha_n[j+1]_q}{[n+1]_q}}{\frac{\alpha_n[j+1]_q}{[n+1]_q}}\\=&\frac{[k+1]_q}{[j+1]_q}\frac{[n+1]_q-q^{n-k}[j+1]_q}{[n-k]_q}\geq 1\end{aligned}$$

which indicates

$$m_{j,n,j}(x,q)\geq m_{j+1,n,j}(x,q)\geq m_{j+2,n,j}(x,q)\geq\cdots\geq m_{n,n,j}(x,q).$$

Case 2: Let $k\leq j$.

$$\begin{aligned}\frac{m_{k,n,j}(x)}{m_{k-1,n,j}(x)}=&\frac{[n-k+1]_q}{[k]_q}\cdot\frac{x}{\alpha_n-q^{n-k+1}x}\geq\frac{[n-k+1]_q}{[k]_q}\cdot\frac{\frac{\alpha_n[j]_q}{[n+1]_q}}{\alpha_n-q^{n-k+1}\frac{\alpha_n[j]_q}{[n+1]_q}}\\=&\frac{[n-k+1]_q}{[k]_q}\frac{[j]_q}{[n+1]_q-q^{n-k+1}[j]_q}\geq 1.\end{aligned}$$

which implies

$$m_{j,n,j}(x,q)\geq m_{j-1,n,j}(x,q)\geq m_{j-2,n,j}(x,q)\geq\cdots\geq m_{0,n,j}(x,q).$$

Since $m_{j,n,j}(x,q)=1$, the proof of lemma is finished. □

Lemma 2.3. *Let $q\in(0,1)$, $j\in\{1,2,\cdots\}$ and $x\in\left[\frac{\alpha_n[j]_q}{[n+1]_q},\frac{\alpha_n[j+1]_q}{[n+1]_q}\right]$.*

(i) *If $k\in\{j+2,j+3,\cdots,n-1\}$ is such that $[k+1]_q-\sqrt{q^k[k+1]_q}\geq[j+1]_q$, then $\overline{M}_{k,n,j}(x,q)\geq\overline{M}_{k+1,n,j}(x,q)$*

(ii) *If $k\in\{1,2,\cdots,j-2\}$ is such that $[k]_q+\sqrt{q^k[k]_q}\leq[j]_q$, then $\widehat{M}_{k,n,j}(x)\geq\widehat{M}_{k-1,n,j}(x)$.*

Proof. (i) Let $k\in\{j+2,j+3,\cdots,n-1\}$ with $[k+1]_q-\sqrt{q^k[k+1]_q}\geq[j+1]_q$. Then we have

$$\frac{\overline{M}_{k,n,j}(x,q)}{\overline{M}_{k+1,n,j}(x,q)}=\frac{[k+1]_q}{[n-k]_q}\cdot\frac{\alpha_n-q^{n-k}x}{x}\cdot\frac{\frac{\alpha_n[k]_q}{[n+1]_q}-x}{\frac{\alpha_n[k+1]_q}{[n+1]_q}-x}.$$

Since the function $h(x)=\frac{\alpha_n-q^{n-k}x}{x}\cdot\frac{\frac{\alpha_n[k]_q}{[n+1]_q}-x}{\frac{\alpha_n[k+1]_q}{[n+1]_q}-x}$ is nonincreasing, it follows that

$$h(x)\geq h\left(\frac{\alpha_n[j+1]_q}{[n+1]_q}\right)=\frac{[n+1]_q-q^{n-k}[j+1]_q}{[j+1]_q}\cdot\frac{[k]_q-[j+1]_q}{[k+1]_q-[j+1]_q}$$

Then, since the condition $[k+1]_q-\sqrt{q^k[k+1]_q}\geq[j+1]_q$ is congruent to $[k+1]_q-\sqrt{[k+1]_q^2-[k]_q[k+1]_q}\geq[j+1]_q$ and this inequality is equivalent to $[k+1]_q\left([k]_q-[j+1]_q\right)\geq[j+1]_q\left([k+1]_q-[j+1]_q\right)$. Therefore, we obtain

$$\frac{\overline{M}_{k,n,j}(x,q)}{\overline{M}_{k+1,n,j}(x,q)}\geq 1.$$

(ii) Let $k \in \{1, 2, \cdots, j-2\}$ and $[k]_q + \sqrt{q^k [k]_q} \le [j]_q$. Then, we have

$$\frac{\widehat{M}_{k,n,j}(x)}{\widehat{M}_{k-1,n,j}(x)} = \frac{[n-k+1]_q}{[k]_q} \cdot \frac{x}{\alpha_n - q^{n-k+1}x} \cdot \frac{x - \frac{\alpha_n [k]_q}{[n+1]_q}}{x - \frac{\alpha_n [k-1]_q}{[n+1]_q}}.$$

Then, since the function $r(x) = \frac{x}{\alpha_n - q^{n-k+1}x} \cdot \frac{x - \frac{\alpha_n [k]_q}{[n+1]_q}}{x - \frac{\alpha_n [k-1]_q}{[n+1]_q}}$ is nondecreasing on the interval $x \in \left[\frac{\alpha_n [j]_q}{[n+1]_q}, \frac{\alpha_n [j+1]_q}{[n+1]_q}\right]$, we get

$$r(x) \ge r\left(\frac{\alpha_n [j]_q}{[n+1]_q}\right) = \frac{[j]_q}{[n+1]_q - q^{n-k+1}[j]_q} \cdot \frac{[j]_q - [k]_q}{[j]_q - [k-1]_q}.$$

Since the condition $[k]_q + \sqrt{q^k [k]_q} \le [j]_q$ implies $[j]_q \left([j]_q - [k]_q\right) \ge [k]_q \left([j]_q - [k-1]_q\right)$, we obtain

$$\frac{\widehat{M}_{k,n,j}(x)}{\widehat{M}_{k-1,n,j}(x)} \ge 1.$$

Therefore, we prove the lemma. □

Lemma 2.4. *Let indicate*

$$s_{n,k}(x,q) = \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{x}{\alpha_n}\right)^k \prod_{s=1}^{n-k} \left(1 - q^s \frac{x}{\alpha_n}\right)^{n-k},$$

$q \in (0,1)$, $j \in \{0, 1, 2, \cdots\}$ and for all $x \in \left[\frac{\alpha_n [j]_q}{[n+1]_q}, \frac{\alpha_n [j+1]_q}{[n+1]_q}\right]$ we get

$$\bigvee_{k=0}^{n} s_{n,k}(x,q) = s_{n,j}(x,q)$$

Proof. Firstly, we demonstrate that for fixed $n \in \mathbb{N}$ and $0 \le k < k+1 \le n$, we get

$$0 \le s_{n,k+1}(x,q) \le s_{n,k}(x,q) \quad \text{if and only if} \quad x \in \left[0, \frac{\alpha_n [j+1]_q}{[n+1]_q}\right].$$

Let estimate the following inequality

$$0 \le \begin{bmatrix} n \\ k+1 \end{bmatrix}_q \left(\frac{x}{\alpha_n}\right)^{k+1} \left(1 - \frac{x}{\alpha_n}\right)_q^{n-k-1} \le \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{x}{\alpha_n}\right)^k \left(1 - \frac{x}{\alpha_n}\right)_q^{n-k}$$

after some simplifications,we can reduce the above inequality to

$$0 \le x \le \frac{\alpha_n [k+1]_q}{[n+1]_q}$$

Therefore, if we take $k = 0, 1, \cdots, n$ in the ineqaulity above, we get

$$s_{n,1}(x,q) \le s_{n,0}(x,q), \quad \text{if and only if} \quad x \in \left[0, \frac{\alpha_n}{[n+1]_q}\right],$$

$$s_{n,2}(x,q) \le s_{n,1}(x,q), \quad \text{if and only if} \quad x \in \left[0, \frac{\alpha_n [2]_q}{[n+1]_q}\right],$$

$$s_{n,3}(x,q) \le s_{n,2}(x,q), \quad \text{if and only if} \quad x \in \left[0, \frac{\alpha_n [3]_q}{[n+1]_q}\right],$$

and

$$s_{n,k+1}(x,q) \le s_{n,k}(x,q), \quad \text{if and only if} \quad x \in \left[0, \frac{\alpha_n [k+1]_q}{[n+1]_q}\right],$$

and at last

$$s_{n,n-2}(x,q) \le s_{n,n-3}(x,q), \quad \text{if and only if} \quad x \in \left[0, \frac{\alpha_n [n-2]_q}{[n+1]_q}\right],$$
$$s_{n,n-1}(x,q) \le s_{n,n-2}(x,q), \quad \text{if and only if} \quad x \in \left[0, \frac{\alpha_n [n-1]_q}{[n+1]_q}\right],$$
$$s_{n,n}(x,q) \le s_{n,n-1}(x,q), \quad \text{if and only if} \quad x \in \left[0, \frac{\alpha_n [n]_q}{[n+1]_q}\right].$$

Eventually, we obtain

$$\text{if} \quad x \in \left[0, \frac{\alpha_n}{[n+1]_q}\right] \text{ then } \quad s_{n,k}(x,q) \le s_{n,0}(x,q), \text{for all} \quad k = 0,1,\cdots,n;$$
$$\text{if} \quad x \in \left[\frac{\alpha_n}{[n+1]_q}, \frac{\alpha_n[2]_q}{[n+1]_q}\right] \text{ then } \quad s_{n,k}(x,q) \le s_{n,1}(x,q), \text{for all} \quad k = 0,1,\cdots,n;$$
$$\text{if} \quad x \in \left[\frac{\alpha_n[2]_q}{[n+1]_q}, \frac{\alpha_n[3]_q}{[n+1]_q}\right] \text{ then } \quad s_{n,k}(x,q) \le s_{n,2}(x,q), \text{for all} \quad k = 0,1,\cdots,n;$$

and in general

$$\text{if} \quad x \in \left[\frac{\alpha_n[n]_q}{[n+1]_q}, \alpha_n\right] \text{ then } \quad s_{n,k}(x,q) \le s_{n,n}(x,q), \text{for all} \quad k = 0,1,\cdots,n,$$

which completes the proof of lemma. □

3. Degree of approximation by $C_{n,q}^{(M)}(f)(x)$

In this section, we obtain the main results about the nonlinear q-Bernstein-Chlodowsky operator of max-product kind using the Shisha-Mond Theorem given for nonlinear max-product type operators in [5, 6].

Theorem 3.1. *Let $f : [0,\alpha_n] \to \mathbb{R}_+$ be a bounded and continuous function and $C_{n,q}^{(M)}(f)(x)$ are the max-product q-Bernstein-Chlodowsky operators given in (5). Then, we get the following estimation*

$$\left|C_{n,q}^{(M)}(f)(x) - f(x)\right| \le 4\left(1 + \frac{2}{q^{n+1}}\right)\omega_1\left(f; \frac{\alpha_n}{\sqrt{[n+1]_q}}\right) \tag{13}$$

which $n \in \mathbb{N}$, $q \in (0,1)$, $x \in [0,\alpha_n]$ and

$$\omega_1(f;\delta) = \sup\{|f(x) - f(y)|; x,y \in [0,\alpha_n], |x-y| \le \delta\}.$$

Proof. Since $C_{n,q}^{(M)}(e_0)(x) = 1$, by using the Shisha-Mond Theorem

$$\left|C_{n,q}^{(M)}(f)(x) - f(x)\right| \le \left(1 + \frac{1}{\delta_n} C_{n,q}^{(M)}(\varphi_x)(x)\right)\omega_1(f;\delta_n), \tag{14}$$

where $\varphi_x(t) = |t-x|$. Estimation of the following term is enough for the proof of lemma:

$$A_{n,q}(x) := C_{n,q}^{(M)}(\varphi_x)(x) = \frac{\bigvee_{k=0}^{n} s_{n,k}(x,q)\left|\frac{\alpha_n[k]_q}{[n]_q} - x\right|}{\bigvee_{k=0}^{n} s_{n,k}(x,q)}$$

Let $x \in \left[\frac{\alpha_n[j]_q}{[n+1]_q}, \frac{\alpha_n[j+1]_q}{[n+1]_q}\right]$, where $j \in \{0,1,\cdots,n\}$ is fixed and arbitrary. By Lemma 2.4, we get

$$A_{n,q}(x) = \bigvee_{k=0}^{n} M_{k,n,j}(x,q).$$

Initially, for $j = 0$ we obtain $A_{n,q}(x) \leq \alpha_n/[n]_q$ for all $x \in \left[0, \frac{\alpha_n}{[n+1]_q}\right]$, so we can claim that $j = \{1, 2, \cdots, n\}$. We will find an upper estimate for each $M_{k,n,j}(x)$, where $j \in \{0, 1, \cdots, n\}$ is fixed, $x \in \left[\frac{\alpha_n[j]_q}{[n+1]_q}, \frac{\alpha_n[j+1]_q}{[n+1]_q}\right]$ and $k \in \{0, 1, \cdots, n\}$. Under the circumstances, the proof will be divided into 3 cases:

$$1)k \in \{j-1, j, j+1\} \quad 2)k \geq j+2 \quad \text{and} \quad 3)k \leq j-2$$

Case 1) If $k = j$ then $M_{j,n,j}(x,q) = \left|\frac{\alpha_n j}{[n]_q} - x\right|$. Since $x \in \left[\frac{\alpha_n[j]_q}{[n+1]_q}, \frac{\alpha_n[j+1]_q}{[n+1]_q}\right]$, one can see that $M_{j,n,j}(x,q) \leq \frac{\alpha_n}{[n+1]_q}$.

If $k = j+1$ then $M_{j+1,n,j}(x,q) = m_{j+1,n,j}(x,q)\left(\frac{\alpha_n[j+1]_q}{[n]_q} - x\right)$. From Lemma 2.2, we have $m_{j+1,n,j}(x,q) \geq 1$, it refers to

$$\begin{aligned} M_{j+1,n,j}(x,q) \leq & \frac{\alpha_n[j+1]_q}{[n]_q} - x \leq \frac{\alpha_n[j+1]_q}{[n]_q} - \frac{\alpha_n[j]_q}{[n+1]_q} \\ = & \frac{\alpha_n\left([j+1]_q[n+1]_q - [j]_q[n]_q\right)}{[n]_q[n+1]_q} \leq \frac{3\alpha_n}{[n+1]_q}. \end{aligned}$$

If $k = j-1$ then $M_{j-1,n,j}(x,q) = m_{j-1,n,j}(x,q)\left(x - \frac{\alpha_n[j-1]_q}{[n]_q}\right)$. By Lemma 2.2, we have $m_{j-1,n,j}(x,q) \geq 1$, it refers to

$$\begin{aligned} M_{j-1,n,j}(x) \leq & x - \frac{\alpha_n[j-1]_q}{[n]_q} \leq \frac{\alpha_n[j+1]_q}{[n+1]_q} - \frac{\alpha_n[j-1]_q}{[n]_q} \\ = & \frac{\alpha_n\left([j+1]_q[n]_q - [j-1]_q[n+1]_q\right)}{[n]_q[n+1]_q} \leq \frac{2\alpha_n}{[n+1]_q}. \end{aligned}$$

Case 2) Subcase (a) Let take $[k]_q - \sqrt{[k+1]_q} < [j]_q$ and using Lemma 2.2, we obtain

$$\begin{aligned} \overline{M}_{k,n,j}(x,q) = & m_{k,n,j}(x,q)\left(\frac{\alpha_n[k]_q}{[n+1]_q} - x\right) \leq \frac{\alpha_n[k]_q}{[n+1]_q} - x \\ \leq & \frac{\alpha_n[k]_q}{[n+1]_q} - \frac{\alpha_n[j]_q}{[n+1]_q} \leq \frac{\alpha_n[k]_q}{[n+1]_q} - \frac{\alpha_n\left([k]_q - \sqrt{[k+1]_q}\right)_q}{[n+1]_q} \\ = & \frac{\alpha_n\sqrt{[k+1]_q}}{[n+1]_q} \leq \frac{\alpha_n}{\sqrt{[n+1]_q}}. \end{aligned}$$

Subcase (b) Let $[k+1]_q - \sqrt{q^k[k+1]_q} \geq [j+1]_q$. Since the function $g(k) = [k+1]_q - \sqrt{q^k[k+1]_q}$ is nondecreasing on the interval $x \in \left[\frac{\alpha_n[j]_q}{[n+1]_q}, \frac{\alpha_n[j+1]_q}{[n+1]_q}\right]$, it follows that there exist $\overline{k} = \{0, 1, 2, \cdots, n\}$ of maximum value such that

$$[\overline{k}+1]_q - \sqrt{q^{\overline{k}}[\overline{k}+1]_q} < [j+1]_q$$

. Let take $k^* = \overline{k}+1$, for all $k \geq k^*$ one get

$$[k+1]_q - \sqrt{q^k[k+1]_q} \geq [j+1]_q.$$

Let substitute

$$[j]_q \geq [\overline{k}+1]_q - q^j - \sqrt{q^{\overline{k}}[\overline{k}+1]_q},$$

then one obtain

$$\begin{aligned}
\overline{M}_{k^*,n,j}(x,q) =& m_{k^*,n,j}(x,q)\left(\frac{\alpha_n[k^*]_q}{[n+1]_q}-x\right) \leq \frac{\alpha_n[\overline{k}+1]_q}{[n+1]_q}-x \\
\leq& \frac{\alpha_n[\overline{k}+1]_q}{[n+1]_q}-\frac{\alpha_n j}{[n+1]_q} \leq \frac{\alpha_n[\overline{k}+1]_q}{[n+1]_q}-\frac{\alpha_n\left([\overline{k}+1]_q-g^j-\sqrt{q^{\overline{k}}[\overline{k}+1]_q}\right)}{[n+1]_q} \\
=& \frac{\alpha_n\left(g^j+\sqrt{q^{\overline{k}}[\overline{k}+1]_q}\right)}{[n+1]_q} \leq \frac{\alpha_n\left(1+\sqrt{[\overline{k}+1]_q}\right)}{[n+1]_q} \leq 2\frac{\alpha_n\sqrt{[\overline{k}+1]_q}}{[n+1]_q} \\
\leq& \frac{2\alpha_n}{\sqrt{[n+1]_q}}.
\end{aligned}$$

Moreover, we have $k^* \geq j+2$, Indeed, this is a consequence of the fact that the function g is nondecreasing on the interval $[0,\alpha_n]$ and it is easy to see that $g(j+1)<j$.

By Lemma 2.3 (i) it follows that $\overline{M}_{\overline{k}+1,n,j}(x) \geq \overline{M}_{\overline{k}+2,n,j}(x) \geq \cdots \geq \overline{M}_{n,n,j}(x)$.

Therefore, we obtain $\overline{M}_{k,n,j}(x) \leq \frac{2\alpha_n}{\sqrt{[n+1]_q}}$ for any $k \in \left\{\overline{k}+1,\overline{k}+2,\cdots,n\right\}$. Thus, for the same k's, it follows from Lemma 2.1 that

$$M_{k,n,j}(x) \leq \frac{2\left(1+\frac{2}{q^{n+1}}\right)\alpha_n}{\sqrt{[n+1]}}$$

Case 3) Subcase (a) Let $[k]_q+\sqrt{q^{k-1}[k]_q} \geq [j]_q$. Then, we obtain

$$\begin{aligned}
\widehat{M}_{k,n,j}(x,q) =& m_{k,n,j}(x,q)\left(x-\frac{\alpha_n[k]_q}{[n+1]_q}\right) \leq \frac{\alpha_n[j+1]}{[n+1]_q}-\frac{\alpha_n[k]_q}{[n+1]_q} \\
=& \frac{\alpha_n([j]_q+q^j)}{[n+1]_q}-\frac{\alpha_n[k]_q}{[n+1]_q}.
\end{aligned}$$

By hypotesis, we get

$$\begin{aligned}
\widehat{M}_{k,n,j}(x) \leq& \frac{\alpha_n\left([k]_q+\sqrt{q^{k-1}[k]_q}+q^j\right)}{[n+1]_q}-\frac{\alpha_n[k]_q}{[n+1]_q} \\
=& \frac{\alpha_n\left(\sqrt{q^{k-1}[k]_q}+q^j\right)}{[n+1]_q} \leq \frac{\alpha_n\left(\sqrt{[k]_q}+1\right)}{[n+1]_q} \leq \frac{\alpha_n\left(\sqrt{[j-2]_q}+1\right)}{[n+1]_q} \\
=& \frac{\alpha_n}{\sqrt{[n+1]_q}}\cdot\frac{\sqrt{[j-2]_q}+1}{\sqrt{[n+1]_q}} \leq \frac{\alpha_n}{\sqrt{[n+1]_q}}\cdot\frac{2\sqrt{j}}{\sqrt{[n+1]_q}} \leq \frac{2\alpha_n}{\sqrt{[n+1]_q}}.
\end{aligned}$$

Subcase (b) Now let $[k]_q+\sqrt{q^{k-1}[k]_q} < [j]_q$. Let $\widetilde{k}=\{0,1,2,\cdots,n\}$ be the minimum value such that $[\widetilde{k}]_q+\sqrt{q^{\widetilde{k}-1}[\widetilde{k}]_q} \geq [j]_q$. Then $k_*=\widetilde{k}-1$ satisfies $[\widetilde{k}-1]_q+\sqrt{q^{\widetilde{k}-2}[\widetilde{k}-1]_q} < [j]_q$ and

$$\begin{aligned}
\widehat{M}_{\widetilde{k}-1,n,j}(x,q) =& m_{\widetilde{k}-1,n,j}(x,q)\left(x-\frac{\alpha_n[\widetilde{k}-1]}{[n+1]_q}\right) \leq \frac{\alpha_n[j+1]_q}{[n+1]_q}-\frac{\alpha_n[\widetilde{k}-1]_q}{[n+1]_q} \\
\leq& \frac{\alpha_n\left([j]_q+q^j\right)}{[n+1]_q}-\frac{\alpha_n[\widetilde{k}-1]_q}{[n+1]_q}.
\end{aligned}$$

Also, we have $[\widetilde{k}]_q + \sqrt{q^{\widetilde{k}-1}[\widetilde{k}]_q} \geq [j]_q$ then, we obtain

$$\begin{aligned}\widehat{M}_{\widetilde{k}-1,n,j}(x,q) \leq & \frac{\alpha_n\left([\widetilde{k}]_q + \sqrt{q^{\widetilde{k}-1}[\widetilde{k}]_q} + q^j\right)}{[n+1]_q} + \frac{\alpha_n[\widetilde{k}-1]_q}{[n+1]_q} \\ = & \frac{\alpha_n\left(q^{\widetilde{k}-1} + \sqrt{q^{\widetilde{k}-1}[\widetilde{k}]_q} + q^j\right)}{[n+1]_q} \leq \frac{\alpha_n\left(2+\sqrt{[\widetilde{k}]_q}\right)}{[n+1]_q} \leq \frac{3\alpha_n}{\sqrt{[n+1]_q}}.\end{aligned}$$

Also, in this case we have $j \geq 2$, which implies $k_* \leq j-2$. By Lemma 2.3 (ii), we get $\widehat{M}_{\widetilde{k}-1,n,j}(x,q) \geq \widehat{M}_{\widetilde{k}-2,n,j}(x,q) \geq \cdots \geq \widehat{M}_{0,n,j}(x,q)$. Therefore, we obtain

$$\widehat{M}_{k,n,j}(x,q) \leq \frac{3\alpha_n}{\sqrt{[n+1]_q}} \quad \text{for any} \quad k \leq j-2 \quad \text{and} \quad x \in \left[\frac{\alpha_n[j]_q}{[n+1]_q}, \frac{\alpha_n[j+1]_q}{[n+1]_q}\right].$$

Hence, in subcases(a) and subcases(b) we have $\widehat{M}_{k,n,j}(x,q) \leq \frac{3\alpha_n}{\sqrt{n+1}}$. From (9) and (11) it is obvious that $M_{k,n,j}(x,q) \leq \widehat{M}_{k,n,j}(x,q)$ so we obtain $M_{k,n,j}(x) \leq \frac{3\alpha_n}{\sqrt{n+1}}$. Consequently, collecting all the above estimates, we obtain

$$M_{k,n,j}(x) \leq \frac{6\alpha_n}{\sqrt{[n+1]_q}} \; \forall x \in \left[\frac{\alpha_n[j]_q}{[n+1]_q}, \frac{\alpha_n[j+1]_q}{[n+1]_q}\right], \; k = \{0,1,2,\cdots,n\}$$

which implies that

$$A_{n,q}(x) \leq \frac{2\left(1+\frac{2}{q^{n+1}}\right)\alpha_n}{\sqrt{[n+1]_q}} \forall x \in [0,\alpha_n], n \in \mathbb{N}$$

and indicating $\delta_n = \frac{2\left(1+\frac{2}{q^{n+1}}\right)\alpha_n}{\sqrt{[n+1]_q}}$ in (14), we get the estimate

$$\left|C_{n,q}^{(M)}(f)(x) - f(x)\right| \leq 4\left(1+\frac{2}{q^{n+1}}\right)\omega_1\left(f; \frac{\alpha_n}{\sqrt{[n+1]_q}}\right), \forall n \in \mathbb{N}, x \in [0,\alpha_n].$$

□

4. Conclusions

In this study, nonlinear max-product type q-Bernstein-Chlodowsky operators are defined and some upper estimates of approximation error for some subclasses of functions are obtained.

References

[1] Bernstein, S., (1912), Démonstration du théoréme de Weierstrass, fondeé sur le calcul des probabilités, Commun. Soc. Math. Kharkov, 13 (2), pp. 1-2.
[2] Ostrovska, S., (2003), q-Bernstein polynomials and their iterates, J. Approx. Theory., 123, pp. 232–255.
[3] Phillips, G. M., (2000), A generalization of the Bernstein polynomials based on the q-integers, Anziam J., 42, pp. 79–86.
[4] Oruc, H. and Tuncer, N., (2002), On the convergence and iterates of q-Bernstein polynomials, J. Approx. Theory, 117, pp. 301–313.
[5] Bede, B. and Gal, S. G., (2010), Approximation by nonlinear Bernstein and Favard-Szasz-Mirakjan operators of max-product kind, J. Concr. Appl. Math., 8, pp. 193–207.
[6] Bede, B., Coroianu, L. and Gal, S. G., (2009), Approximation and shape preserving properties of the Bernstein operator of max-product kind, Int. J. Math. Math. Sci., 26 pp.

[7] Anastassiou, G. A. and Duman, O., (2016), Intelligent Mathematics II: Applied Mathematics and Approximation Theory, Springer, DOI 10.1007/978-3-319-30322-2.
[8] Derriennic, M. M., (2005), Modified Bernstein polynomials and Jacobi polynomials in q-calculus, Rend. Circ. Mat. Palermo Serie II, 76, pp. 269–290.
[9] Finta, Z. and Gupta, V., (2009), Approximation by q-Durrmeyer operators. J. Applied Math. and Computing, 29, pp. 401–415.
[10] Lupas, A., (1987), A q-analogue of the Bernstein operator. Seminar on Numerical and Statistical Calculus, Cluj-Napoca, pp. 85–92,
[11] Phillips, G. M., (1997), Bernstein polynomials based on the q-integers, Ann. Numer. Math., 4, pp. 511–518.
[12] Finta, Z. and Gupta, V., (2009), Approximation by q-Durrmeyer operators, J. Applied Math. and Computing, 29, pp. 401–415.
[13] Yüksel Güngör, Ş. and Ispir, N., (2018), Approximation by Bernstein-Chlodowsky operators of max-product kind, Mathematical Communications, 23, pp. 205-225.
[14] Yüksel Güngör, Ş. and Ispir, N., (2016), Quantitative Estimates for Generalized Szász Operators of Max-Product Kind, Results in Mathematics, 70 (3), pp. 447-456.
[15] Bede, B., Nobuhara, H., Fodor, J. and Hirota, K., (2006), Max-product Shepard approximation operators, J. Adv. Comput. Intell. Intell. Inf., 10, pp. 494-497.
[16] Bede, B. and Coroianu, L., (2010), Approximation and shape preserving properties of the nonlinear Bleimann-Butzer-Hahn operators of max-product kind, Comment. Math. Univ. Carolin., 3 (51), pp. 397-415.
[17] Coroianu, L. and Gal, S., (2011), Classes of functions with improved estimates in approximation by the max-product Bernstein operator, Anal. Appl., 3 (9), pp. 249-274.
[18] Acar, E., Karahan, D. and Kirci Serenbay, S., (2020), Approximation for the Bernstein operator of max-product kind in symmetric range, Khayyam J. Math., 6 (2), pp. 257–273.
[19] Rao, N., Wafi, A. and Acu, A. M., (2019), q-Szász–Durrmeyer Type Operators Based on Dunkl Analogue, Complex Anal. Oper. Theory, 13, pp. 915–934 .
[20] Rao, N. and Wafi, A., (2018), (p, q)-bivariate-Bernstein-Chlodowsky operators, Filomat, 32 (2), pp. 369–378.
[21] Wafi, A., Rao, N. and Deepmala, (2019), Approximation properties of (p,q)-variant of Stancu-Schurer operators, Boletim da Sociedade Paranaense de Matematica, 37 (4), pp. 137-151.
[22] Rao, N. and Wafi, A., (2019), Chlodowsky Szász-Kantorovich operators via Dunkl analogue, Applications and Applied Mathematics, 14 (1), pp. 370–381.

Ecem Acar graduated from Çukurova University. She is currently pursuing her Ph.D in the department of Mathematics at Harran University and her research interests are in approximation theory, operator theory, cellular automata. Presently, she is working as a Researcher in the Department of Mathematics at Harran University.

Sevilay Kırcı Serenbay graduated from Ankara University. She completed her Ph.D. in the Department of Mathematics at Ankara University. Her area of research is approximation theory, operator theory, integral operator theory. She is presently working as a professor and The Head of the Department of Mathematics at Harran University.