TWMS J. App. and Eng. Math. V.14, N.1, 2024, pp. 122-133

# SOFT QUASILINEAR INNER PRODUCT SPACES

M. Ş. GÖNCİ[1], H. BOZKURT[1*], §

Abstract. In this paper, we introduced the notion of soft quaslinear inner product space which is a generalization of the quasilinear inner product spaces and this recent concept gave us the opportunity to work with an approach similar to linear functional analysis. We studied on completeness of soft quasilinear inner product space and we described the concept of the soft Hilbert quasilinear space. Finally, we presented some examples and theorems related to orthogonality and orthonormality in soft quasilinear inner product spaces.

Keywords: Soft quasilinear inner product space, soft Hibert space, orthogonality, orthonormality.

AMS Subject Classification: 54F05, 47H04, 46C99, 46K15.

## 1. Introduction

In [1], Molodtsov presented the concept of soft set as a new mathematical tool for dealing with ambiquities. Then Maji and et al. [2] offered several operations on soft sets. After, Das and Samanta presented the notion of soft element in [3], soft real number in [4], the soft point in [5]. Later, Das and et al. defined notions of soft linear spaces and soft normed linear spaces by using the notion of soft element in [6]. Also, Sezer and Atagün introduced a fundamental type of soft linear spaces which expands the notion of linear spaces in [7]. In [8], Das and Samanta presented the idea on soft inner product on soft linear spaces. Then, research works on soft set theory in other various fields progressed. Yazar and et al. again described the soft vector spaces and soft normed spaces in a point of view in [9] and presented some new features of soft inner product spaces in [10].

On the other hand, Aseev presented the concept of quasilinear spaces and normed quasilinear spaces. Thanks to these new notions, he acquired some conclusions. Then, Levent and Yılmaz [11], [12] worked on some features of quasilinear spaces. In [9], they worked on quasilinear inner product spaces. Recently, we [13] have introduced notions of quasilinear inner product space and Hilbert quasilinear space. In [14], [15] and [16], we

[1] Department of Mathematics, Batman University, Batman, 72100, Batman, Turkey.
e-mail: m.siringonci@gmail.com; ORCID: https://orcid.org/0000-0003-1630-0318.
e-mail: hacer.bozkurt@batman.edu.tr; ORCID: https://orcid.org/0000-0002-2216-2516.
[*] Corresponding author.

§ Manuscript received: December 09, 2021; accepted: March 29, 2022.
TWMS Journal of Applied and Engineering Mathematics, Vol.14, No.1 

studied various features of quasilinear inner product spaces. Then, we [17] defined the concept of soft quasilinear space and soft normed quasilinear spaces based on [3], [6], and [18]. Moreover, we [17] studied on some properties of soft structures of quasilinear spaces with illustrating exercises.

Essentially, in this research, we introduced the notion of soft quasilinear inner product space which extends the concept of quaslinear inner product spaces. After, we researched some features of soft quasilinear inner product spaces with examples. We gave the definition of soft Hilbert spaces and studied on the orthogonality and orthonormality in soft Hilbert quasilinear spaces.

## 2. Preliminaries

In this part, we will recall some concepts reletad to soft set theory and some basic notions such as soft quasilinear spaces and soft normed quasilinear spaces.

Let $Q$ be an universe and $P$ be a set of parameters, $P(Q)$ indicates the power set of $Q$ and $B$ be a non-empty subset of $P$.

**Definition 2.1.** [1] *A pair $(G, P)$ is called a soft set over $Q$, where $G$ is a mapping defined by $G : P \to P(Q)$. In other words, a soft set over $Q$ is a parametrized family of subsets of the universe $Q$. For $\lambda \in P$, $G(\lambda)$ may be considered as the set of $\lambda$-approximate elements of the soft set $(G, P)$.*

**Definition 2.2.** [6] *A soft set $(G, P)$ over $Q$ is said to be an absolute soft set represented by $\widetilde{Q}$, if for every $\lambda \in P$, $G(\lambda) = Q$. A soft set $(G, P)$ over $Q$ is said to be a null soft set represented by $\Phi$, if for every $\lambda \in P$, $G(\lambda) = \emptyset$.*

**Definition 2.3.** [4] *Let $Q$ be an non-empty set and $P$ be a nonempty parameter set. Then a function $q : P \to Q$ is said to be soft element of $Q$. A soft element $q$ of $Q$ is said belongs to a soft set $G$ of $Q$, which is denoted by $q \in Q$, if $q(\lambda) \in G(\lambda)$, $\lambda \in P$. So, for a soft set $G$ of $Q$ with respect to the index set $P$, we get $G(\lambda) = \left\{q(\lambda), \lambda \widetilde{\in} P\right\}$. A soft set $(G, P)$ for which $G(\lambda)$ is a singleton set, $\forall \lambda \in P$ can be determined with a soft element by simply determined the singleton set with the element that it contains $\forall \lambda \in P$.*

The set of all soft sets $(G, P)$ over $Q$ will be described by $S\left(\widetilde{Q}\right)$ for which $G(\lambda) \neq \emptyset$, for all $\lambda \in P$ and the collection of all soft elements of $(G, P)$ over $Q$ will be denoted by $SE\left(\widetilde{Q}\right)$.

**Definition 2.4.** [17] *Let $Q$ be a quasilinear space and $P$ be a parameter set. Let $G$ be a soft set over $(Q, P)$. $G$ is said to be a soft quasilinear space of $Q$ if $Q(\lambda)$ is a quasilinear subspace of $Q$ for every $\lambda \in P$.*

[17] We use the notation $\widetilde{q}, \widetilde{w}, \widetilde{z}$ to indicate soft quasi vectors of a soft quasilinear space and $\widetilde{a}, \widetilde{b}, \widetilde{c}$ to specify soft real numbers. If a soft quasi element $\widetilde{q}$ has an inverse i.e. $\widetilde{q} - \widetilde{q} = \widetilde{\theta}$ such that $\widetilde{q}(\lambda) - \widetilde{q}(\lambda) = \widetilde{\theta}(\lambda)$ for every $\lambda \in P$ then it is called regular. If a soft quasi element $\widetilde{q}$ has no inverse, then it is called singular. Also, $\widetilde{Q}_r$ express for the set of all soft regular elements in $\widetilde{Q}$ and $\widetilde{Q}_s$ imply the sets of all soft singular elements in $\widetilde{Q}$.

**Definition 2.5.** [17] *Let $\widetilde{Q}$ be the absolute soft quasilinear space i.e. $\widetilde{Q}(\lambda) = Q$ for every $\lambda \in P$. Then a mapping $\|.\| : SE\left(\widetilde{Q}\right) \longrightarrow \mathbb{R}(P)$ is said to be soft norm on the soft quasilinear space $\widetilde{Q}$, if $\|.\|$ satisfies the following conditions:*

i) $\|\widetilde{q}\|\widetilde{\geq}0$ if $\widetilde{q}\neq\overline{\theta}$ for every $\widetilde{q}\in\widetilde{Q}$,
ii) $\|\widetilde{q}+\widetilde{w}\|\widetilde{\leq}\|\widetilde{q}\|+\|\widetilde{w}\|$ for every $\widetilde{q},\widetilde{w}\in\widetilde{Q}$,
iii) $\|\widetilde{\alpha}\cdot\widetilde{q}\|=|\widetilde{\alpha}|\cdot\|\widetilde{q}\|$ for every $\widetilde{q}\in\widetilde{Q}$ and for every soft scalar $\widetilde{\alpha}$,
iv) if $\widetilde{q}\widetilde{\preceq}\widetilde{w}$, then $\|\widetilde{q}\|\widetilde{\leq}\|\widetilde{w}\|$ for every $\widetilde{q},\widetilde{w}\in\widetilde{Q}$,
v) if for any $\varepsilon\widetilde{>}0$ there exists an element $\widetilde{q}_\varepsilon\in\widetilde{Q}$ such that, $\widetilde{q}\widetilde{\preceq}\widetilde{w}+\widetilde{q}_\varepsilon$ and $\|\widetilde{q}_\varepsilon\|\widetilde{\leq}\varepsilon$ then $\widetilde{q}\widetilde{\preceq}\widetilde{w}$ for any soft elements $\widetilde{q},\widetilde{w}\in\widetilde{Q}$.

A soft quasilinear space $\widetilde{Q}$ with a soft norm$\|.\|$ on $\widetilde{Q}$ is called soft normed quasilinear space and is indicated by $\left(\widetilde{Q},\|.\|\right)$ or $\left(\widetilde{Q},\|.\|,P\right)$.

**Theorem 2.1.** *If a soft norm $\|.\|$ on soft normed quasilinear space $\widetilde{Q}$ satisfied the condition " $\xi\in Q$, and $\lambda\in P$, $\{\|\widetilde{q}\|(\lambda)=\xi\}$ is a singleton set.". If for every $\lambda\in P$, $\|.\|_\lambda:Q\to\mathbb{R}^+$ be a mapping such that for every $\xi\in Q$, $\|\xi\|_\lambda=\|\widetilde{q}\|(\lambda)$, where $\widetilde{q}\in\widetilde{Q}$ such that $\widetilde{q}(\lambda)=\xi$. Then for every $\lambda\in P$, $\|.\|_\lambda$ is a norm on quasilinear space $Q$.*

*Proof.* This norm is well-defined from above condition. Here, we obtain norm conditions of $\|.\|_\lambda$ from soft norm axioms of $\|.\|$. □

[17] Let $\widetilde{Q}$ be a soft normed quasilinear space. Then, soft Hausdorff or soft norm metric on $\widetilde{Q}$ is defined by

$$h_Q(\widetilde{q},\widetilde{w})=\inf\left\{\widetilde{r}\widetilde{\geq}0:\widetilde{q}\widetilde{\preceq}\widetilde{w}+\widetilde{q}_1^r\ ,\ \widetilde{w}\widetilde{\preceq}\widetilde{q}+\widetilde{q}_2^r,\|\widetilde{q}_i^r\|\widetilde{\leq}\widetilde{r}\right\}.$$

## 3. Main Results

Let $\widetilde{Q}$ be the absolute soft quasilinear space i.e. $\widetilde{Q}(\lambda)=Q,\forall\lambda\in P$. We use the notation $\widetilde{q},\widetilde{w},\widetilde{z}$ to describe soft quasi vectors of a soft quasilinear space and $\widetilde{a},\widetilde{b},\widetilde{c}$ to describe soft real numbers.

Before giving the description of soft quasilinear inner product space, let's present some new concepts and examples necessary to define the notion of soft quasilinear inner product space.

**Definition 3.1.** *Let $\widetilde{Q}$ be a soft quasilinear space, $\widetilde{W}\subseteq\widetilde{Q}$ and $\widetilde{q}\in\widetilde{W}$. The set*

$$F_{\widetilde{q}}^{\widetilde{W}}=\{\widetilde{m}\in\widetilde{W}_r:\widetilde{m}\widetilde{\preceq}\widetilde{q}\},$$

*is called floor in $\widetilde{W}$ of $\widetilde{q}$. If $\widetilde{W}=\widetilde{Q}$ then we will say only floor of $\widetilde{q}$ and written shortly $F_{\widetilde{q}}$ instead of $F_{\widetilde{q}}^{\widetilde{Q}}$.*

**Definition 3.2.** *Let $\widetilde{Q}$ be a soft quasilinear space, $\widetilde{Q}$ is called a solid floored soft quasilinear space whenever*

$$\widetilde{q}=\sup\{\widetilde{m}\in\widetilde{W}_r:\widetilde{m}\widetilde{\preceq}\widetilde{q}\}$$

*for every $\widetilde{q}\in\widetilde{Q}$. Otherwise, $\widetilde{Q}$ is called a non-solid floored soft quasilinear space.*

**Example 3.1.** *Let us consider the Example 12 in [17] and the soft set $(G,(\Omega_C(\mathbb{R}))_r)$ over quasilinear space $\Omega_C(\mathbb{R})$. Let $\widetilde{q}$ be a soft quasi element of $(G,(\Omega_C(\mathbb{R}))_r)$ as the following*

$$\widetilde{q}(\lambda)=[-\lambda,\lambda]\in\Omega_C(\mathbb{R})$$

*for every $\lambda\in(\Omega_C(\mathbb{R}))_r$. The floor of $\widetilde{q}$ be defined as follows:*

$$\begin{aligned}F_{\widetilde{q}}&=\left\{\widetilde{w}\in(\Omega_C(\mathbb{R}))_r:\widetilde{w}\widetilde{\preceq}\widetilde{q}\right\}\\&=\left\{\widetilde{w}(\lambda)\in(\Omega_C(\mathbb{R}))_r(\mathbb{R}):\widetilde{w}(\lambda)\widetilde{\subseteq}\widetilde{q}(\lambda)\right\}.\end{aligned}$$

*For example,* $\lambda = \{1\} \in (\Omega_C(\mathbb{R}))_r$ *then* $\widetilde{q}(\{1\}) = [-1, 1] \in \Omega_C(\mathbb{R})$. *If* $\widetilde{w}(\{1\}) = \{0\}$, $\widetilde{z}(\{1\}) = \left\{\frac{1}{2}\right\}$ *and* $\widetilde{v}\{1\} = \left\{\frac{3}{2}\right\}$ *then* $\widetilde{w}, \widetilde{z} \in F_{\widetilde{q}}$ *but,* $\widetilde{v} \notin F_{\widetilde{q}}$.

**Theorem 3.1.** *Absolute soft quasilinear space* $\left(\widetilde{\Omega_C(\mathbb{R})}\right)$ *is a solid floored.*

*Proof.* Since $\Omega_C(\mathbb{R})$ is solid floored quasilinear space, we get $x = \sup F_x$ for every $x \in \Omega_C(\mathbb{R})$. Let $\widetilde{q}, \widetilde{w}$ be soft quasi vectors of the absolute soft quasilinear space $\widetilde{\Omega_C(\mathbb{R})}$. Then, we get

$$\widetilde{q}(\lambda) = \sup F_{\widetilde{q}(\lambda)} = \sup\left\{\widetilde{w}(\lambda) \in (\Omega_C(\mathbb{R}))_r : \widetilde{w}(\lambda)\widetilde{\subseteq}\widetilde{q}(\lambda)\right\}$$

for all $\lambda \in P$ since $\Omega_C(\mathbb{R})$ is solid floored quasilinear space. So, $\left(\widetilde{\Omega_C(\mathbb{R})}\right)$ is a soft solid floored quasilinear space. □

**Definition 3.3.** *Let* $\widetilde{Q}$ *be a soft quasilinear space. Consolidation of floor of* $\widetilde{Q}$ *is the smallest solid floored soft quasilinear space* $\widehat{\left(\widetilde{Q}\right)}$ *containing* $\widetilde{Q}$*, namely, if there exists different solid floored soft quasilinear space* $\widetilde{W}$ *including* $\widetilde{Q}$*, then* $\widehat{\left(\widetilde{Q}\right)}\widetilde{\subseteq}\widetilde{W}$ .

Let us now give an expanded description of soft inner product quasilinear space.

**Definition 3.4.** *Let* $\widetilde{Q}$ *be the absolute soft quasilinear space i.e.* $\widetilde{Q}(\lambda) = Q$, $\forall\lambda \in P$. *Then a mapping*

$$\langle . \rangle : SE\left(\widetilde{Q}\right) \times SE\left(\widetilde{Q}\right) \to \Omega(\mathbb{R})(P)$$

*is said to be a soft quasi inner product on the soft quasilinear space* $\widetilde{Q}$*, if* $\langle . \rangle$ *satisfies the folllowing conditions:*

*i)* $\langle \widetilde{q}, \widetilde{w} \rangle \in (\Omega(\mathbb{R}))_r \equiv \widetilde{\mathbb{R}}$ *if* $\widetilde{q}, \widetilde{w} \in \widetilde{Q}_r$,

*ii)* $\langle \widetilde{q} + \widetilde{w}, \widetilde{z} \rangle \widetilde{\subseteq} \langle \widetilde{q}, \widetilde{z} \rangle + \langle \widetilde{w}, \widetilde{z} \rangle$ *for all* $\widetilde{q}, \widetilde{w}, \widetilde{z} \in \widetilde{Q}$,

*iii)* $\langle \widetilde{\alpha} \cdot \widetilde{q}, \widetilde{w} \rangle = \widetilde{\alpha} \cdot \langle \widetilde{q}, \widetilde{w} \rangle$ *for all* $\widetilde{q}, \widetilde{w} \in \widetilde{Q}$ *and for every soft scalar* $\widetilde{\alpha}$,

*iv)* $\langle \widetilde{q}, \widetilde{w} \rangle = \langle \widetilde{w}, \widetilde{q} \rangle$ *for all* $\widetilde{q}, \widetilde{w} \in \widetilde{Q}$,

*v)* $\langle \widetilde{q}, \widetilde{w} \rangle \widetilde{\supseteq} \overline{0}$ *if* $\widetilde{q} \in \widetilde{Q}_r$ *and* $\langle \widetilde{q}, \widetilde{q} \rangle = \left\{\overline{0}\right\} \Leftrightarrow \widetilde{q} = \{\theta\}$,

*vi)* $\|\langle \widetilde{q}, \widetilde{w} \rangle\|_{\Omega(\mathbb{R})} = \sup\left\{\|\langle x, y \rangle\| : x \in F_{\widetilde{q}}^{\widehat{(\widetilde{Q})}}, y \in F_{\widetilde{w}}^{\widehat{(\widetilde{Q})}}\right\}$,

*vii)* $\langle \widetilde{q}, \widetilde{w} \rangle \widetilde{\subseteq} \langle \widetilde{z}, \widetilde{v} \rangle$ *if* $\widetilde{q}\widetilde{\preceq}\widetilde{z}$ *and* $\widetilde{w}\widetilde{\preceq}\widetilde{v}$ *for all* $\widetilde{q}, \widetilde{w}, \widetilde{z}, \widetilde{v} \in \widetilde{Q}$,

*viii)* $\forall\widetilde{\epsilon}\widetilde{\geq}\overline{0}$, $\exists\widetilde{q_\epsilon} \in \widetilde{Q}$ *such that* $\widetilde{q}\widetilde{\preceq}\widetilde{w} + \widetilde{q_\epsilon}$ *and* $\langle \widetilde{q_\epsilon}, \widetilde{q_\epsilon} \rangle \widetilde{\subseteq} S_{\widetilde{\epsilon}}(\theta)$ *then* $\widetilde{q}\widetilde{\preceq}\widetilde{w}$.

*A soft quasilinear space* $\widetilde{Q}$ *with a soft quasi inner product* $\langle . \rangle$ *on* $\widetilde{Q}$ *is called a soft quasilinear inner product space and denoted by* $\left(\widetilde{Q}, \langle . \rangle, P\right)$.

**Remark 3.1.** *If* $\widetilde{Q}$ *is a soft linear space, then above conditions are determined by conditions of the real soft inner product spaces. Moreover, a regular subspace* $\widetilde{Q}_r$ *of a soft quasilinear inner product space* $\widetilde{Q}$ *is a soft (linear) inner product space with the same inner product.*

**Example 3.2.** *Let* $Q = \Omega_C(\mathbb{R})$. *Then* $Q$ *is a quasilinear inner product space with respect to the inner product* $\langle X, Y \rangle = \{xy : x \in X, y \in Y\}$ *for every* $X, Y \in Q$ [14]. *Let* $\widetilde{q}, \widetilde{w}$ *be soft quasi elements of the absolute soft quasilinear space* $\widetilde{Q} = \widetilde{\Omega_C(\mathbb{R})}$ *such that*

$$\begin{aligned} \widetilde{q} \quad &: \quad P \to P(Q) \\ \lambda \quad &\to \quad \widetilde{q}(\lambda) = q^\lambda \end{aligned}$$

*and*

$$\begin{aligned} \widetilde{w} &: P \to P(Q) \\ \lambda &\to \widetilde{w}(\lambda) = w^{\lambda}. \end{aligned}$$

*Then* $\widetilde{q}(\lambda), \widetilde{w}(\lambda) \in Q$. *The mapping*

$$\begin{aligned} \langle . \rangle &: SE\left(\widetilde{Q}\right) \times SE\left(\widetilde{Q}\right) \to \Omega\left(\mathbb{R}\right)(P) \\ (\widetilde{q}, \widetilde{w}) &= \langle \widetilde{q}, \widetilde{w} \rangle (\lambda) \\ &= \left\langle q^{\lambda}, w^{\lambda} \right\rangle \\ &= \left\{ x^{\lambda} y^{\lambda} : x^{\lambda} \in q^{\lambda}, y^{\lambda} \in w^{\lambda} \right\} \\ &= \langle \widetilde{q}(\lambda), \widetilde{w}(\lambda) \rangle \ \ , (\lambda \in P) \end{aligned}$$

*is a soft quasilinear inner product on the soft quasilinear space* $\widetilde{Q}$. *Also,* $\widetilde{\Omega(\mathbb{R})}$ *is soft quasilinear inner product with same soft inner product. If we admit* $Q = \Omega_C(\mathbb{R}^n)$, *then is a quasilinear inner product space with respect to the inner product* $\langle X, Y \rangle = \{\langle x, y \rangle_{\mathbb{R}^n} : x \in X, y \in Y\}$ *for every* $X, Y \in Q$ [14]. *Additionaly, the mapping*

$$\begin{aligned} \langle . \rangle_1 &: SE\left(\widetilde{\Omega_C(\mathbb{R}^n)}\right) \times SE\left(\widetilde{\Omega_C(\mathbb{R}^n)}\right) \to \Omega\left(\mathbb{R}\right)(P) \\ (\widetilde{q}, \widetilde{w}) &= \langle \widetilde{q}, \widetilde{w} \rangle_1 (\lambda) \\ &= \left\langle q^{\lambda}, w^{\lambda} \right\rangle \\ &= \left\{ \left\langle x^{\lambda}, y^{\lambda} \right\rangle_{\mathbb{R}^n} : x^{\lambda} \in q^{\lambda}, y^{\lambda} \in w^{\lambda} \right\} \\ &= \langle \widetilde{q}(\lambda), \widetilde{w}(\lambda) \rangle \ \ , (\lambda \in P) \end{aligned}$$

*is a soft quasilinear inner product on the soft quasilinear space* $\widetilde{\Omega_C(\mathbb{R}^n)}$.

**Proposition 3.1.** *Let* $\{\langle . \rangle_{\lambda} : \lambda \in P\}$ *be a cluster of inner product on quasilinear space* $Q$. *Then the mapping*

$$\begin{aligned} \langle . \rangle &: SE\left(\widetilde{Q}\right) \times SE\left(\widetilde{Q}\right) \to \Omega\left(\mathbb{R}\right)(P) \\ (\widetilde{q}, \widetilde{w}) &\to \langle \widetilde{q}, \widetilde{w} \rangle (\lambda) = \langle \widetilde{q}(\lambda), \widetilde{w}(\lambda) \rangle_{\lambda}, \end{aligned}$$

$\forall \lambda \in P, \forall \widetilde{q}, \widetilde{w} \in \widetilde{Q}$ *is a soft quasi inner product on the soft quasilinear space* $\widetilde{Q}$.

*Proof.* $\forall \widetilde{q}, \widetilde{w} \in \widetilde{Q}_r$ so, $q, w \in \widetilde{Q}_r \equiv \widetilde{\mathbb{R}}(P)$ such that $\widetilde{q}(\lambda) = q$ and $\widetilde{w}(\lambda) = w$. $\langle \widetilde{q}, \widetilde{w} \rangle (\lambda) = \langle \widetilde{q}(\lambda), \widetilde{w}(\lambda) \rangle_{\lambda} = \langle q, w \rangle_{\lambda} \in (\Omega(\mathbb{R}))_r$.

For every $\widetilde{q}, \widetilde{w}, \widetilde{z} \in \widetilde{Q}$, we have

$$\begin{aligned} \langle \widetilde{q} + \widetilde{w}, \widetilde{z} \rangle (\lambda) &= \langle \widetilde{q}(\lambda) + \widetilde{w}(\lambda), \widetilde{z}(\lambda) \rangle_{\lambda} \\ &\ \widetilde{\subseteq} \langle \widetilde{q}(\lambda), \widetilde{w}(\lambda) \rangle_{\lambda} + \langle \widetilde{w}(\lambda), \widetilde{z}(\lambda) \rangle_{\lambda} \\ &= \langle \widetilde{q}, \widetilde{z} \rangle (\lambda) + \langle \widetilde{w}, \widetilde{z} \rangle (\lambda). \end{aligned}$$

For every $\widetilde{q}, \widetilde{w} \in \widetilde{Q}$ and for every soft scalar $\widetilde{\alpha}$, we have

$$\begin{aligned} \langle \widetilde{\alpha} \cdot \widetilde{q}, \widetilde{w} \rangle (\lambda) &= \langle \widetilde{\alpha}(\lambda) \cdot \widetilde{q}(\lambda), \widetilde{w}(\lambda) \rangle_{\lambda} \\ &= \widetilde{\alpha}(\lambda) \cdot \langle \widetilde{q}(\lambda), \widetilde{w}(\lambda) \rangle_{\lambda} \\ &= \widetilde{\alpha} \cdot \langle \widetilde{q}, \widetilde{w} \rangle (\lambda). \end{aligned}$$

We get

$$\langle \widetilde{q}, \widetilde{w} \rangle (\lambda) = \langle \widetilde{q}(\lambda), \widetilde{w}(\lambda) \rangle_\lambda = \langle \widetilde{w}(\lambda), \widetilde{q}(\lambda) \rangle_\lambda = \langle \widetilde{w}, \widetilde{q} \rangle (\lambda)$$

for every $\widetilde{q}, \widetilde{w} \in \widetilde{Q}$. We obtain $\langle \widetilde{q}, \widetilde{q} \rangle \widetilde{\geq} \overline{0}, \forall \widetilde{q} \in \widetilde{Q_r}$ from $\langle \widetilde{q}, \widetilde{q} \rangle (\lambda) = \langle \widetilde{q}(\lambda), \widetilde{q}(\lambda) \rangle_\lambda \geq \{0\}, \forall \lambda \in P, \forall \widetilde{q} \in \widetilde{Q}_r$. Also, $\langle \widetilde{q}, \widetilde{q} \rangle (\lambda) = \overline{0}$ if and only if $\langle \widetilde{q}(\lambda), \widetilde{q}(\lambda) \rangle_\lambda = \{\overline{0}\}$. Thus $\widetilde{q} = \theta$ since $\widetilde{q}(\lambda) = \overline{0}$ for every $\lambda \in P$ . Further, we get

$$\begin{aligned} \|\langle \widetilde{q}, \widetilde{w} \rangle\|_{\Omega(\mathbb{R})} &= \sup\{|\widetilde{z}(\lambda)| : \widetilde{z}(\lambda) \in \langle \widetilde{q}(\lambda), \widetilde{w}(\lambda) \rangle\} \\ &= \sup\{|xy| : x \in \widetilde{q}(\lambda), y \in \widetilde{w}(\lambda)\} \\ &= \sup\{\|\langle x, y \rangle\| : x \in F_{\widetilde{q}(\lambda)}, y \in F_{\widetilde{w}(\lambda)}\} \\ &= \sup\left\{\|\langle x, y \rangle\| : x \in F_{\widetilde{q}}^{(\widetilde{Q})}, y \in F_{\widetilde{w}}^{(\widetilde{Q})}\right\}. \end{aligned}$$

If $\widetilde{q} \widetilde{\preceq} \widetilde{w}$ and $\widetilde{w} \widetilde{\preceq} \widetilde{v}$ then $\widetilde{q}(\lambda) \widetilde{\leq} \widetilde{w}(\lambda)$ and $\widetilde{z}(\lambda) \widetilde{\leq} \widetilde{v}(\lambda)$ for all $\widetilde{q}, \widetilde{w}, \widetilde{z}, \widetilde{v} \in \widetilde{Q}$. From [19], we have $\langle \widetilde{q}(\lambda), \widetilde{z}(\lambda) \rangle_\lambda \widetilde{\subseteq} \langle \widetilde{w}(\lambda), \widetilde{v}(\lambda) \rangle_\lambda$. Thus, we obtain $\langle \widetilde{q}, \widetilde{z} \rangle \widetilde{\preceq} \langle \widetilde{w}, \widetilde{v} \rangle$ .

Let $\forall \widetilde{\epsilon} \widetilde{\geq} \overline{0}$, $\exists \widetilde{q_\epsilon} \in \widetilde{Q}$ such that $\widetilde{q} \widetilde{\preceq} \widetilde{w} + \widetilde{q_\epsilon}$ and $\langle \widetilde{q_\epsilon}, \widetilde{q_\epsilon} \rangle \widetilde{\subseteq} S_{\widetilde{\epsilon}}(\overline{\theta})$. Then $\widetilde{q}(\lambda) \widetilde{\leq} \widetilde{w}(\lambda) + \widetilde{q_\epsilon}(\lambda)$ and $\langle \widetilde{q_\epsilon}(\lambda), \widetilde{q_\epsilon}(\lambda) \rangle_\lambda \widetilde{\subseteq} S_{\widetilde{\epsilon}}(\overline{\theta})(\lambda)$ for every $\lambda \in P, \forall \widetilde{q}, \widetilde{w} \in \widetilde{Q}$. Thus, we get $\widetilde{q}(\lambda) \widetilde{\leq} \widetilde{q}(\lambda)$ since $Q$ is a quasilinear inner prodcut space with inner product $\{\langle . \rangle_\lambda : \lambda \in P\}$. Thus, $\widetilde{q} \widetilde{\preceq} \widetilde{w}$. □

**Corollary 3.1.** *Every inner product $\langle . \rangle_Q$ on a quasilinear space $Q$ can be extended to a soft quasilinear inner product on the soft quasilinear space $\widetilde{Q}$.*

*Proof.* If we take a mapping $\langle . \rangle : SE\left(\widetilde{Q}\right) \times SE\left(\widetilde{Q}\right) \to \Omega(\mathbb{R})(P)$ by $(\widetilde{q}, \widetilde{w}) \to \langle \widetilde{q}, \widetilde{w} \rangle (\lambda) = \langle \widetilde{q}(\lambda), \widetilde{w}(\lambda) \rangle_Q$, $\forall \lambda \in P, \forall \widetilde{q}, \widetilde{w} \in \widetilde{Q}$ then using the same procedure as in Proposition 3.1, we can proved that $\langle . \rangle$ is a soft inner product on soft quasilinear space $\widetilde{Q}$. □

**Theorem 3.2.** *If a soft quasilinear inner product $\langle . \rangle$ satisfies the*

$$\left\{\langle \widetilde{q}, \widetilde{w} \rangle (\lambda) : \widetilde{q}, \widetilde{w} \in \widetilde{Q} \textit{ such that } \widetilde{q}(\lambda) = q,\ \widetilde{w}(\lambda) = w\right\} \tag{1}$$

*is a single-valued set for $(q, w) \in Q \times Q$ and $\lambda \in P$ then $\langle . \rangle_\lambda : Q \times Q \to \Omega(\mathbb{R})$ be a mapping such that for all $(q, w) \in Q \times Q$, $\langle q, w \rangle_\lambda = \langle \widetilde{q}, \widetilde{w} \rangle (\lambda)$ is an inner product on quasilinear space $Q$.*

*Proof.* This inner product is well-defined from above condition. Here, we obtain quasilinear inner product conditions of $\langle . \rangle_\lambda$ from soft quasilinear inner product axioms of $\langle . \rangle$. □

**Proposition 3.2.** *Let $\left(\widetilde{Q}, \langle . \rangle, P\right)$ be a soft quasilinear inner product space $\widetilde{q}, \widetilde{w}, \widetilde{z} \in \widetilde{Q}$ and $\widetilde{\alpha}, \widetilde{\beta}$ be soft scalars. Then*

*a)* $\left\langle \widetilde{\alpha} \cdot \widetilde{q} + \widetilde{\beta} \cdot \widetilde{w}, \widetilde{z} \right\rangle \widetilde{\subseteq} \widetilde{\alpha} \cdot \langle \widetilde{q}, \widetilde{z} \rangle + \widetilde{\beta} \cdot \langle \widetilde{w}, \widetilde{z} \rangle$,

*b)* $\langle \widetilde{q}, \widetilde{\alpha} \cdot \widetilde{w} \rangle = \widetilde{\alpha} \cdot \langle \widetilde{q}, \widetilde{w} \rangle$,

*c)* $\left\langle \widetilde{q}, \widetilde{\alpha} \cdot \widetilde{w} + \widetilde{\beta} \cdot \widetilde{z} \right\rangle \widetilde{\subseteq} \widetilde{\alpha} \cdot \langle \widetilde{q}, \widetilde{w} \rangle + \widetilde{\beta} \cdot \langle \widetilde{q}, \widetilde{z} \rangle$.

*Proof.* We have, $\left\langle \widetilde{\alpha} \cdot \widetilde{x} + \widetilde{\beta} \cdot \widetilde{y}, \widetilde{z} \right\rangle \widetilde{\subseteq} \langle \widetilde{\alpha} \cdot \widetilde{x}, \widetilde{z} \rangle + \left\langle \widetilde{\beta} \cdot \widetilde{y}, \widetilde{z} \right\rangle = \widetilde{\alpha} \cdot \langle \widetilde{x}, \widetilde{z} \rangle + \widetilde{\beta} \cdot \langle \widetilde{y}, \widetilde{z} \rangle$, $\langle \widetilde{x}, \widetilde{\alpha} \cdot \widetilde{y} \rangle = \langle \widetilde{\alpha} \cdot \widetilde{y}, \widetilde{x} \rangle = \widetilde{\alpha} \cdot \langle \widetilde{y}, \widetilde{x} \rangle = \widetilde{\alpha} \cdot \langle \widetilde{x}, \widetilde{y} \rangle$ and $\left\langle \widetilde{x}, \widetilde{\alpha} \cdot \widetilde{y} + \widetilde{\beta} \cdot \widetilde{z} \right\rangle \widetilde{\subseteq} \langle \widetilde{x}, \widetilde{\alpha} \cdot \widetilde{y} \rangle + \left\langle \widetilde{x} + \widetilde{\beta} \cdot \widetilde{z} \right\rangle = \widetilde{\alpha} \cdot \langle \widetilde{x}, \widetilde{y} \rangle + \widetilde{\beta} \cdot \langle \widetilde{x}, \widetilde{z} \rangle$. □

**Theorem 3.3.** *Let $\left(\widetilde{Q}, \langle . \rangle, P\right)$ be a soft quasilinear inner product space satisfying (1). Let $\widetilde{q}, \widetilde{w} \in \widetilde{Q}$. Then*

$$\|\langle \widetilde{q}, \widetilde{w} \rangle\| \widetilde{\leq} \|\widetilde{q}\| \, \|\widetilde{w}\|. \tag{2}$$

*Proof.* Let $\widetilde{Q}$ be a soft quasilinear inner product space with inner product $\langle . \rangle$. From vi) condition of Definition 3.4 and Remark 3.1, we get

$$\begin{aligned}\|\langle \widetilde{q}, \widetilde{w}\rangle\|_{\Omega(\mathbb{R})} &= \sup\left\{\|\langle x, y\rangle\|_{\Omega(\mathbb{R})} : x \in F_{\widetilde{q}}^{\widehat{(\widetilde{Q})}}, y \in F_{\widetilde{w}}^{\widehat{(\widetilde{Q})}}\right\} \\ &= \sup\left\{|\langle x, y\rangle| : x \in F_{\widetilde{q}}^{\widehat{(\widetilde{Q})}}, y \in F_{\widetilde{w}}^{\widehat{(\widetilde{Q})}}\right\} \\ &\widetilde{\leq} \sup\left\{\|x\|_{\widehat{(\widetilde{Q})}} \|y\|_{\widehat{(\widetilde{Q})}} . : x \in F_{\widetilde{q}}^{\widehat{(\widetilde{Q})}}, y \in F_{\widetilde{w}}^{\widehat{(\widetilde{Q})}}\right\} \\ &\widetilde{\leq} \sup\left\{\|x\|_{\widehat{(\widetilde{Q})}} : x \in F_{\widetilde{q}}^{\widehat{(\widetilde{Q})}}\right\} \sup\left\{\|y\|_{\widehat{(\widetilde{Q})}} . : y \in F_{\widetilde{w}}^{\widehat{(\widetilde{Q})}}\right\} \\ &= \|\widetilde{q}\|_{\widetilde{Q}} \|\widetilde{w}\|_{\widetilde{Q}}\end{aligned}$$

for every $\widetilde{q}, \widetilde{w} \in \widetilde{Q}$. □

**Theorem 3.4.** *Let $\left(\widetilde{Q}, \langle . \rangle, P\right)$ be a soft quasilinear inner product space satisfying (1). Let us define $\|.\| : \widetilde{Q} \to \mathbb{R}(P)$ by*

$$\|\widetilde{q}\| = \sqrt{\|\langle \widetilde{q}, \widetilde{w}\rangle\|_{\Omega(\mathbb{R})}}$$

*for all $\widetilde{q} \in \widetilde{Q}$. Then $\|.\|$ is a soft quasi norm on $\widetilde{Q}$.*

*Proof.* For all $\widetilde{q} \in \widetilde{Q}$, we get $\|\widetilde{q}\| = \sqrt{\|\langle \widetilde{q}, \widetilde{q}\rangle\|_{\Omega(\mathbb{R})}} \widetilde{\geq} \widetilde{0}$ and $\|\widetilde{q}\| = \widetilde{0}$ if and only if $\sqrt{\|\langle \widetilde{q}, \widetilde{q}\rangle\|_{\Omega(\mathbb{R})}} = \widetilde{0}$ if and only if $\widetilde{q} = \theta$.

We obtain,

$$\|\widetilde{q} + \widetilde{w}\|^2 = \|\langle \widetilde{q} + \widetilde{w}, \widetilde{q} + \widetilde{w}\rangle\|_{\Omega(\mathbb{R})} \widetilde{\leq} \|\langle \widetilde{q}, \widetilde{q}\rangle + \langle \widetilde{q}, \widetilde{w}\rangle + \langle \widetilde{w}, \widetilde{q}\rangle + \langle \widetilde{w}, \widetilde{w}\rangle\| \widetilde{\leq} (\|\widetilde{q}\| + \|\widetilde{w}\|)^2$$

from (2) for every $\widetilde{q}, \widetilde{w} \in \widetilde{Q}$.

We have,

$$\|\widetilde{\alpha} \cdot \widetilde{q}\| = \sqrt{\|\langle \widetilde{\alpha} \cdot \widetilde{q}, \widetilde{\alpha} \cdot \widetilde{q}\rangle\|_{\Omega(\mathbb{R})}} = \sqrt{\|\widetilde{\alpha}\widetilde{\alpha} \cdot \langle \widetilde{q}, \widetilde{q}\rangle\|_{\Omega(\mathbb{R})}} = |\widetilde{\alpha}| \sqrt{\|\langle \widetilde{q}, \widetilde{q}\rangle\|_{\Omega(\mathbb{R})}} = |\widetilde{\alpha}| \|\widetilde{q}\|$$

for all $\widetilde{q} \in \widetilde{Q}$ and for every soft scalar $\widetilde{\alpha}$.

If $\widetilde{q} \widetilde{\leq} \widetilde{w}$ for every $\widetilde{q}, \widetilde{w} \in \widetilde{Q}$, then $\langle \widetilde{q}, \widetilde{q}\rangle \widetilde{\subseteq} \langle \widetilde{w}, \widetilde{w}\rangle$. Thus, we get

$$\|\widetilde{q}\|^2 = \|\langle \widetilde{q}, \widetilde{q}\rangle\|_{\Omega(\mathbb{R})} \widetilde{\leq} \|\langle \widetilde{w}, \widetilde{w}\rangle\|_{\Omega(\mathbb{R})} = \|\widetilde{w}\|^2 .$$

Let for any $\widetilde{\epsilon} \widetilde{\geq} \widetilde{0}$ there exists an soft quasi element $\widetilde{q}_\epsilon \in \widetilde{Q}$ such that $\widetilde{q} \widetilde{\preceq} \widetilde{w} + \widetilde{q}_{\widetilde{\epsilon}}$ and $\|\widetilde{q}_{\widetilde{\epsilon}}\| \widetilde{\leq} \widetilde{\epsilon}$. Then, we have$\|\langle \widetilde{q}_\epsilon, \widetilde{q}_\epsilon\rangle\|_{\Omega(\mathbb{R})} \widetilde{\leq} \widetilde{\epsilon}^2$ and $\langle \widetilde{q}_\epsilon, \widetilde{q}_\epsilon\rangle_{\Omega(\mathbb{R})} \widetilde{\leq} S_{\widetilde{\epsilon}^2}(\theta)$. So, we obtain $\widetilde{q} \widetilde{\preceq} \widetilde{w}$ since for any $\widetilde{\epsilon} \widetilde{\geq} \widetilde{0}$ there exists an soft quasi element $\widetilde{q}_\epsilon \in \widetilde{Q}$ such that $\widetilde{q} \widetilde{\preceq} \widetilde{w} + \widetilde{q}_{\widetilde{\epsilon}}$ and $\langle \widetilde{q}_\epsilon, \widetilde{q}_\epsilon\rangle_{\Omega(\mathbb{R})} \widetilde{\leq} S_{\widetilde{\epsilon}^2}(\theta)$. Thus, $\|.\|$ is a satisfies all soft quasi norm conditions. □

**Remark 3.2.** *If $\left\{\langle \widetilde{q}, \widetilde{w}\rangle(\lambda) : \widetilde{q}, \widetilde{w} \in \widetilde{Q} \text{ such that } \widetilde{q}(\lambda) = q,\ \widetilde{w}(\lambda) = w\right\}$ is a single-valued set for $q, w \in Q$ and for a $\lambda \in P$ then*

$$\left\{\|\widetilde{q}\|(\lambda) = \sqrt{\|\langle \widetilde{q}, \widetilde{q}\rangle\|_{\Omega(\mathbb{R})}}(\lambda) : \widetilde{q}(\lambda) = q\right\} \tag{3}$$

*is a single-valued so singleton set.*

**Remark 3.3.** *Same as the quasilinear inner product spaces, the soft quasi norm of a soft inner product quasilinear space does not have to satisfy the parallelogram law.*

**Theorem 3.5.** *Regular subspace of a soft quasilinear inner product space has always satisfy the parallelogram law since this quasi inner product is classical soft inner product.*

*Proof.* If $\widetilde{Q}$ is a soft quasilinear inner product space, then regular subspace of $\widetilde{Q}$ that is $\widetilde{Q}_r$ is a soft linear space. Inner product of this ragular subspace is satisfy parallellogram law because of this inner product is classical soft inner product □

**Example 3.3.** *Let $Q = \Omega_C(\mathbb{R})$. Then $\widetilde{\Omega_C(\mathbb{R})}$ is a soft quasilinear inner product space with soft inner product given by Example 3.2. The soft quasi norm induced by the soft quasi inner product is*

$$\begin{aligned} \|\widetilde{q}\|(\lambda) &= \sqrt{\|\langle \widetilde{q}(\lambda), \widetilde{q}(\lambda)\rangle\|_{\Omega(\mathbb{R})}} \\ &= \sqrt{\left\|\left\{(a^{\lambda})^2\right\} : a^{\lambda} \in \widetilde{q}(\lambda)\right\|_{\Omega(\mathbb{R})}} \\ &= \sup_{\lambda \in \widetilde{x}(\lambda)} \left\|a^{\lambda}\right\|. \end{aligned}$$

*Let us consider the soft quasi elements $\widetilde{q}, \widetilde{w} \in \widetilde{\Omega_C(\mathbb{R})}$ such that for every $\lambda \in P$, $\widetilde{q}(\lambda) = \widetilde{w}(\lambda) = [0,1] \in \Omega_C(\mathbb{R})$. Then $\|\widetilde{q}\|(\lambda) = \|\widetilde{w}\|(\lambda) = \widetilde{1}$, $\|\widetilde{q} + \widetilde{w}\|(\lambda) = \widetilde{2}$ and $\|\widetilde{q} - \widetilde{w}\|(\lambda) = \widetilde{1}$. But $\|\widetilde{q} + \widetilde{w}\|^2 + \|\widetilde{q} - \widetilde{w}\|^2 \neq 2\|\widetilde{q}\|^2 + 2\|\widetilde{w}\|^2$.*

**Example 3.4.** *Let $Q = I\mathbb{R}^2$. Then $I\mathbb{R}^2$ is a normed quasilinear space with respect to the norm $\|q\| = \|(q_1, q_2)\| = \left(\sum_{i=1}^{2} \|q_i\|_{I\mathbb{R}}^2\right)^{1/2}$ for a $q \in I\mathbb{R}^2$* [15]. *Also, $I\mathbb{R}^2$ quasilinear inner product space with $\langle q, w\rangle = \sum_{i=1}^{2} \langle q_i, w_i\rangle_{I\mathbb{R}}$ for every $q, w \in I\mathbb{R}^2$. Let $\widetilde{q}$ and $\widetilde{w}$ be soft quasi elements of the absolute soft quasilinear space $\widetilde{I\mathbb{R}^2}$. Then $\widetilde{q}\ (\lambda) = \left(q_1^{\lambda}, q_2^{\lambda}\right)$, $\widetilde{w}\ (\lambda) = \left(w_1^{\lambda}, w_2^{\lambda}\right)$ are elements of $I\mathbb{R}^2$. The mapping*

$$\begin{aligned} \langle . \rangle &: SE\left(\widetilde{I\mathbb{R}^2}\right) \times SE\left(\widetilde{I\mathbb{R}^2}\right) \to \Omega(\mathbb{R})(P) \\ (\widetilde{q}, \widetilde{w}) &= \langle \widetilde{q}, \widetilde{w}\rangle(\lambda) \\ &= \sum_{i=1}^{2} \left\langle q_i^{\lambda}, w_i^{\lambda}\right\rangle_{I\mathbb{R}} \end{aligned}$$

*is a soft quasilinear inner product on the soft quasilinear space $\widetilde{I\mathbb{R}^2}$. This inner product satisfies all soft quasi inner product conditions. Further, the soft quasi norm of obtained from above soft quasi inner product is*

$$\|\widetilde{q}\|_{\widetilde{I\mathbb{R}^2}} = \|\widetilde{q}\|_{\widetilde{I\mathbb{R}^2}}(\lambda) = \left(\sum_{i=1}^{2} \|q_i\|_{I\mathbb{R}}^2\right)^{1/2}.$$

If $\left(\widetilde{Q}, \|.\|, P\right)$ is a soft normed quasilinear space, then

$$h(\widetilde{q}, \widetilde{w}) = \inf\left\{\widetilde{\epsilon} \widetilde{\geq} \widetilde{0} : \widetilde{q} \widetilde{\preceq} \widetilde{w} + \widetilde{q}_1^{\epsilon},\ \widetilde{w} \widetilde{\preceq} \widetilde{q} + \widetilde{q}_2^{\epsilon},\ \left\|\widetilde{q}_i^{\epsilon}\right\|_{\widetilde{Q}} \widetilde{\leq} \widetilde{\epsilon}\right\}$$

is define a soft Hausdorff metric. The equality $h(\widetilde{q}, \widetilde{w}) = \|\widetilde{q} - \widetilde{w}\|$ may not be satisfied for every $\widetilde{q}, \widetilde{w} \in \widetilde{Q}$. However, the inequality $h(\widetilde{q}, \widetilde{w}) \widetilde{\leq} \|\widetilde{q} - \widetilde{w}\|$ is satisfied. Because of this situation, in place of analyzing topological properties of soft normed quasilinear spaces, analyzing according to the metric reproduced from this soft norm is more useful.

**Definition 3.5.** *A soft quasilinear inner product space which satisfy (1) said to be a complete if it is complete with respect to the soft Hausdorff metric defined by soft quasi inner product. A complete soft quasilinear inner product space is said to be a soft quasilinear Hilbert space.*

**Example 3.5.** *The soft quasilinear inner product space* $(\widetilde{\Omega(\mathbb{R})}, \langle . \rangle, P)$ *defined as in Example 3.2 is a soft Hilbert quasilinear space where the parameter set* $P$. *We know that* $(\widetilde{\Omega(\mathbb{R})}$ *is soft quasilinear inner product space with inner product* $\langle \widetilde{q}, \widetilde{w} \rangle(\lambda) = \langle \widetilde{q}(\lambda), \widetilde{w}(\lambda) \rangle_{\Omega(\mathbb{R})}$ *for every* $\lambda \in P$. *Let* $\{\widetilde{q}_n\}$ *be a Cauchy sequence in the soft Hausdorff metric space* $\left(\widetilde{(\Omega(\mathbb{R})}, h\right)$. *Then every* $\widetilde{\epsilon}\widetilde{\geq}\widetilde{0}$, $\exists N \in \mathbb{N}$ *such that for every* $m, n \geq N$, *we get*

$$\widetilde{q}_n \widetilde{\preceq} \widetilde{q}_m + \widetilde{q}_{1n}^{\epsilon},\ \widetilde{q}_m \widetilde{\preceq} \widetilde{q}_n + \widetilde{q}_{2n}^{\epsilon},\ \|\widetilde{q}_{in}^{\epsilon}\| \widetilde{\leq} \widetilde{\epsilon}.$$

*Thus,*

$$\widetilde{q}_n(\lambda) \widetilde{\leq} \widetilde{q}_m(\lambda) + \widetilde{q}_{1n}^{\epsilon}(\lambda),\ \widetilde{q}_m(\lambda) \widetilde{\leq} \widetilde{q}_n(\lambda) + \widetilde{q}_{2n}^{\epsilon}(\lambda),\ \|\widetilde{q}_{in}^{\epsilon}(\lambda)\| \widetilde{\leq} \widetilde{\epsilon}(\lambda)$$

*for each* $\lambda \in P$ *and* $m, n \geq N$. *Therefore,* $\{\widetilde{q}_n(\lambda)\}$ *is a Cauchy sequence of element in* $\Omega(\mathbb{R})$. *Since* $\Omega(\mathbb{R})$ *is complete there exists a sequence* $\widetilde{q}(\lambda) \in \Omega(\mathbb{R})$ *and* $M_\lambda$ *such that*

$$\widetilde{q}_n(\lambda) \widetilde{\leq} \widetilde{q}(\lambda) + \widetilde{q}_{1n}^{\epsilon}(\lambda),\ \widetilde{q}(\lambda) \widetilde{\leq} \widetilde{q}_n(\lambda) + \widetilde{q}_{2n}^{\epsilon}(\lambda),\ \|\widetilde{q}_{in}^{\epsilon}(\lambda)\| \widetilde{\leq} \widetilde{\epsilon}(\lambda)$$

*for every* $\lambda \in P$ *and* $n \geq M_\lambda$. *For every* $\lambda \in P$ *and positive integer* $K$ *such that* $K = \max\{M_\lambda : \lambda \in P\}$, *we obtain* $\|\widetilde{q}_{in}^{\epsilon}\| \widetilde{\leq} \widetilde{\epsilon}$ *and* $\widetilde{q}_n \widetilde{\preceq} \widetilde{q} + \widetilde{q}_{1n}^{\epsilon}$, $\widetilde{q} \widetilde{\preceq} \widetilde{q}_n + \widetilde{q}_{2n}^{\epsilon}$. *This gives* $\widetilde{q}_n \to \widetilde{q} \in \widetilde{(\Omega(\mathbb{R})}$. *Hence,* $(\widetilde{\Omega(\mathbb{R})}, \langle . \rangle, P)$ *is a soft quasilinear Hilbert space.*

**Proposition 3.3.** *Let* $\left(\widetilde{Q}, \langle . \rangle, P\right)$ *be a soft quasilinear inner product space satisfying (1) and* $\widetilde{q}, \widetilde{w} \in \widetilde{Q}$. *If* $\widetilde{q}_n \to \widetilde{q}$ *and* $\widetilde{w}_n \to \widetilde{w}$ *then* $\langle \widetilde{q}_n, \widetilde{w}_n \rangle \to \langle \widetilde{q}, \widetilde{w} \rangle$ *as* $n \to \infty$.

*Proof.* Since $\widetilde{q}_n \to \widetilde{q}$ and $\widetilde{w}_n \to \widetilde{w}$, for any $\widetilde{\epsilon}\widetilde{\geq}\widetilde{0}$ there exists a $\exists N \in \mathbb{N}$ such that the condition

$$\widetilde{q}_n \widetilde{\preceq} \widetilde{q} + \widetilde{q}_{1n}^{\epsilon},\ \widetilde{q} \widetilde{\preceq} \widetilde{q}_n + \widetilde{q}_{2n}^{\epsilon},\ \|\widetilde{q}_{in}^{\epsilon}\| \widetilde{\leq} \widetilde{\epsilon}$$

holds for every $n > N$. Again, for any $\widetilde{\epsilon}\widetilde{\geq}\widetilde{0}$ there exists a $\exists M \in \mathbb{N}$ such that the condition

$$\widetilde{w}_n \widetilde{\preceq} \widetilde{w} + \widetilde{w}_{1n}^{\epsilon},\ \widetilde{w} \widetilde{\preceq} \widetilde{w}_n + \widetilde{w}_{2n}^{\epsilon},\ \|\widetilde{w}_{in}^{\epsilon}\| \widetilde{\leq} \widetilde{\epsilon}$$

holds for every $n > M$. Since $\widetilde{\Omega_C(\mathbb{R})}$ is a soft quasilinear inner product space, we get

$$\langle \widetilde{q}_n, \widetilde{w}_n \rangle \widetilde{\subseteq} \langle \widetilde{q} + \widetilde{q}_{1n}^{\epsilon}, \widetilde{w} + \widetilde{w}_{1n}^{\epsilon} \rangle \widetilde{\subseteq} \langle \widetilde{q}, \widetilde{w} \rangle + \langle \widetilde{q}, \widetilde{w}_{1n}^{\epsilon} \rangle + \langle \widetilde{q}_{1n}^{\epsilon}, \widetilde{w} \rangle + \langle \widetilde{q}_{1n}^{\epsilon}, \widetilde{w}_{1n}^{\epsilon} \rangle$$

and

$$\langle \widetilde{q}, \widetilde{w} \rangle \widetilde{\subseteq} \langle \widetilde{q}_n + \widetilde{q}_{2n}^{\epsilon}, \widetilde{w}_n + \widetilde{w}_{2n}^{\epsilon} \rangle \widetilde{\subseteq} \langle \widetilde{q}_n, \widetilde{w}_n \rangle + \langle \widetilde{q}_n, \widetilde{w}_{2n}^{\epsilon} \rangle + \langle \widetilde{q}_{2n}^{\epsilon}, \widetilde{w}_n \rangle + \langle \widetilde{q}_{2n}^{\epsilon}, \widetilde{w}_{2n}^{\epsilon} \rangle.$$

If we take $\widetilde{z}_{1n}^{\epsilon} = \langle \widetilde{q}, \widetilde{w}_{1n}^{\epsilon} \rangle + \langle \widetilde{q}_{1n}^{\epsilon}, \widetilde{w} \rangle + \langle \widetilde{q}_{1n}^{\epsilon}, \widetilde{w}_{1n}^{\epsilon} \rangle$ and $\widetilde{z}_{2n}^{\epsilon} = \langle \widetilde{q}_n, \widetilde{w}_{2n}^{\epsilon} \rangle + \langle \widetilde{q}_{2n}^{\epsilon}, \widetilde{w}_n \rangle + \langle \widetilde{q}_{2n}^{\epsilon}, \widetilde{w}_{2n}^{\epsilon} \rangle$, Then $\widetilde{z}_{1n}^{\epsilon}, \widetilde{z}_{2n}^{\epsilon} \in \Omega(\mathbb{R})$. From (2) inequality, we obtain $\|\widetilde{z}_{1n}^{\epsilon}\| \to \infty$ and $\|\widetilde{z}_{2n}^{\epsilon}\| \to \infty$ for $n \to \infty$. Hence, we obtain for any $\widetilde{\epsilon}\widetilde{\geq}\widetilde{0}$ there exists a $\exists K \in \mathbb{N}$ such that the condition

$$\langle \widetilde{q}_n, \widetilde{w}_n \rangle \widetilde{\subseteq} \langle \widetilde{q}, \widetilde{w} \rangle + \widetilde{z}_{1n}^{\epsilon},\ \langle \widetilde{q}, \widetilde{w} \rangle \widetilde{\subseteq} \langle \widetilde{q}_n, \widetilde{w}_n \rangle + \widetilde{z}_{2n}^{\epsilon} \text{ and } \|\widetilde{z}_{in}^{\epsilon}\| \widetilde{\leq} \widetilde{\epsilon}$$

holds for every $n > K$ for $K = \max\{N, M\}$. This gives $\langle \widetilde{q}_n, \widetilde{w}_n \rangle \to \langle \widetilde{q}, \widetilde{w} \rangle$ as $n \to \infty$. □

**Definition 3.6.** *A soft quasi vector* $\widetilde{q}$ *of soft quasilinear inner product space* $\widetilde{Q}$ *is said to be orthogonal to soft quasi element* $\widetilde{w} \in \widetilde{Q}$ *if*

$$\|\langle \widetilde{q}, \widetilde{w} \rangle\|_{\Omega(\mathbb{R})} = \overline{0}.$$

*It is also denoted by* $\widetilde{q}\perp\widetilde{w}$. *Let* $\widetilde{M}$ *be a non-null soft quasi subset of soft quasilinear inner product space* $\widetilde{Q}$ *such that* $\widetilde{M}(\lambda) \neq \emptyset$ *for every* $\lambda \in P$. *If a soft quasi vector* $\widetilde{q}$ *of soft quasilinear inner product space* $\widetilde{Q}$ *orthogonol to every soft quasi vectors of* $\widetilde{M}$, *then we say that* $\widetilde{q}$ *is orthogonal to* $\widetilde{M}$ *and we write* $\widetilde{q}\perp\widetilde{M}$.

**Definition 3.7.** *A non-null orthonormal soft quasi subset* $\widetilde{M}$ *of soft quasilinear inner product space* $\widetilde{Q}$ *such that* $\widetilde{M}(\lambda) \neq \emptyset$ *for every* $\lambda \in P$ *is a orthogonal soft quasi subset in* $\widetilde{Q}$ *whose soft quasi vectors have norm* $\overline{1}$; *that is, for all* $\widetilde{q}, \widetilde{w} \in \widetilde{M}$

$$\|\langle \widetilde{q}, \widetilde{w}\rangle\|_{\Omega(\mathbb{R})} = \begin{cases} \overline{0}, & \widetilde{q} = \widetilde{w} \\ \overline{1}, & \widetilde{q} \neq \widetilde{w} \end{cases}.$$

**Example 3.6.** *Regard as the soft quasilinear inner product space* $(\widetilde{\Omega_C(\mathbb{R})}, \langle . \rangle, P)$ *defined as in Example 3.2. Let us consider soft quasi elements* $\widetilde{q}, \widetilde{w} \in (\widetilde{\Omega_C(\mathbb{R})}$ *such that for every* $\lambda \in P$, $\widetilde{q}(\lambda) = [0,1]$ *and* $\widetilde{w}(\lambda) = \{0\}$. *So, we get* $\|\langle \widetilde{q}, \widetilde{w}\rangle\|_{\Omega(\mathbb{R})} = \overline{0}$ *since* $\|\langle \widetilde{q}(\lambda), \widetilde{w}(\lambda)\rangle\|_{\Omega(\mathbb{R})} = \|\langle [0,1], \{0\}\rangle\|_{\Omega(\mathbb{R})} = 0$. *Further, consider the soft quasilinear inner product qpace* $(\widetilde{\Omega_C(\mathbb{R}^2)}, \langle . \rangle, P)$. *Take soft quasi elements* $\widetilde{q}_1, \widetilde{q}_2, \widetilde{q}_3, \widetilde{q}_4 \in (\widetilde{\Omega_C(\mathbb{R}^2)}$ *such that for every* $\lambda \in P$,

$$\begin{aligned} \widetilde{q}_1(\lambda) &= \{(0,a) : 0 \leq a \leq 1\}, \\ \widetilde{q}_2(\lambda) &= \{(a,0) : 0 \leq a \leq 1\}, \\ \widetilde{q}_3(\lambda) &= \{(0,-a) : 0 \leq a \leq 1\}, \\ \widetilde{q}_4(\lambda) &= \{(-a,0) : 0 \leq a \leq 1\}. \end{aligned}$$

*Then we obtain* $\|\langle \widetilde{q}_1, \widetilde{q}_2\rangle\|_{\Omega(\mathbb{R})} = \|\langle \widetilde{q}_3, \widetilde{q}_4\rangle\|_{\Omega(\mathbb{R})} = \|\langle \widetilde{q}_1, \widetilde{q}_4\rangle\|_{\Omega(\mathbb{R})} = \|\langle \widetilde{q}_2, \widetilde{q}_3\rangle\|_{\Omega(\mathbb{R})} = \overline{0}$. *So,* $\widetilde{q}_1\perp\widetilde{q}_2$, $\widetilde{q}_3\perp\widetilde{q}_4$, $\widetilde{q}_1\perp\widetilde{q}_4$ *and* $\widetilde{q}_2\perp\widetilde{q}_3$. *If we take* $\widetilde{M} = \{\widetilde{q}_1, \widetilde{q}_2\}$ *or* $\widetilde{M} = \{\widetilde{q}_3, \widetilde{q}_4\}$, *then* $\widetilde{M}$ *is orthonormal soft quasi subset of* $(\widetilde{\Omega_C(\mathbb{R}^2)}$.

**Theorem 3.6.** *If* $\widetilde{q}, \widetilde{w} \in \overline{Q}$ *and* $\widetilde{q}\perp\widetilde{w}$, *then* $\|\widetilde{q} + \widetilde{w}\|^2 \widetilde{\leq} \|\widetilde{q}\|^2 + \|\widetilde{w}\|^2$ *and* $\|\widetilde{q} - \widetilde{w}\|^2 \widetilde{\leq} \|\widetilde{q}\|^2 + \|\widetilde{w}\|^2$.

*Proof.* We obtain

$$\|\widetilde{q} + \widetilde{w}\|^2 = \|\langle \widetilde{q} + \widetilde{w}, \widetilde{q} + \widetilde{w}\rangle\| \widetilde{\leq} \|\langle \widetilde{q}, \widetilde{q}\rangle\| + \|\langle \widetilde{q}, \widetilde{w}\rangle\| + \|\langle \widetilde{w}, \widetilde{q}\rangle\| + \|\langle \widetilde{w}, \widetilde{w}\rangle\| = \|\widetilde{q}\|^2 + \|\widetilde{w}\|^2.$$

Similarly, we get

$$\|\widetilde{q} - \widetilde{w}\|^2 = \|\langle \widetilde{q} - \widetilde{w}, \widetilde{q} - \widetilde{w}\rangle\| \widetilde{\leq} \|\langle \widetilde{q}, \widetilde{q}\rangle\| + \|\langle \widetilde{q}, \widetilde{w}\rangle\| + \|\langle \widetilde{w}, \widetilde{q}\rangle\| + \|\langle \widetilde{w}, \widetilde{w}\rangle\| = \|\widetilde{q}\|^2 + \|\widetilde{w}\|^2.$$

□

**Definition 3.8.** *Let* $\widetilde{M}$ *be a non-null soft quasi subset of soft quasilinear inner product space* $\widetilde{Q}$ *such that* $\widetilde{M}(\lambda) \neq \emptyset$ *for every* $\lambda \in P$. *Then the set of all soft quasi vectors of* $\widetilde{Q}$, *orthogonal to* $\widetilde{M}$ *is called the orthogonal complement of* $\widetilde{M}$. *We denote this set by* $\widetilde{M}^{\perp}$.

**Theorem 3.7.** *Let* $\widetilde{Q}$ *be a soft quasilinear inner product space and* $\widetilde{M}$ *be a non-null soft quasi subset of* $\widetilde{Q}$. *Then* $\widetilde{M}^{\perp}$ *is a closed subspace of* $\widetilde{Q}$.

*Proof.* Let $\widetilde{q}$ and $\widetilde{w}$ soft quasi vectors of $\widetilde{Q}$ orthogonal to $\widetilde{M}$ , that is $\widetilde{q}, \widetilde{w} \in \widetilde{M}^{\perp}$. For every soft scalars $\widetilde{\alpha}, \widetilde{\beta} \in \mathbb{R}(\mathbb{R})$ and soft quasi vector $\widetilde{z} \in \widetilde{M}$, we have

$$\begin{aligned} & \left\| \left\langle \widetilde{\alpha} \cdot \widetilde{q} + \widetilde{\beta} \cdot \widetilde{w}, \widetilde{z} \right\rangle \right\| \widetilde{\leq} \left\| \langle \widetilde{\alpha} \cdot \widetilde{q}, \widetilde{z} \rangle \right\| + \left\| \left\langle \widetilde{\beta} \cdot \widetilde{w}, \widetilde{z} \right\rangle \right\| \\ = \ & |\widetilde{\alpha}| \left\| \langle \widetilde{q}, \widetilde{z} \rangle \right\| + \left| \widetilde{\beta} \right| \left\| \langle \widetilde{w}, \widetilde{z} \rangle \right\| \\ = \ & \overline{0}. \end{aligned}$$

Since the soft quasi norm is positive, we obtain $\widetilde{\alpha} \cdot \widetilde{q} + \widetilde{\beta} \cdot \widetilde{w} \in \widetilde{M}^{\perp}$. Hence, $\widetilde{M}^{\perp}$ is a subspace of $\widetilde{Q}$. Let's show that $\widetilde{M}^{\perp}$ is closed. Let $\{\widetilde{q}_n\}$ be a soft quasi vector of $\widetilde{M}^{\perp}$ and $\{\widetilde{q}_n\} \to \widetilde{q}$ for some $\widetilde{q} \in \widetilde{Q}$. Then for any $\widetilde{\epsilon} \widetilde{\geq} \overline{0}$ there exists a $N \in \mathbb{N}$ such that the following condition hold for every $n \geq N$;

$$\widetilde{q}_n \widetilde{\preceq} \widetilde{q} + \widetilde{q}_{1n}^{\epsilon}, \ \widetilde{q} \widetilde{\preceq} \widetilde{q}_n + \widetilde{q}_{2n}^{\epsilon}, \ \|\widetilde{q}_{in}^{\epsilon}\| \widetilde{\leq} \widetilde{\epsilon}.$$

Since every soft quasilinear inner product space is a soft normed quasilinear space with soft quasi norm obtained soft quasi inner product, we get

$$\widetilde{q}_n \widetilde{\preceq} \widetilde{q} \text{ and } \widetilde{q} \widetilde{\preceq} \widetilde{q}_n.$$

For every $\widetilde{z} \in \widetilde{M}$, we obtain $\overline{0} = \|\langle \widetilde{q}_n, \widetilde{z} \rangle\| \widetilde{\leq} \|\langle \widetilde{q}, \widetilde{z} \rangle\|$ and $\|\langle \widetilde{q}, \widetilde{z} \rangle\| \widetilde{\leq} \|\langle \widetilde{q}_n, \widetilde{z} \rangle\| = \overline{0}$. Thus, $\|\langle \widetilde{q}, \widetilde{z} \rangle\| = \overline{0}$. This gives $\widetilde{q} \in \widetilde{M}^{\perp}$. So, $\widetilde{M}^{\perp}$ is a closed soft quasi subspace of $\widetilde{Q}$. □

## 4. Conclusions

In this work, the notion of inner product on a soft quasilinear space is introduced. Some basic properties of soft quasilinear inner product spaces are explored with examples. Also, the concept of soft Hilbert quasilinear space is defined. The properties of orthogonal and orthonormal sets on soft quasilinear inner product spaces are examined and some related theorems are proved.

## References

[1] Molodtsov, D., (1999), Soft set-theory first results, Computational and Appllied Mathematics, 37, pp. 19-31.
[2] Maji, P. K., Biswas, R. and Ropy, A. R., (2003), Soft set theory, Comput. Math. Appl., 45, pp. 555-562.
[3] Das, S. and Samanta, S. K., (2013), On soft metric spaces, J. Fuzzy Math., 21, pp. 707-734.
[4] Das, S. and Samanta, S. K., (2012), Soft real sets, soft real numbers and their properties, J. Fuzzy Math., Inform., 6(2), pp. 551-576.
[5] Das, S. and Samanta, S. K., (2013), Soft metric, Annals of Fuzzy Math. and Inform., 6(1), pp. 77-94.
[6] Das, S., Majumdar, P. and Samanta, S. K., (2015), On soft linear spaces and soft normed linear spaces, Annals of Fuzzy Math. and Inform., 9(1), pp. 91-109.
[7] Sezer, A. S. and Atagün, A. O., (2014), A new kind of vector space: soft vector space, Souheast Asian Bull. of Math., 40. pp. 753-770.
[8] Das, S. and Samanta, S. K., (2013), On soft inner product spaces, Annals of Fuzzy Math. and Inform., 6(1), pp. 151-170.
[9] Yazar, M. I., Bilgin, T., Bayramov, S. and Gündüz, Ç., (2014), A new view on soft normed spaces, International Mathematical Forum, 9(24), pp. 1149-1159.
[10] Yazar, M. I., Aras, Ç. G. and Bayramov, S., (2019), Results on Hilbert spaces, TWMS J. App. Eng. Math., 9(1), pp. 159-164.
[11] Levent, H. and Yilmaz, Y., (2018), Hahn- Banach extension theorem for interval-valued functions and existence of quasilinear functionals, New Trends in Mathematical Sciences, 6(2), pp. 19-28.
[12] Levent, H., and Yilmaz, Y., (2018), Translation, modulation and dilation systems set-valued signal processing, Carpathian Mathematical Publications, 10(1), pp. 143-164.

[13] Bozkurt, H. Yılmaz, Y., (2016), Some new results on inner product qusilinear spaces, Cogent Mathematics, Article ID 1194801(3), 10 pages.
[14] Yılmaz, Y., Bozkurt, H. and Çakan, S., (2016), On orthonormal sets in inner product quasilinear spaces, Creat. Math. and Inform., 25(2), pp. 237-247.
[15] Bozkurt, H. and Yılmaz, Y., (2016), New inner product quasilinear spaces on interval numbers, Journal of Function Spaces, Article ID 2619271, 9 pages.
[16] Bozkurt, H. and Yılmaz, Y., (2016), Some New Properties of Inner Product Quasilinear Spaces. Bulletin of Mathematical Analysis and Applications, 8(1), pp. 37-45.
[17] Bozkurt, H., (2020), Soft quasilinear spaces and soft normed quasilinear spaces, Adıyaman University Journal of Science, 10(2), pp. 506-523.
[18] Aseev, S. M., (1986), Quasilinear operators and their application in the theory of multivalued mappings, Proceedings of the Steklov Institute of Mathematics, 2, pp. 23-52.
[19] Bozkurt, H., (2016), Inner product quasilinear spaces and some generalizations, PhD Thesis, İnönü University, Turkey.

**Mehmet Şirin Gönci** is currently graduate student at Batman University. He works as a mathematics teacher at Yenişehir Secondary School affiliated to the Ministry of National Education. His area of interest includes Functional Analysis.

**Hacer Bozkurt** received her master's degree from the Department of Mathematics at Sakarya University in 2010. She also completed her doctoral programme in the department of Mathematics at İnönü University in 2016. Her area of interest includes Functional Analysis and Fixed Point Theory. She has been working in the Department of Mathematics at Batman University as an assistant professor.