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CR-SUBMANIFOLDS OF A NEARLY TRANS-HYPERBOLIC SASAKIAN MANIFOLD WITH RESPECT TO SEMI SYMMETRIC NON-METRIC CONNECTION

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Abstract. The present paper deals with the study of CR-submanifolds of Nearly Trans-Hyperbolic Sasakian manifold with respect to semi symmetric Non-metric connection. Nijenhuis tensor, integrability conditions for some distributions on CR-submanifolds of a nearly trans-hyperbolic Sasakian manifold with respect to semi symmetric non-metric connection are discussed.

Keywords: CR-submanifolds, Nearly Trans-Hyperbolic Sasakian manifold, semi symmetric non-metric connection, Gauss and Weingarten equations, parallel distributions.

AMS Subject Classification: 53C40, 53C15.

1. Introduction

A. Bejancu introduced the notion of CR-submanifolds of a Kaehlar manifold [1]. Later, CR-submanifold have been studied by Kobayashi [14], Shahid et al. [18, 17], Yano and Kon [20] and others. Upadhyay and Dube [14] have studied almost contact hyperbolic (f,g,η,ξ)- structure, Dube and Mishra [6] have considered hypersurfaces immersed in an almost hyperbolic Hermitian manifold. Also Dube and Niwas [5] worked with almost r-contact hyperbolic structure in a product manifold. Gherghe studied harmonicity on a nearly trans-Sasakian manifold [7]. Bhatt and Dube [3] studied CR-submanifolds of trans-hyperbolic Sasakian manifold. Joshi and Dube [13] studied semi-invariant submanifold of an almost r-contact hyperbolic metric manifold. Gill and Dube have also worked on CR-submanifolds of a trans-hyperbolic Sasakian manifold [8]. Hui and Mandal [10] studied pseudo parallel contact CR-submanifolds of Kenmotsu manifold. Hui and Roy [11] have studied warped product CR-submanifolds of Sasakian manifolds with respect to certain

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connections. Also Pal, Shahid and Hui [15] worked with CR-submanifolds of $(LCS)_n$-manifolds with respect to quarter symmetric non-metric connection. Hui, Atceken, Pal and Mishra [12] have considered on contact CR-submanifolds of $(LCS)_n$-manifolds. Let ∇ be a linear connection and T be a torsion tensor in an n-dimensional differentiable manifold $\overline{M}$ [4]. The connection ∇ is symmetric if the torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is metric if there is a Riemannian metric g in $\overline{M}$ such that ∇g=0, otherwise it is non-metric. It is well known that a linear connection is symmetric and if and only if it is the Levi-civita connection. In [9], S. Golab introduced the idea of a semi-symmetric and quarter symmetric linear connections.

2. Preliminaries

Let $\overline{M}$ be an n-dimensional almost hyperbolic contact metric manifold with the almost hyperbolic contact metric structure (ϕ,ξ,η,g) where ϕ is a tensor of type (1,1), a vector field ξ, called structure vector field and η is a dual 1-form of ξ satisfying the following

$$\phi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = -1, \quad \phi \ o \ \xi = 0, \quad \eta \ o \ \phi = 0$$

$$g(\phi X, \phi Y) = -g(X, Y) - \eta(X)\eta(Y)$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X)$$

for all vector fields X, $Y \in T\overline{M}$ [16]. A linear connection is said to be a semi-symmetric connection if its torsion tensor T is of the form

$$T(X, Y) = \eta(Y)X - \eta(X)Y,$$

where η is 1-form and ϕ is a tensor field of the type (1,1).

Definition 2.1. *An m dimensional Riemannian submanifold M of $\overline{M}$ is called a CR-submanifold if ξ is tangent to M and there exists on M a differentiable distribution D: $x \rightarrow D_x \subset T_x(M)$ such that*
(i) The distribution D_x is invariant under ϕ, i.e., $\phi D_x \subset D_x$ for each $x \in M$;
(ii) The orthogonal complementary distribution $D^\perp$: $x \rightarrow D_x^\perp \subset T_x(M)$ of the distribution D on M is anti-invariant under ϕ, i.e., $\phi D_x^\perp(M) \subset T_x^\perp(M)$ for all $x \in M$, where $T_x(M)$ and $T_x^\perp(M)$ are tangent space and normal space of M at $x \in M$ respectively.
If dim$D_x^\perp$=0 (resp., dimD_x=0), then CR-submanifold is called an invariant (resp., anti-invariant). The distribution D (resp., $D^\perp$) is called the horizontal (resp., vertical) distribution. The pair (D, $D^\perp$) is called ξ-hoizontal (resp., ξ-invariant) if $\xi_x \in D_x$(resp., $\xi_x \in D_x^\perp$) for $x \in M$.

For any vector field X tangent to M, we put

$$X = PX + QX, \tag{1}$$

where PX and QX belong to the distribution D and $D^\perp$ respectively.
For any vector field N normal to M, we put

$$\phi N = BN + CN, \tag{2}$$

where BN (resp., CN) denotes the tangential (resp., normal) component of ϕN.
Now, we remark that owing to the existence of the 1-form η, we can define a semi symmetric non-metric connection in any almost contact metric manifold by

$$\overline{\nabla}_X Y = \overline{\overline{\nabla}}_X Y + \eta(Y)X \tag{3}$$

such that $(\overline{\nabla}_X g)(Y,Z) = -\eta(Y)g(X,Z) - \eta(Z)g(X,Y)$ for any X, $Y \in TM$, where $\overline{\overline{\nabla}}$ is the induced connection with respect to g on M.
An almost hyperbolic contact metric structure (ϕ, ξ, η, g) on $\overline{M}$ is called trans-hyperbolic Sasakian[2] if and only if

$$(\overline{\overline{\nabla}}_X \phi)Y = \alpha[g(X,Y)\xi - \eta(Y)\phi X] + \beta[g(\phi X, Y)\xi - \eta(Y)\phi X] \tag{4}$$

for all X, Y tangents to $\overline{M}$ and α, β are functions on $\overline{M}$. On a trans-hyperbolic Sasakian manifold M, we have

$$(\overline{\overline{\nabla}}_X \xi) = -\alpha(\phi X) + \beta[X - \eta(X)\xi]$$

By using (3) and (4), we get

$$(\overline{\nabla}_X \phi)Y = \alpha[g(X,Y)\xi - \eta(Y)\phi X] + \beta[g(\phi X, Y)\xi - \eta(Y)\phi X] - \eta(Y)\phi X \tag{5}$$

Similarly we have

$$(\overline{\nabla}_Y \phi)X = \alpha[g(Y,X)\xi - \eta(X)\phi Y] + \beta[g(\phi Y, X)\xi - \eta(X)\phi Y] - \eta(X)\phi Y \tag{6}$$

on adding (5) and (6), we obtain

$$(\overline{\nabla}_X \phi)Y + (\overline{\nabla}_Y \phi)X = \alpha[2g(X,Y)\xi - \eta(X)\phi Y - \eta(Y)\phi X] - (\beta+1)[\eta(Y)\phi X + \eta(X)\phi Y] \tag{7}$$

This is the condition for $\overline{M}$ (ϕ, ξ, η, g) with a semi symmetric non-metric connection to be nearly trans-hyperbolic Sasakian manifold.
We denote by g the metric tensor of $\overline{M}$ as well as that induced on M. Let $\overline{\nabla}$ be the semi symmetric non-metric connection on $\overline{M}$ and ∇ be the induced connection on M with respect to the unit normal N.

Theorem 2.1. *(i). If M is ξ-horizontal , X, $Y \in D$ and D is parallel with respect to ∇, then the connection induced on CR-submanifold of a nearly trans-hyperbolic Sasakian manifold with respect to a semi symmetric non-metric connection is also a semi symmetric non-metric connection.*
(ii).If M is ξ-vertical , X, $Y \in D^{\perp}$ and $D^{\perp}$ is parallel with respect to ∇, then the connection induced on a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold with respect to a semi symmetric non-metric connection is also a semi symmetric non-metric connection.
(iii).The Gauss formula with respect to the semi symmetric non-metric connection is of the form

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y)$$

Proof. Let ∇ be the induced connection with respect to the unit normal N on a CR-submanifold of a nearly trans-hypebolic Sasakian manifold from a semi symmetric non-metric connection $\overline{\nabla}$, then

$$\overline{\nabla}_X Y = \nabla_X Y + m(X,Y), \tag{8}$$

where m is a tensor field of the type of the type (0, 2) on CR-submanifold M. If $\nabla^{å}$ be the induced connection on CR-subamanifold from Riemannian connection $\overline{\overline{\nabla}}$, then

$$\overline{\overline{\nabla}}_X Y = \nabla^{å}{}_X Y + h(X,Y), \tag{9}$$

where h is a second fundamental form. By the definition of the semi symmetric non-metric connection, we have

$$\overline{\nabla}_X Y = \overline{\overline{\nabla}}_X Y + \eta(Y)X$$

Now using (8) and (9) in above equation, we have

$$\nabla_X Y + m(X,Y) = \nabla^{å}{}_X Y + h(X,Y) + \eta(Y)X$$

using (1), the above equation can be written as

$$P\nabla_X Y + Q\nabla_X Y + m(X,Y) = P\nabla^{å}{}_X Y + Q\nabla^{å}{}_X Y + h(X,Y) + \eta(Y)PX + \eta(Y)QX \quad (10)$$

From (10), we have tangential and normal components

$$h(X,Y) = m(X,Y) \quad (11)$$

$$P\nabla_X Y - \eta(Y)PX = P\nabla^{å}{}_X Y \quad (12)$$

$$Q\nabla_X Y - \eta(Y)QX = Q\nabla^{å}{}_X Y \quad (13)$$

Using (11), the Gauss formula for a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold with a semi symmetric non-metric connection is

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad (14)$$

This proves (iii).
In view of (12), if M is ξ-horizontal, X, $Y \in D$ and D is parallel with respect to ∇, then the connection induced on a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold with respect to a semi-symmetric non-metric connection is also a semi symmetric non-metric connection.
Similarly, using (13), if M is ξ-vertical, X, $Y \in D^{\perp}$ and $D^{\perp}$ is parallel with repect to ∇, then the connection induced on a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold with respect to a semi symmetric non-metric connection.
Weingarten formula is given by

$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N + \eta(N)X \quad (15)$$

for X, $Y \in TM$, $N \in T^{\perp}M$, $h : TM \times TM \to TM^{\perp}$ (resp., $A_N : TM \to TM$) is the second fundamental form (resp., tensor) of M in $\overline{M}$ and $\nabla^{\perp}$ denotes the operator of the normal connection. Moreover, we have

$$g(h(X,Y),N) = g(A_N X, Y)$$

□

3. Main Results

Lemma 3.1. *Let M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a semi symmetric non-metric connection. Then*

$$P(\nabla_X \phi PY) + P(\nabla_Y \phi PX) - P(A_{\phi QY}X) - P(A_{\phi QX}Y) = \phi P\nabla_X Y + \phi P\nabla_Y X$$

$$+2\alpha g(X,Y)P\xi - \alpha\eta(X)\phi PY - \alpha\eta(Y)\phi PX - (\beta+1)[\eta(X)\phi PY + \eta(Y)\phi PX]$$

$$Q(\nabla_X \phi PY) + Q(\nabla_Y \phi PX) - Q(A_{\phi QY}X) - Q(A_{\phi QX}Y) = 2\alpha g(X,Y)Q\xi - \alpha\eta(X)QY$$
$$-\alpha\eta(Y)QX + 2Bh(X,Y) \quad (16)$$

$$h(X,\phi PY) + h(Y,\phi PX) + \nabla^{\perp}_X \phi QY + \nabla^{\perp}_Y \phi QX = \phi(Q\nabla_X Y) + \phi(Q\nabla_Y X) + 2Ch(X,Y)$$
$$-(\beta+1)[\eta(Y)\phi QX + \eta(X)\phi QY] \quad (17)$$

for X, $Y \in TM$.

Proof. Differentiating covariantly

$$\overline{\nabla}_X \phi Y = (\overline{\nabla}_X \phi)Y + \phi(\overline{\nabla}_X Y)$$

and by (1), (2), (14) and (15)

$$\nabla_X \phi PY + h(X, \phi PY) - A_{\phi QY}X + \nabla^{\perp}_X \phi QY - \phi[\nabla_X Y + h(X, Y)] = \alpha[g(X, Y)\xi$$

$$-\eta(Y)\phi X] + \beta g(\phi X, Y)\xi - (\beta + 1)\eta(Y)\phi X.$$

Similarly, we have

$$\nabla_Y \phi PX + h(Y, \phi PX) - A_{\phi QX}Y + \nabla^{\perp}_Y \phi QX - \phi[\nabla_Y X + h(Y, X)] = \alpha[g(X, Y)\xi$$

$$-\eta(X)\phi Y] + \beta g(\phi Y, X)\xi - (\beta + 1)\eta(X)\phi Y.$$

On adding above equations, we have

$$\nabla_X \phi PY + \nabla_Y \phi PX + h(X, \phi PY) + h(Y, \phi PX) - A_{\phi QY}X - A_{\phi QX}Y + \nabla^{\perp}_X \phi QY+$$

$$\nabla^{\perp}_Y \phi QX - \phi\nabla_X Y - \phi\nabla_Y X + 2\phi h(X, Y) = \alpha[2g(X, Y)\xi - \eta(X)\phi Y - \eta(Y)\phi X] - (\beta + 1)$$

$$[\eta(X)\phi Y + \eta(Y)\phi X]$$

Using (1) and (2) and equating horizontal, vertical and normal components. The lemma follows. □

Lemma 3.2. *Let M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a semi-symmetric non-metric connection. Then*

$$2(\overline{\nabla}_X \phi)Y = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X, Y] + \alpha[2g(X, Y)\xi - \eta(X)\phi Y$$

$$-\eta(Y)\phi X] - (\beta + 1)[\eta(Y)\phi X + \eta(X)\phi Y]$$

for any X, $Y \in D$.

Proof. Using (14), we have

$$\overline{\nabla}_X \phi Y - \overline{\nabla}_Y \phi X = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X).$$

Also, we have

$$\overline{\nabla}_X \phi Y - \overline{\nabla}_Y \phi X = (\overline{\nabla}_X \phi)Y - (\overline{\nabla}_Y \phi)X + \phi[X, Y]$$

From above equations, we get

$$(\overline{\nabla}_X \phi)Y - (\overline{\nabla}_Y \phi)X = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X, Y] \tag{18}$$

For a nearly trans-hyperbolic Sasakian manifold with a semi symmetric non-metric connection, we have

$$(\overline{\nabla}_X \phi)Y + (\overline{\nabla}_Y \phi)X = \alpha[2g(X, Y)\xi - \eta(X)\phi Y - \eta(Y)\phi X] - (\beta + 1)[\eta(Y)\phi X + \eta(X)\phi Y] \tag{19}$$

Combining (18) and (19), the lemma follows. □

In particular, we have the following corollary.

Corollary 3.1. *Let M be a ξ-vertical CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a semi symmetric non-metric connection. Then*

$$2(\overline{\nabla}_X \phi)Y = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X, Y] + 2\alpha g(X, Y)\xi$$

for any X, $Y \in D$.

Similarly, by Weingarten formula, we can easily get the following lemma.

Lemma 3.3. *Let M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a semi symmetric non-metric connection. Then*

$$2(\overline{\nabla}_Y\phi)Z = A_{\phi Y}Z - A_{\phi Z}Y + \nabla^{\perp}_Y\phi Z - \nabla^{\perp}_Z\phi Y - \phi[Y,Z] + \alpha[2g(Y,Z)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y]$$

$$-(\beta+1)[\eta(Z)\phi Y + \eta(Y)\phi Z]$$

for any Y, $Z \in D$.

Corollary 3.2. *Let M be a ξ-horizontal CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a semi symmetric non-metric connection. Then*

$$2(\overline{\nabla}_Y\phi)Z = A_{\phi Y}Z - A_{\phi Z}Y + \nabla^{\perp}_Y\phi Z - \nabla^{\perp}_Z\phi Y - \phi[Y,Z] + 2\alpha g(Y,Z)\xi$$

for any Y, $Z \in D^{\perp}$.

Lemma 3.4. *Let M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a semi symmetric non-metric connection. Then*

$$2(\overline{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla^{\perp}_X\phi Y - h(Y,\phi X) - \nabla_Y\phi X - \phi[X,Y] + \alpha[2g(X,Y)\xi - \eta(X)\phi Y$$

$$-\eta(Y)\phi X] - (\beta+1)[\eta(Y)\phi X + \eta(X)\phi Y]$$

for any $X \in D$, $Y \in D^{\perp}$.

Proof. From Gauss and Weingarten equations for $X \in D$ and $Y \in D^{\perp}$ respectively we get

$$\overline{\nabla}_X\phi Y - \overline{\nabla}_Y\phi X = -A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla_Y\phi X - h(Y,\phi X) \tag{20}$$

Also we have

$$\overline{\nabla}_X\phi Y - \overline{\nabla}_Y\phi X = (\overline{\nabla}_X\phi)Y - (\overline{\nabla}_Y\phi)X + \phi[X,Y] \tag{21}$$

From (20) and (21), we get

$$(\overline{\nabla}_X\phi)Y - (\overline{\nabla}_Y\phi)X = -A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y] \tag{22}$$

Also for nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a semi symmetric non-metric connection, we have

$$(\overline{\nabla}_X\phi)Y + (\overline{\nabla}_Y\phi)X = \alpha[2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y] - (\beta+1)[\eta(Y)\phi X + \eta(X)\phi Y] \tag{23}$$

Adding (22) and (23), we get

$$2(\overline{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla^{\perp}_X\phi Y - h(Y,\phi X) - \nabla_Y\phi X - \phi[X,Y] + \alpha[2g(X,Y)\xi - \eta(X)\phi Y$$

$$-\eta(Y)\phi X] - (\beta+1)[\eta(Y)\phi X + \eta(X)\phi Y]$$

Hence the lemma. □

4. Parallel Distributions

Definition 4.1. *The horizontal (resp., vertical) distributions D (resp., $D^{\perp}$) is said to be parallel [1] with respect to the semi-symmetric non-metric connection ∇ on M if $\nabla_X Y \in D$ (resp., $\nabla_Z W \in D^{\perp}$) for any X, $Y \in D$ (resp., W, $Z \in D^{\perp}$).*

Now, we have the following proposition.

Proposition 4.1. *Let M be a ξ-vertical CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a semi symmetric non-metric connection. Then*

$$h(X, \phi Y) = h(Y, \phi X)$$

for all X, $Y \in D$.

Proof. By the parallelness of horizontal distributions D, we have

$$\nabla_X \phi Y \in D, \nabla_Y \phi X \in D \quad for\ any \quad X, Y \in D$$

For $Y \in D$, using the fact that $QX=QY=0$, (16) gives

$$Bh(X, Y) = g(X, Y)Q\xi \quad for\ any \quad X, Y \in D$$

Therefore in view of (2), we have

$$\phi h(X, Y) = g(X, Y)Q\xi + Ch(X, Y) \quad for\ any \quad X, Y \in D$$

From (17), we have

$$h(X, \phi Y) + h(Y, \phi X) = 2\phi h(X, Y) - 2g(X, Y)Q\xi \quad for\ any \quad X, Y \in D \tag{24}$$

Now, putting $X = \phi\ X \in D$ and $Y = \phi\ Y \in D$ in (24), we get respectively

$$h(\phi X, \phi Y) + h(Y, X) = 2\phi h(\phi X, Y) - 2g(\phi X, Y)Q\xi \tag{25}$$

$$h(\phi Y, \phi X) + h(X, Y) = 2\phi h(X, \phi Y) - 2g(X, \phi Y)Q\xi \tag{26}$$

Hence from (24) and (25), we have

$$\phi h(X, \phi Y) - \phi h(Y, \phi X) = g(X, \phi Y)Q\xi - g(\phi X, Y)Q\xi \tag{27}$$

Operating ϕ on both sides of (27) and using $\phi\xi=0$, we get

$$h(X, \phi Y) = h(Y, \phi X)$$

for all X, $Y \in D$. □

Now, for the distribution $D^{\perp}$, we have the following proposition.

Proposition 4.2. *Let M be a ξ-vertical CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a semi symmetric non-metric connection. If the distribution $D^{\perp}$ is parallel with a semi symmetric non-metric connection on M. Then*

$$(A_{\phi Y} Z + A_{\phi Z} Y) \in D^{\perp} \qquad for\ any\ Y,\ Z \in D^{\perp}.$$

Proof. From Weingarten formula, we have

$$\overline{\nabla}_Y \phi Z = -A_{\phi Z} Y + \nabla^{\perp}_Y \phi Z$$

and

$$\overline{\nabla}_Z \phi Y = -A_{\phi Y} Z + \nabla^{\perp}_Z \phi Y \qquad for\ any\ Y,\ Z \in D^{\perp}.$$

From above Weingarten equations, we have

$$-A_{\phi Z} Y + \nabla^{\perp}_Y Z - A_{\phi Y} Z + \nabla^{\perp}_Z \phi Z = (\overline{\nabla}_Y \phi)Z + (\overline{\nabla}_Z \phi)Y + \phi(\overline{\nabla}_Y Z + \overline{\nabla}_Z Y)$$

Using (7) and (14), we obtain

$$-A_{\phi Z}Y - A_{\phi Y}Z = \alpha[2g(Y,Z)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y] - (\beta+1)[\eta(Y)\phi Z + \eta(Z)\phi Y]$$
$$+\phi\nabla_Y Z + \phi\nabla_Z Y + 2\phi h(Y,Z) \qquad for\ any\ Y,\ Z \in D^{\perp}. \tag{28}$$

Taking inner product with $X{\in}D$ in (28), we get

$$g(A_{\phi Z}Y, X) + g(A_{\phi Y}Z, X) = g(\nabla_Y Z, \phi X) + g(\nabla_Z Y, \phi X)$$

If the distributions $D^{\perp}$ is parallel then $\nabla_Y Z \in D^{\perp}$ and $\nabla_Z Y{\in}D^{\perp}$ for any $Y,\ Z \in D^{\perp}$. Thus we have

$$g(A_{\phi Z}Y, X) + g(A_{\phi Y}Z, X) = 0$$

$$g(A_{\phi Z}Y + A_{\phi Y}Z, X) = 0$$

Which implies that $A_{\phi Z}Y{+}A_{\phi Y}Z{\in}D^{\perp}$ for any $Y,\ Z \in D^{\perp}$. □

Definition 4.2. *A CR-submanifold with a semi symmetric non-metric connection is said to be mixed totally geodesic if $h(X,Z) = 0$ for all $X \in D$ and $Z \in D^{\perp}$.*

Lemma 4.1. *Let M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a semi symmetric non-metric connection. Then M is mixed totally geodesic if and only if $A_N X \in D$ for all $X \in D$.*

Definition 4.3. *A normal vector field $N{\neq}0$ with a semi symmetric non-metric connection is called D-parallel normal section if $\nabla^{\perp}_X N{=}0$ for all $X \in D$.*

Now, we have the following proposition.

Proposition 4.3. *Let M be a mixed totally geodesic ξ-vertical CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a semi symmetric non-metric connection. Then the normal section $N \in \phi D^{\perp}$ is D-parallel if and only if $\nabla_X \phi N \in D$ for all $X \in D$.*

5. Integrability conditions of distributions

In this section,we calculate the Nijenhuis tensor $N(X,Y)$ on a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a semi symmetric non-metric connection.

Lemma 5.1. *Let $\overline{M}$ be a nearly trans-hyperbolic Sasakian manifold with a semi symmetric non-metric connection. Then*

$$(\overline{\nabla}_{\phi X}\phi)Y = 2\alpha g(\phi X, Y)\xi - \alpha\eta(Y)\phi X - (\beta+1)\eta(Y)X + (\beta+1)\eta(Y)\eta(X)\xi + \eta(X)\overline{\nabla}_Y\xi$$
$$+\phi(\overline{\nabla}_Y\phi)(X) + ((\overline{\nabla}_Y\eta)X)\xi \tag{29}$$

for any $X,\ Y \in T\overline{M}$.

Proof. From the definition of nearly trans-hyperbolic Sasakian manifold with a semi symmetric non-metric connection $\overline{M}$, we have

$$(\overline{\nabla}_{\phi X}\phi)Y = 2\alpha g(\phi X, Y)\xi - \alpha\eta(Y)\phi X - (\beta+1)\eta(Y)X + (\beta+1)\eta(Y)\eta(X)\xi - (\overline{\nabla}_Y\phi)\phi X \tag{30}$$

Also we have

$$(\overline{\nabla}_Y\phi)\phi X = -\eta(X)\overline{\nabla}_Y\xi - \phi(\overline{\nabla}_Y\phi)X - ((\overline{\nabla}_Y\eta)X)\xi \tag{31}$$

Now using (31) in (30), we get

$$(\overline{\nabla}_{\phi X}\phi)Y = 2\alpha g(\phi X, Y)\xi - \alpha\eta(Y)\phi X - (\beta+1)\eta(Y)X + (\beta+1)\eta(Y)\eta(X)\xi + \eta(X)\overline{\nabla}_Y\xi$$

$$+\phi(\overline{\nabla}_Y\phi)X+((\overline{\nabla}_Y\eta)X)\xi$$

for any X, $Y \in T\overline{M}$, which completes the proof of the lemma. □

On a nearly trans-hyperbolic Sasakian manifold with a semi symmetric non-metric connection $\overline{M}$, Nijenhuis tensor is given by

$$N(X,Y)=(\overline{\nabla}_{\phi X}\phi)Y-(\overline{\nabla}_{\phi Y}\phi)X-\phi(\overline{\nabla}_X\phi)Y+\phi(\overline{\nabla}_Y\phi)X \tag{32}$$

for any X, $Y \in T\overline{M}$. From (29) and (32), we get

$$N(X,Y)=4\alpha g(\phi X,Y)\xi-\alpha[\eta(Y)\phi X-\eta(X)\phi Y]-(\beta+1)[\eta(Y)X-\eta(X)Y]$$
$$+\eta(X)\overline{\nabla}_Y\xi-\eta(Y)\overline{\nabla}_X\xi-2g(X,\phi Y)\xi+2\phi(\overline{\nabla}_Y\phi)X-2\phi(\overline{\nabla}_X\phi)Y \tag{33}$$

In view of (7), we have

$$\phi(\overline{\nabla}_X\phi)Y=-\alpha\eta(Y)\phi X-\alpha\eta(X)\phi Y-(\beta+1)[\eta(Y)X+\eta(X)Y]+2(\beta+1)\eta(X)\eta(Y)\xi$$
$$-\phi(\overline{\nabla}_Y\phi)X$$

Using (33), we obtain

$$N(X,Y)=4\alpha g(\phi X,Y)+\alpha\eta(Y)\phi X+3\alpha\eta(X)\phi Y-(\beta+1)\eta(Y)X+3(\beta+1)\eta(X)Y$$
$$-2g(X,\phi Y)\xi+4\phi(\overline{\nabla}_Y\phi)X-4(\beta+1)\eta(X)\eta(Y)\xi+\eta(X)\overline{\nabla}_Y\xi-\eta(Y)\overline{\nabla}_X\xi \tag{34}$$

for any X, $Y \in T\overline{M}$.

Proposition 5.1. *Let M be a ξ-vertical CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a semi symmetric non-metric connection.Then the distribution D is integrable if the following conditions are satisfied*

$$S(X,Z)\in D, \qquad h(X,Z)=h(\phi X,Z)$$

for any X, $Z \in D$.

Proof. The torsion tensor $S(X,Y)$ of the almost contact metric structure (ϕ,ξ,η,g) is given by

$$S(X,Y)=N(X,Y)+2d\eta(X,Y)\xi=N(X,Y)-2g(\phi X,Y)\xi \tag{35}$$

Thus, we have

$$S(X,Y)=[\phi X,\phi Y]-\phi[\phi X,Y]-\phi[X,\phi Y]-2g(\phi X,Y)\xi$$

for any X, $Y \in TM$. Suppose that the distribution D is integrable. so for X, $Y \in D$, $Q[X,Y]$=0. If $S(X,Y)\in D$, then from (34) and (35), we have

$$4\alpha g(\phi X,Y)Q\xi+4(\phi Q\nabla_Y\phi X+\phi h(Y,\phi X)+Q\nabla_Y X+h(X,Y))=0$$

for any X, $Y \in D$ and $\xi \in D^\perp$. Replacing Y by ϕZ for $Z \in D$,we get

$$4\alpha g(\phi X,\phi Z)Q\xi+4(\phi Q\nabla_{\phi Z}\phi X+\phi h(\phi Z,\phi X)+Q\nabla_{\phi Z}X+h(X,\phi Z))=0 \tag{36}$$

Interchanging X and Z for X, $Z \in D$ in (36), we have

$$4\alpha g(\phi Z,\phi X)Q\xi+4(\phi Q\nabla_{\phi X}\phi Z+\phi h(\phi X,\phi Z)+Q\nabla_{\phi X}Z+h(Z,\phi X))=0$$

Subtracting above equations, we get

$$\phi Q[\phi X,\phi Z]+Q[X,\phi Z]+h(Z,\phi X]-h(X,\phi Z]=0$$

for any X , $Z \in D$ and the assertion follows. □

Now, we prove the following proposition.

Proposition 5.2. *Let M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a semi symmetric non metric connection. Then*

$$A_{\phi Y}Z - A_{\phi Z}Y = \frac{1}{3}\phi P[Y,Z] + \frac{2}{3}\alpha[\eta(Y)Z - \eta(Z)Y]$$

for any Y, $Z \in D^{\perp}$.

Proof. For Y, $Z \in D^{\perp}$ and $X \in TM$, we have

$$2g(A_{\phi Z}Y, X) = 2g(h(X,Y), \phi Z)$$
$$2g(A_{\phi Z}Y, X) = g(h(X,Y), \phi Z) + g(h(X,Y), \phi Z)$$
$$2g(A_{\phi Z}Y, X) = g(\overline{\nabla}_X Y + \overline{\nabla}_Y X, \phi Z)$$
$$2g(A_{\phi Z}Y, X) = -g(\phi(\overline{\nabla}_X Y + \overline{\nabla}_Y X), Z)$$
$$2g(A_{\phi Z}Y, X) = -g[(\overline{\nabla}_Y \phi X + \overline{\nabla}_X \phi Y) - \alpha(2g(X,Y)\xi - \eta(X)\phi Y - \eta(Y)\phi X) + (\beta+1) (\eta(Y)\phi X + \eta(X)\phi Y), Z]$$
$$2g(A_{\phi Z}Y, X) = -g(\overline{\nabla}_Y \phi X, Z) - g(\overline{\nabla}_X \phi Y, Z) + 2\alpha g(X,Y)\eta(Z)$$
$$2g(A_{\phi Z}Y, X) = g(\overline{\nabla}_Y Z, \phi X) + g(A_{\phi Y}Z, X) + 2\alpha g(X,Y)\eta(Z).$$

The above equation is true for all $X \in TM$, therefore transvecting the vector field X both sides, we obtain

$$2A_{\phi Z}Y = A_{\phi Y}Z - \phi\overline{\nabla}_Y Z + 2\alpha\eta(Z)Y \tag{37}$$

Interchanging the vector fields Y and Z, we get

$$2A_{\phi Y}Z = A_{\phi Z}Y - \phi\overline{\nabla}_Z Y + 2\alpha\eta(Y)Z \tag{38}$$

From (37) and (38), we get

$$A_{\phi Y}Z - A_{\phi Z}Y = \frac{1}{3}\phi P[Y,Z] + \frac{2}{3}\alpha[\eta(Y)Z - \eta(Z)Y] \tag{39}$$

for any Y, $Z \in D^{\perp}$, which completes the proof. □

Proposition 5.3. *Let M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a semi symmetric non-metric connection. Then the distribution $D^{\perp}$ is integrable if and only if*

$$A_{\phi Y}Z - A_{\phi Z}Y = \frac{2}{3}\alpha[\eta(Y)Z - \eta(Z)Y] \tag{40}$$

for any Y, $Z \in D^{\perp}$.

Proof. Suppose that the distribution $D^{\perp}$ is integrable. Then $[Y,Z] \in D^{\perp}$ for any Y, $Z \in D^{\perp}$. Since P is a projection operator on D, so $P[Y,Z]$=0. Thus from (39) we get (40). Conversly, we suppose that (40) holds. Then using (39), we have $\phi P[Y,Z]$=0 for any Y, $Z \in D^{\perp}$. Since rank ϕ=$2n$. Therefore, either $P[Y,Z]$=0 or $P[Y,Z]$=$k\xi$. But $P[Y,Z]$=$k\xi$ is not possible as P is a projection operator on D. Thus $P[Y,Z]$=0, which is equivalent to $[Y,Z] \in D^{\perp}$ for any Y, $Z \in D^{\perp}$ and hence $D^{\perp}$ is integrable. □

Corollary 5.1. *Let M be a ξ-horizontal CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a semi symmetric non-metric connection. Then the distribution $D^{\perp}$ is integrable if and only if*

$$A_{\phi Y}Z - A_{\phi Z}Y = 0$$

for any Y, $Z \in D^{\perp}$.

6. Conclusions

The notion of CR-submanifolds of a nearly trans-hyperbolic Sasakian manifold with a semi symmetric non metric connection investigated which shows that the existence of a parallel distribution relating to ξ-vertical CR-submanifolds of a nearly trans-hyperbolic Sasakian manifold with a semi symmetric non metric connection. Further we have tried to find the condition under which the distributions required by CR-submanifolds of a nearly trans-hyperbolic Sasakian manifold with a semi symmetric non metric connection are parallel are obtained. D-parallel normal section have been also studied.

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