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NUMERICAL SOLUTION FOR ANTI-PERSISTENT PROCESS BASED STOCHASTIC INTEGRAL EQUATIONS

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Abstract. In this article, we propose the shifted Legendre polynomial solutions for anti-persistent process based stochastic integral equations. The operational matrices for stochastic integration and fractional stochastic integration are efficiently generated using the properties of shifted Legendre polynomials. In addition, the original problem can be reduced to a system of simultaneous equations with $(N+1)$ unknowns in the function approximation. By solving the given stochastic integral equations, we obtain numerical solutions. The proposed method's convergence is derived in terms of the error function's expectation, and the upper bound of the error in L^2 norm is also discussed in detail. The applicability of this methodology is demonstrated using numerical examples and the solution's quality is statistically validated by comparing it with the exact solution.

Keywords: Stochastic Ito Volterra integral equation, Shifted Legendre polynomial, Stochastic operational matrix, Convergence analysis, Error estimation.

AMS Subject Classification: Primary 65C30, 60G42, 60H35, 60H10, 65C20; Secondary 60H20, 68U20.

1. Introduction

The stochastic differential equation, stochastic integral equation, and other stochastic models are created by adding a random element, which is sometimes referred to as the "noise term,"to deterministic models. In domains as diverse as biology, medicine, population dynamics, mechanics, and finance, such models have been used to study different physical or biological phenomena [3, 6, 33, 35]. To study the random effects, we use the following anti-persistent process based stochastic integral equation model [12, 25].

$$X(t) = f(t) + \int_0^t k_1(s,t)N_1(s,X(s))ds + \int_0^t k_2(s,t)N_2(s,X(s))dB^H(s), \tag{1}$$

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where $f(t), k_i(s,t), N_i(s,X)$ are linear or nonlinear known functions for $t \in [0,1], H < \frac{1}{2}, i = 1,2$, and $X(t)$ is the unknown stochastic process to be determined. All the abovementioned functions, including X(t), are stochastic processes defined in the probability space. Here, $B^H(t)$ is the fractional Brownian motion whose detailed information is discussed in the following section. Brownian motion is a process with independent increments. It cannot reflect the long-range dependence implied by the observed market data in the financial market. It is believed that the fractional Brownian motion, a generalisation of normal Brownian motion with the Hurst index as an additional parameter, expresses this long-range dependence. (1) is used to investigate the behavior of the stock price with a risky asset X(t) where the spot price $f(t) = X_0$ at time 0, $k_1(s,t) = \mu(s), k_2(s,t) = \sigma(s)$, $N_1 = N_2 = X(s)$ and $B^H(t)$ is a fractional Brownian motion with $B^H(0) = 0$. The generated model is linear and valid in the range [0, T] where T is the maturity of the option [1, 2, 5, 18, 10].

It is not easy to deal with the nonlinear terms N_1 and N_2 in terms of the unknown stochastic process $X(s)$. Several numerical methods, along with their variations, have been applied to solve these stochastic equations [8, 21]. Good approximation approaches [13, 19, 20, 27, 28, 29, 30, 31] based on the orthogonal basis of polynomials to find an approximate solution have piqued mathematicians' interest in recent years. The Jacobi polynomial, which occurs as the eigenfunctions of a singular Sturm-Liouville problem, is one such polynomial [9]. The solutions to the aforementioned problem are a set of polynomials such as Legendre, Chebyshev, and other spherical polynomials in the interval [-1,1] and are generated from Jacobi polynomials by assigning appropriate values to their parameters.

The authors also used wavelet theory to find the approximate solution of Stochastic differential equations driven by fractional Brownian motion. A new class of orthogonal wavelet, namely the Chebyshev cardinal wavelet approach coupled with operational matrices of differentiation and integration can be seen in [14, 15].The convergence and error analysis of the proposed method are established in the Sobelov space. Also, in [15], a new procedure is established for constructing the variable order fractional Brownian motion. Legendre wavelets coupled with the Galerkin method approach and Chebyshev wavelet, together with the Galerkin method, are some excellent work, as evidenced by [16, 17]. The stochastic operational matrix and the operational matrices of integration of Legendre wavelets can be found in [16]. A new stochastic operational matrix of second kind Chebyshev wavelets has been derived. The properties of second kind Chebyshev wavelets together with the operational matrices of integration and stochastic Ito integration transform the considered problem into a system of equations which nurtures the researchers' interest [17]. The numerical method based on hat functions [12] has been implemented to solve nonlinear stochastic Ito Volterra integral equations driven by fractional Brownian motion. The error mean produced by that method was observed to be higher for the test problems taken by the researchers.

Not only motivated by the aforementioned works, but also due to the limitedness in the availability of literature on the study of nonlinear stochastic differential equations driven by fractional Brownian motion, we have employed the shifted version of Legendre polynomials called shifted Legendre polynomials to obtain an approximate solution of (1) in the interval $[0,1]$ and $H < \frac{1}{2}$. The usage of shifted Legendre polynomials has provided fruitful results for various other works undertaken in [34, 23]. The operational matrices of integration are coupled with the salient properties of these polynomials to convert the given problem into a system of simultaneous algebraic equations. Solving these equations provides the required numerical solution.

The overview of this paper comprises the following. The fundamental definitions and theorems required for our subsequent study are given in the following section named Mathematical Background, followed by the fundamentals of shifted Legendre polynomials and their properties. The various operational matrices required for the proposed method are also derived. In the next section, we give a detailed presentation of the convergence theorems and the error estimates. The accuracy and applicability of the scheme are tested on a few examples and comparative results are also presented in the section on Numerical Examples. The superiority of this method is also highlighted in the same section. Concluding remarks are given in the final section.

2. Mathematical Background

We share information about our proposed study in this section. Brownian motion, which is a fundamental example of a stochastic process, is employed. The underlying probability space $(\Omega, \mathcal{F}, P)$ can be constructed on the space $\Omega = C_0(R_+)$ of continuous real-valued functions on R_+ starting at 0. We've also used Gronwall's inequality to show that the error function approaches zero for large values of the parameters. The concept of a sequence $\{Xn\}$ convergeing in the given space, where the function is defined, is also examined. Our subsequent development is primarily based on the basic properties of the Ito integral and Ito isometry [11, 22, 32].

2.1. **Fractional Brownian Motion.** The Hurst parameter is a measure of the long term memory of time series. It relates the time series autocorrelations and the rate at which they drop as the lag between pairs of values increases. In other words, the Hurst parameter (H) indicates how rough the motion is. The roughness increases when H decreases, and vice-versa. The process is stated to have a Brownian motion when H is 0.5, when it is less than 0.5, the increments have a negative correlation (anti-persistent time series), and when it is higher than 0.5, the increments have a positive correlation (persistent time series). Throughout this article, $B^H(t)$ represents a fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$ [29, 30, 31, 32] . It can be represented as $B^H(t) = \int_0^t \bar{K}(t,s)dB(s),\ t \geq 0$, where kernel $\bar{K}(s,t)$ is given by

$$\bar{K}(t,s) = C_H \left[\frac{t^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}}(t-s)^{H-\frac{1}{2}} - \left(H - \frac{1}{2}\right) \int_s^t \frac{u^{H-\frac{3}{2}}}{s^{H-\frac{1}{2}}}(u-s)^{H-\frac{1}{2}} du \right],$$

with $C_H = \sqrt{\frac{2H}{(1-2H)\beta(1-2H, H+\frac{1}{2})}}$. Hence, there exists a Gaussian stochastic process with $H \in (0,1)$, where $E\left(B^H(t)\right) = 0,\ cov\left(B^H(s), B^H(t)\right) = \frac{1}{2}s^{2H} + t^{2H} - |t-s|^{2H}$, which is called the fractional Brownian motion [11, 22].

Theorem 2.1. [7]*If $k_2(s,t)x(s)$ is bounded by a constant M, $k_1(s,t)x(s)$ is a measurable function, and satisfy the following assumptions*
(i) Lipschitz condition: $|k_1(s,t)x(s) - k_1(s,t)\hat{x}(s)| \leq L_0|x(s) - \hat{x}(s)|$,
(ii) Linear growth condition: $|k_i(s,t)x(s)| \leq L_i(1 + |x(s)|), i = 1,2$ then, (1) *has a unique solution.*

3. Shifted Legendre Polynomials

3.1. **Preliminaries and properties.** The Legendre polynomials, $P_n(z)$, are the solutions of Legendre's Differential Equations [4]. The orthogonal property of Legendre polynomials is defined as $\int_{-1}^1 P_n(z)P_m(z)dz = \frac{2}{2n+1}\delta_{nm}$, where δ_{nm} is the Kronecker delta. The shifted Legendre polynomials $L_n(t)$ are derived from $P_n(z)$ by replacing z with 2t-1, which in turn

refines the interval to [0,1]. The orthogonal property of $L_n(t)$ with Kronecker delta in [0,1] is defined by $\int_0^1 L_n(t)L_m(t)dt = \frac{1}{2n+1}\delta_{nm}$. Then
(i) The recurrence relation of $L_n(t)$ is defined as:

$$L_{i+1}(t) = \frac{(2i+1)(2t-1)}{i+1}L_i(t) - \frac{i}{i+1}L_{i-1}(t), i = 1, 2..., \qquad (2)$$
$$L_0(t) = 1, L_1(t) = 2t-1.$$

(ii) The analytic form of the shifted Legendre polynomial $L_n(t)$ of degree n is given by:

$$L_n(t) = \sum_{i=0}^{n}(-1)^{n+i}\frac{(n+i)!}{(n-i)!}\frac{t^i}{(i!)^2}, L_n(0) = (-1)^n, L_n(1) = 1.$$

(iii) The shifted Legendre vector L(t) is normally defined as:

$$L(t) = [L_0(t) \quad L_1(t) \quad . \quad . \quad . \quad L_N(t)]^T. \qquad (3)$$

(iv) The matrix form of L(t) which is of degree N can be represented as:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & \\ (-1)^{1+0}\frac{(1+0)!}{(1-0)!(0!)^2} & (-1)^{1+1}\frac{(1+1)!}{(1-1)!(1!)^2} & \cdots & 0 & \\ (-1)^{2+0}\frac{(2+0)!}{(2-0)!(0!)^2} & (-1)^{2+1}\frac{(2+1)!}{(2-1)!(1!)^2} & \cdots & 0 & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^{N+0}\frac{(N+0)!}{(N-0)!(0!)^2} & (-1)^{N+1}\frac{(N+1)!}{(N-1)!(1!)^2} & \cdots & (-1)^{N+N}\frac{(N+N)!}{(N-N)!(N!)^2} & \end{bmatrix}\begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^N \end{bmatrix}.$$

Thus

$$L(t) = DY(t). \qquad (4)$$

The dual matrix Q_1 is given by

$$Q_1 = \int_0^1 L(t)L^T(t)dt = \int_0^1 DY(t)(DY(t))^T dt = D\left(\int_0^1 Y(t)Y^T(t)dt\right)D^T = DHD^T, \qquad (5)$$

where H is the Hilbert matrix which is of order (N+1) given by

$$H = \int_0^1 Y(t)Y^T(t)dt = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{N+1} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{N+2} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{N+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N+1} & \frac{1}{N+2} & \frac{1}{N+3} & \cdots & \frac{1}{2N+1} \end{bmatrix}.$$

Theorem 3.1. *Any arbitrary function $u(t) \in L^2[0,1]$ can be approximated in terms of $L_n(t)$ as*

$$u(t) = \sum_{n=0}^{\infty} u_n L_n(t). \qquad (6)$$

One can identify u_j as $u_j = (2j+1)\int_0^1 u(x)L_j(x)dx, j = 0, 1, ...$.

If we approximate $u(t)$ by the first $N+1$ terms, we can write
$u(t) \simeq \sum_{n=0}^{N} u_n L_n(t) = U^T L(t) = L^T(t)U$, where U is the shifted Legendre coefficient vector given by $U = [u_0 \quad u_1 \quad . \quad . \quad . \quad u_N]^T$.
We approximate the kernel function by truncating the Taylor series of degree N in the form

$k(s,t)=\sum_{m=0}^{N}\sum_{n=0}^{N}k_{mn}s^m t^n$, where $k_{mn}=\frac{1}{m!n!}\frac{\partial^{m+n}k(0,0)}{\partial s^m \partial t^n}$, $n,m=0,1,...N$.
The matrix form of the above expression is given by $k(s,t)=Y^T(t)KY(s)$.
Additionally, the kernel function $k(s,t)$ can be expanded approximately by $L_m(s)$ and $L_n(t)$ of degree N in the form $k_N(s,t)=\sum_{m=0}^{N}\sum_{n=0}^{N}L_{k_{mn}}L_m(s)L_n(t)$ and the matrix form of $k(s,t)$ in terms of $L(s)$ and $L^T(t)$ is
$k(s,t)=L(s)K_L L^T(t), K_L=L_{k_{mn}}$.

3.2. **Operational Matrices.** In the subsequent parts of the section, we construct the operational matrices as follows[26, 36]: We define the product matrix, $Q(t)$ as

$$Q(t)=L(t)L^T(t), \tag{7}$$

which is a square matrix of order $(N+1)$.
Let $U=[u_0 \quad u_1 \quad . \quad . \quad . \quad u_N]^T\in\mathbb{R}^{N+1}$. Then,

$$Q(t)U\simeq \hat{U}L(t). \tag{8}$$

$\hat{U}$ is called the product operational matrix of shifted Legendre polynomial, calculated using

$$Q(t)U=D\left[\sum_{i=0}^{N}u_iL_i(t) \quad \sum_{i=0}^{N}u_itL_i(t) \quad . \quad . \quad . \quad \sum_{i=0}^{N}u_it^nL_i(t)\right]^T. \tag{9}$$

Now, the function $t^kL_i(t)$ is to be approximated as
$t^kL_i(t)\simeq L^T(t)C_{k,i}$, $C_{k,i}=[C_0^{k,i} \quad C_1^{k,i} \quad . \quad . \quad . \quad C_N^{k,i}]^T$.
From (5) we have, $\int_0^t t^kL_i(t)L(t)dt\simeq\left[\int_0^t L(t)L^T(t)dt\right]C_{k,j}=Q_1C_{k,j}$.
Therefore, for each i and k , we get

$$\begin{aligned}C_{k,i}&\simeq Q_1^{-1}\int_0^t t^kL(t)L_i(t)dt,\\ &=Q_1^{-1}\left[\int_0^t t^kL_0(t)L_i(t)dt \quad \int_0^t t^kL_1(t)L_i(t)dt \quad . \quad . \quad . \quad \int_0^t t^kL_N(t)L_i(t)dt\right]^T.\end{aligned}$$

Now the term $\sum_{i=0}^{N}u_it^kL_i(t)$ can be computed as follows:

$$\begin{aligned}\sum_{i=0}^{N}u_it^kL_i(t)&\simeq\sum_{i=0}^{N}u_iL^T(t)C_{k,i},=\sum_{i=0}^{N}u_i\sum_{j=0}^{N}L_j(t)C_j^{k,i},=\sum_{j=0}^{N}L_j(t)\sum_{i=0}^{N}u_iC_j^{k,i},\\ &=L^T(t)\left[\sum_{i=0}^{N}u_iC_0^{k,i} \quad \sum_{i=0}^{N}u_iC_1^{k,i} \quad . \quad . \quad . \quad \sum_{i=0}^{N}u_iC_N^{k,i}\right]^T\\ &=L^T(t)[C_{k,0} \quad C_{k,1} \quad . \quad . \quad . \quad C_{k,N}]U,\end{aligned}$$

$$=L^T(t)\hat{C}_k. \tag{10}$$

where $\hat{C}_k=[C_{k,0} \quad C_{k,1} \quad . \quad . \quad . \quad C_{k,N}]U, \qquad k=0,1,2\ldots N$.
Now, we define a new matrix $\hat{L}=[\hat{C}_0 \quad \hat{C}_1 \quad . \quad . \quad . \quad \hat{C}_N]$.
From (9) and (10), we obtain $\hat{U}=D\hat{L}^T$.
By (2), $\int_0^t L_n(s)ds=\frac{1}{2(2n+1)}[L_{n+1}(t)-L_{n-1}(t)]$.
Let P be the integration matrix of polynomials.

$$\int_0^t L(s)ds=PL(t)-\frac{1}{2(2n+1)}L_{n+1}(t), \tag{11}$$

where $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots & 0 & 0 \\ \frac{-1}{6} & 0 & \frac{1}{6} & 0 & \dots & 0 & 0 \\ 0 & \frac{-1}{10} & 0 & \frac{1}{10} & \dots & 0 & 0 \\ 0 & 0 & \frac{-1}{14} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{2(2n-3)} \\ 0 & 0 & 0 & 0 & \dots & \frac{-1}{2(2n-1)} & 0 \end{bmatrix}$.

The integration of the vector $L(t)$ given in (11) needs to be approximated as

$$\int_0^t L(s)ds \simeq PL(t). \tag{12}$$

Hence, any function $u(t)$ can be approximated as

$$\int_0^t u(s)ds \simeq \int_0^t U^T L(s)ds = U^T PL(t). \tag{13}$$

3.3. **The fractional stochastic integration operational matrix of shifted Legendre polynomials.** For the the vector $L(t)$, we define the fractional stochastic operational matrix of integration P_H as

$$\int_0^t L(s)dB^H(s) = P_H L(t), \tag{14}$$

$$\int_0^t L(s)dB^H(s) = \int_0^t DY(s)dB^H(s), \tag{15}$$

$$= D \begin{pmatrix} \int_0^t dB^H(s) \\ \int_0^t sdB^H(s) \\ \vdots \\ \int_0^t s^N dB^H(s) \end{pmatrix},$$

$$= D \left[W(t)Y(t) - \begin{pmatrix} 0 \\ \int_0^t B^H(s)ds \\ \vdots \\ N\int_0^t s^{N-1}B^H(s)ds \end{pmatrix} \right].$$

$$= DV_H(t) = D(\Upsilon_i), i = 0, 1, ..., N$$

where $\Upsilon_i = t^i B^H(t) - i\int_0^t s^{i-1}B^H(s)ds$, $i = 0, 1, ..., N$.
Evaluating the integral for each i, we get
$\Upsilon_i = t^i B^H(t) - \frac{ti}{4}(2(\frac{t}{2})^{i-1}B^H(\frac{t}{2}) + t^{i-1}B^H(t)) = [(1 - \frac{i}{4})B^H(t) - \frac{i}{2^i}B^H(\frac{t}{2})]t^i$.

We assume that $B^H(0.5)$ and $B^H(0.25)$ are the approximate value of $B^H(t)$ and $B^H(\frac{t}{2})$ respectively for any value of t in [0,1]. Hence $DV_H(t)$ is given by

$$D \begin{pmatrix} B^H(0.5) & 0 & \dots & 0 \\ 0 & \frac{3}{4}B^H(0.5) - \frac{1}{2}B^H(0.25) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (1 - \frac{N}{4})B^H(0.5) - \frac{N}{2^N}B^H(0.25) \end{pmatrix} \begin{pmatrix} 1 \\ t \\ \vdots \\ t^N \end{pmatrix}.$$

Let

$$\Gamma_H = \begin{pmatrix} B^H(0.5) & 0 & \dots & 0 \\ 0 & \frac{3}{4}B^H(0.5) - \frac{1}{2}B^H(0.25) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (1-\frac{N}{4})B^H(0.5) - \frac{N}{2^N}B^H(0.25) \end{pmatrix}.$$

Then,
$DV_H(t) = D\Gamma_H Y(t) = D\Gamma_H D^{-1}L(t) = P_H L(t)$, where $P_H = D\Gamma_H D^{-1}$ and Γ_H is the diagonal matrix of order (N+1) whose diagonal elements are of the form $(1-\frac{j}{4})B^H(0.5) - \frac{j}{2^j}B^H(0.25) \quad j = 0,1,2,...,N.$
By using (14) and(15), the integral of $u(t)$ is defined as

$$\int_0^t u(s)dB^H(s) \simeq \int_0^t U^T L(s)dB^H(s) = U^T P_H L(t). \tag{16}$$

Let

$$\phi_i(t) = N_i(t, X(t)), \quad i = 1,2. \tag{17}$$

Using (1) in (17),

$$\phi_i(t) = N_i(t, f(t) + \int_0^t k_1(s,t)\phi_1(s)ds + \int_0^t k_2(s,t)\phi_2(s)dB^H(s)), \quad i = 1,2. \tag{18}$$

Approximating the above mentioned functions in terms of $L(s)$ and $L^T(t)$ in the following manner

$$f(t) \simeq L^T(t)F, \tag{19}$$

$$k_i(s,t) \simeq L^T(t)K_{iL}^T L(s), \tag{20}$$

$$\phi_i(t) \simeq L^T(t)\Phi_i, \quad i = 1,2\ , \tag{21}$$

where F, Φ_i are $(N+1)$ column vectors and K_{iL} are square matrices of order $(N+1)$.
By substituting (19) - (21) in (18),

$$\begin{aligned} L^T(t)\Phi_i(t) = N_i(t, L^T(t)F &+ \int_0^t L^T(t)K_{1L}^T L(s)L^T(s)\Phi_1(s)ds \\ &+ \int_0^t L^T(t)K_{2L}^T L(s)L^T(s)\Phi_2(s)dB^H(s)), \quad i = 1,2. \end{aligned} \tag{22}$$

By using (8), (12), (14), (22) becomes

$$L^T(t)\Phi_i(t) = N_i(t, L^T(t)F + L^T(t)K_{1L}^T\hat{\Phi_1}PL(t) + L^T(t)K_{2L}^T\hat{\Phi_2}P_H L(t)), \quad i = 1,2. \tag{23}$$

We collocate (23) at $N+1$ points by using the formula $t_s = \frac{2s+1}{2(N+1)}; s = 0,1,\dots,N.$ Therefore, for $i = 1,2$,

$$L^T(t_s)\Phi_i(t) = N_i(t_s, L^T(t_s)F + L^T(t_s)K_{1L}^T\hat{\Phi_1}PL(t_s) + L^T(t_s)K_{2L}^T\hat{\Phi_2}P_H L(t_s)). \tag{24}$$

By collocating (24) at these $(N+1)$ points, we get a nonlinear system of $2(N+1)$ algebraic equations from which the coefficients can be obtained by using Newton's method. Hence the approximate solution of (1) is obtained as

$$X(t) \simeq L^T(t)F + L^T(t)K_{1L}^T\hat{\Phi_1}PL(t) + L^T(t)K_{2L}^T\hat{\Phi_2}P_H L(t). \tag{25}$$

4. Theoretical Analysis

Let $e_N(t) = X(t) - X_N(t)$ be the error function where $X_N(t)$ is the Nth degree approximation of the exact solution X(t). The error bound and convergence theorem for the proposed method in terms of the function approximation and the error function are discussed here.

Theorem 4.1. *Let $f_N(t)$ be the function approximation of $f(t)$ then the error bound is given by $\|f(t) - f_N(t)\|_{L^2} \leq C\hat{F}(2)^{-N}, t \in [0,1]$, where*
$\hat{F} =^{sup}_{t} \left\|f^{(N)}(t)\right\|_{L^2}$, C being a constant.

Proof.

$$\|f(t) - f_N(t)\|^2 = \int_0^1 (f(t) - f_N(t))^2 dt \leq \int_0^1 \left(\frac{1}{N!2^N}\hat{F}dt\right)^2 = \left(\frac{1}{N!2^N}\hat{F}\right)^2$$
$$= (C\hat{F}2^{-N})^2,$$

where $C = \frac{1}{N!}$ and $\hat{F} =^{sup}_{t} \left\|f^{(N)}(t)\right\|, t \in [0,1]$. □

Theorem 4.2. *Let $k_N(s,t)$ be the shifted Legendre approximation of the function $k(s,t)$ then the error bound is $\|k(s,t) - k_N(s,t)\| \leq \hat{C}\hat{K}(2)^{-2N}$ where $\hat{C}$ is a positive constant, $\hat{K} =^{sup}_{(s,t)} \left\|\frac{\partial^{2N}k(s,t)}{\partial s^N \partial t^N}\right\|, (s,t) \in [0,1] \times [0,1]$.*

Proof. Proof of this theorem is based on the assumptions and the steps followed in Theorem 4.1. □

Theorem 4.3. *Let $X_N(t)$ be the approximate solution of the exact solution $X(t)$ with $N_1(s,t)$, $N_2(s,t)$ satisfying the Lipschitz condition*
$\|N_1(s,t_1) - N_1(s,t_2)\| + \|N_2(s,t_1) - N_2(s,t_2)\| \leq L\|t_1 - t_2\|$.
Also assume that
i)$\|\phi_i(t)\| \leq \rho_i, t \in [0,1]$
ii)$\|k_i(s,t)\| \leq M_i$, for every (s,t) defined in the domain $[0,1] \times [0,1]$
iii) $G(N) < 1$ for i=1,2. Then, we have
I)$\|X(t) - X_N(t)\| \leq \frac{\eta(N)+((M_1+\psi(N))\beta_1(N)+\psi(N)\rho_1)+\|W(t)\|((M_2+\gamma(N))\beta_2(N)+\gamma(N)\rho_2)}{1-G(N)}$
II) $X_N(t) \to X(t)$ in L^2 when $E\left(|e_N(t)|^2\right) \to 0$. where
$\eta(N) = C\hat{F}(2)^{-N}$; $\lambda(N) = \hat{C}_1(2)^{-2N}$; $\gamma(N) = \hat{C}_2(2)^{-2N}$; $\beta_i(N) = C\hat{\Phi}_i(2)^{-N}$; $\hat{\Phi}_i = sup\left\|\Phi_i^{(N)}(t)\right\|$ $i = 1,2$.

Proof. Proof of I : Let $\hat{\phi}_i(s)$ be the approximate solution of $\phi_i(s)$ of (17). Then we have $\hat{\Phi}_i(s) = \hat{N}_i(s, X_N(s))$ and $\phi_i^N(s) = N_i(s, X_N(s))$, $i = 1,2$. Hence from the above theorems, we have

$$\left\|\phi_i(s) - \hat{\phi}_i(s)\right\| \leq \left\|\phi_i(s) - \phi_i^N(s)\right\| + \left\|\phi_i^N(s) - \hat{\phi}_i(s)\right\| \leq L\|X(s) - X_N(s)\| + \beta_i(N), \quad (26)$$

Also the approximation of (1) is given as

$$X_N(t) = f_N(t) + \int_0^t k_{1N}(s,t)\hat{\phi}_1(s)ds + \int_0^t k_{2N}(s,t)\hat{\phi}_2(s)dB^H(s).$$

Hence the norm of the error function is given by

$$\|X(t) - X_N(t)\| \leq \|f(t) - f_N(t)\| + \left\|k_1(s,t)\phi_1(s) - k_{1N}(s,t)\hat{\phi}_1(s)\right\|$$
$$+ \left\|B^H(t)\right\| \left\|k_2(s,t)\phi_2(s) - k_{2N}(s,t)\hat{\phi}_2(s)\right\|. \quad (27)$$

By using Theorems 4.1, 4.2 and assumptions (i) and (ii) of Theorem 4.3, for $i = 1, 2$ we have

$$\left\|k_i(s,t)\phi_i(s) - k_{iN}(s,t)\hat{\phi}_i(s)\right\| \leq \|k_i(s,t)\| \left\|\phi_i(s) - \hat{\phi}_i(s)\right\| + \|k_i(s,t) - k_{iN}(s,t)\| \left(\left\|\phi_i(s) - \hat{\phi}_i(s)\right\| + \|\phi_i(s)\|\right), \tag{28}$$

$$\left\|k_i(s,t)\phi_i(s) - k_{iN}(s,t)\hat{\phi}_i(s)\right\| \leq (M_i + \lambda(N))L \|X(t) - X_N(t)\| + (M_i + \lambda(N))\beta_i(N) + \lambda(N)\rho_i. \tag{29}$$

Using (26),(27) and assumption (iii) of Theorem 4.3, we have

$$\|X(t) - X_N(t)\| \leq \frac{\eta(N) + H_1(N) + \left\|B^H(t)\right\| H_2(N)}{1 - G(N)}, \tag{30}$$

where $G(N) = L(M_1 + \lambda(N)) - \left\|B^H(t)\right\| L(M_2 + \gamma(N))$,
$H_1(N) = (M_1 + \lambda(N))\beta_1(N) + \lambda(N)\rho_1$,
$H_2(N) = (M_2 + \gamma(N))\beta_2(N) + \gamma(N)\rho_2$.
Proof of II : $E\left(|e_N(t)|^2\right) = E\left(|X(t) - X_N(t)|^2\right)$.
By using Theorems 4.1 and 4.2, we get,

$$E\left(|X(t) - X_N(t)|^2\right) \leq P(N) + T(N)E\left(|X(t) - X_N(t)|^2\right), \tag{31}$$

where $P(N) = 3\eta^2(N) + 9(M_1+\lambda(N))^2\beta_1^2(N) + 9\lambda^2(N)\rho_1^2 + 9\left|B^H(t)\right|^2((M_2+\gamma(N))^2\beta_2^2(N) + \gamma^2(N)\rho_2^2$ and
$T(N) = 9(M_1 + \lambda(N))^2L^2 + 9\left|B^H(t)\right|^2\gamma^2(N)\rho_2^2)$.
Hence from (29) and Gronwall inequality, we have $E\left(|e_N(t)|^2\right) \to 0$. □

5. Numerical Examples

In this section, two examples are presented to demonstrate the applicability of the proposed method. N and k represent the degree of the approximate function and the number of simulations, respectively.

Example 1:[12]
We consider (1) with $f(t) = \frac{1}{10}, k_1(s,t) = -2Ha^2s^{2H-1}$,
$k_2(s,t) = a,\ N_1(s, X(s)) = X(s)(1 - X^2(s)), N_2(s, X(s)) = (1 - X^2(s))$,
$a = \frac{1}{30}$, $t \in (0, 1)$. The approximate solution for X(t) is obtained by the method described in Section 3. The exact solution of the above example is $X(t) = tanh(aB^H(t) + tanh^{-1}(X_0))$. Table 1 shows the mean $\overline{X}_E$ and standard deviation S_E of the absolute errors of X(t) along with their 0.95 confidence intervals. We also consider k = 500, N = 32 and H = 0.2. The graph of the exact and approximate solutions obtained by using shifted Legendre polynomials at $H = 0.4$ is also shown in Figure 1(a).

Example 2: [12]
We consider (1) with $f(t) = \frac{1}{10}, k_1(s,t) = -Ha^2s^{2H-1}$,
$k_2(s,t) = a, N_1(s, X(s)) = tanh(X(s))sech^2(X(s)), N_2(s, X(s)) = sech(X(s))$,
$a = \frac{1}{30}$, $t \in (0, 1)$. The approximate solution for X(t) is obtained by the method described in Section 3. The exact solution of the above example is $X(t) = arcsinh(aB^H(t) + sinh(X_0))$. Table 2 shows the mean $\overline{X}_E$ and standard deviation S_E of the absolute errors of X(t) along with their 0.95 confidence intervals. We also consider k=500, N=32 and H =

Table 1. Mean, standard deviation and mean confidence interval for error in Example 1 with Hurst parameter 0.2.

			0.95 Confidence interval	
t	$\overline{X}_E$	S_E	Upper bound	Lower bound
0.1	1.5972e-03	1.2125e-03	1.5200e-03	1.6700e-03
0.2	1.9065e-03	1.4775e-03	1.8100e-03	2.0000e-03
0.3	2.0850e-03	1.5383e-03	1.9901e-03	2.1802e-03
0.4	2.2208e-03	1.6952e-03	2.1201e-03	2.3208e-03
0.5	2.2507e-03	1.6744e-03	2.1500e-03	2.3500e-03
0.6	2.3265e-03	1.7272e-03	2.2200e-03	2.4301e-03
0.7	2.4324e-03	1.8247e-03	2.3200e-03	2.5502e-03
0.8	2.4694e-03	1.9437e-03	2.3501e-03	2.5903e-03
0.9	2.5608e-03	1.9722e-03	2.4401e-03	2.6802e-03
1.0	2.6259e-03	2.0592e-03	2.5001e-03	2.7504e-03

0.3. The graph of the exact and approximate solutions obtained by using shifted Legendre polynomials at $H = 0.4$ is depicted in Figure 1(b).

The mean error comparison of the numerical method based on hat functions and the proposed SLP method are shown in Figure 2(a) and Figure 2(b) for examples 1 with H = 0.2 and example 2 with H = 0.3 respectively. It has been observed from the figures that the mean error value of our proposed method is smaller than the mean error of the method based on hat function [12].

Table 2. Mean, standard deviation and mean confidence interval for error in Example 2 with Hurst parameter 0.3.

			0.95 Confidence interval	
t	$\overline{X}_E$	S_E	Upper bound	Lower bound
0.1	1.2620e-03	9.3297e-04	1.2042e-03	1.3198e-03
0.2	1.6577e-03	1.2750e-03	1.5787e-03	1.7367e-03
0.3	1.8295e-03	1.3660e-03	1.7448e-03	1.9141e-03
0.4	2.0130e-03	1.5211e-03	1.9188e-03	2.1073e-03
0.5	2.1612e-03	1.6576e-03	2.0584e-03	2.2639e-03
0.6	2.2957e-03	1.7370e-03	2.1880e-03	2.4033e-03
0.7	2.3539e-03	1.7467e-03	2.2456e-03	2.4621e-03
0.8	2.5151e-03	1.9722e-03	2.3928e-03	2.6373e-03
0.9	2.5839e-03	1.9862e-03	2.4608e-03	2.7070e-03
1.0	2.7173e-03	2.0601e-03	2.5896e-03	2.8450e-03

The key features of our proposed methodology are summarised as follows. The proposed methodology generates a solution that is identical to the exact solution. The technique's superiority is based on the minimal amount of error it causes, which can be seen in

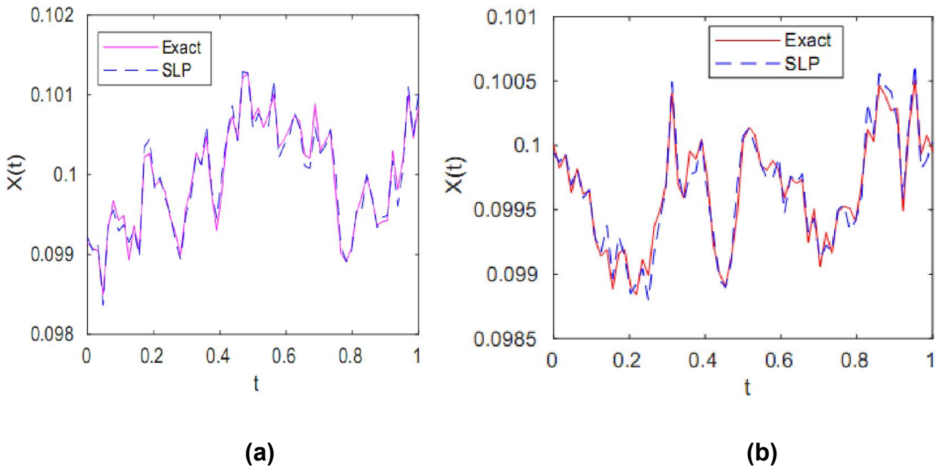

FIGURE 1. The Graph of Exact and Approximate solutions of (a) Example 1 with H = 0.3 and (b) Example 2 with H = 0.4

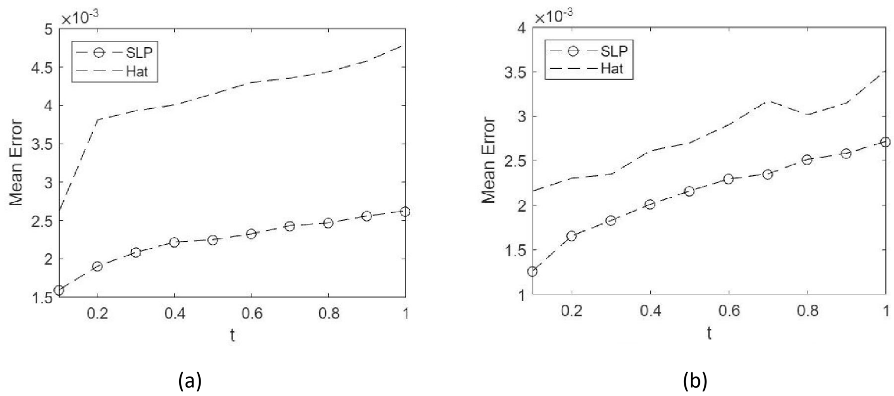

FIGURE 2. Mean Error comparison of (a) Example 1 with H=0.2 and (b) Example 2 with H=0.3

the figures. Also, from the experimental problems, we observe from the figures that the mean error value of our proposed method is smaller than the mean error of the method based on hat functions. The tables also show that the error values are within the upper bound discussed in the theoretical analysis. Since the polynomials used in this case are orthogonal, the construction of operational matrices and the calculation of connection coefficients involved in function approximation are simple. The problems considered for discussion in this paper do not have the exact solution in the case of $H < \frac{1}{2}$. The approximate solution has been expanded to a certain number of terms and it has been compared with the function $f(t)$. The quality of the numerical solution can be realised from tables and figures. The comparison has been carried out with the theoretical errror bound since no other method is available in the literature for solving the problems that are mentioned in the numerical examples.

The various integration matrices and their properties are used to convert the given equations into a system of algebraic equations. The advantage of having the lower triangular and tridiagonal forms allows us to solve the problem more accurately, whereas when dealing with Euler polynomials, it seeks the help of the Bernoulli polynomials, resulting in a massive amount of work even though it reduces to a lower triangular system. Since shifted Legendre polynomials have the weak form of sparse matrices, they are preferred to generalised hat functions, Bernoulli, and Bernstein polynomials in terms of computation difficulty. Certain numerical methods like Euler, Euler - Maruyama, R-K method, and Milstein method require the previous iteration values for pointwise solutions, whereas this method does not require any such assigned values. It has the advantage of generating a more accurate solution with fewer basis functions, and these polynomials are fundamental for dealing with any sort of stochastic differential equation.

6. Conclusion

This paper discusses a fast approximation method for solving a nonlinear stochastic integral equations with fractional Brownian motion, which are common in the physical and biological sciences. Approximating the supplied function in terms of a linear combination of unknown constants and the basis of the polynomials is the essence of the proposed methodology. To solve the given equation, stochastic operational matrices for stochastic integration and fractional stochastic integration have been constructed. The shifted Legendre polynomial matrix is a triangular matrix. As a result, the dual matrix is found to be diagonal. This is a noteworthy characteristic when working with the shifted Legendre polynomial. The proposed methodology has undergone theoretical research, and the method's applicability has been statistically validated by using numerical examples. The magnitude of the absolute error and the mean error are small, as can be seen from the tables and figures. Since the original problem is solved using a set of algebraic equations, this technique is simple to implement and can be applied to solve additional stochastic differential equations. This proposed methodology can be implemented and analysed for higher dimensional equations with the help of Kronecker delta functions.

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