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COEFFICIENT ESTIMATE OF THE SECOND HANKEL DETERMINANT OF LOGARITHMIC COEFFICIENTS FOR THE SUBCLASS OF CLOSE-TO-CONVEX FUNCTION

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ABSTRACT. Let S denote the subclass of the analytic function and univalent functions in D, where D is defined as a unit disk, and having the Taylor representation form of S. In this paper, we will be obtaining the second Hankel determinants, in which the elements are the logarithmic coefficients mainly for the subclass of the close-to-convex functions in S.

Keywords: Univalent functions, Analytic functions, Second Hankel determinant, Logarithmic coefficients, Close-to-convex function.

AMS Subject Classification: 30C50,30C45.

1. Introduction

Let S be a subclass of the analytic function A, normalized by $f(0) = f'(0) - 1 = 0$ in D where D is defined as the unit disk, $|z| < 1$ such that $z \in C$. If the function $f \in A$, then $f(z)$ has the series form,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

There are three main subclasses in S that include starlike functions, convex functions, and close-to-convex functions. We made S represent the class of univalent functions in A. The $f \in A$ becomes a starlike function, when it satisfies the following conditions

$$Re\left(\frac{z\,f'(z)}{f(z)}\right) > 0, \tag{2}$$

for $z \in D$. The class of the starlike function is denoted by S^*. It has an important member as well, as the class S which is the Koebe function that can be defined as follows,

$$k(z) = \frac{z}{(1-z)^2}, \tag{3}$$

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The Koebe function plays significant roles in extreme functions mainly for the furthermost issues for the class S^* and S classes. A function $f \in A$ is called a convex function, when it meets the following conditions,

$$Re\left(1+\frac{z f''(z)}{f'(z)}\right)>0, \tag{4}$$

for $z \in D$. This class is denoted by CV in class S. The function $f \in A$ is said to be a close-to-convex function if there exists a real number, α, where $|\alpha| < \pi/2$ and the function $g(z)$ is convex which meets the following conditions,

$$Re\left(e^{i\alpha}\frac{f'(z)}{g'(z)}\right)>0, \tag{5}$$

where $z \in D$ [12]. In 1916, Alexander indicated that the connection between the starlike function and convex function where the condition if the function $h(z) \in S^*$ then $h(z) = zg'(z)$ where $g(z) \in CV$. (As cited by [18]). Therefore, the condition (5) can be formed as follows,

$$Re\left(e^{i\alpha}\frac{zf'(z)}{h(z)}\right)>0, \tag{6}$$

where $z \in D$. From there, we know that the starlike function and the convex functions are close-to-convex functions. We can summarize it by proper inclusion $CV \subseteq S^* \subseteq K \subseteq S$. We denote the class of close-to-convex function as K. The classes of starlike function, convex function and close-to-convex function have a representation that uses the Caratheodory class P, which is an analytic function p in D by having the following form,

$$p(z)=1+\sum_{n=1}^{\infty} c_n z^n, \tag{7}$$

where $z \in D$ and having a positive real part in D. These classes can be expressed by coefficients of functions in P. The logarithmic coefficients of function f, can be written as,

$$\log\frac{f(z)}{z}=2\sum_{n=1}^{\infty}\gamma_n z^n, \tag{8}$$

where $z \in D$. The logarithmic coefficients show a critical role in Milin's conjecture (as cited by [13]). It shows that for the class S, the sharp estimates for the single logarithmic coefficient, $|\gamma_1| \leq 1$, $|\gamma_2| \leq 0.635$ and are unknown for $n = 3, 4, \cdots$. There are several results of logarithmic coefficients estimates that have been established, for example, the results in [1], [2], [4], [7], [8], [9], [15], [16], [20] and [21].

The Hankel matrices and determinants play a significant role and have several applications in a number of branches of mathematics. Many researchers have established the Hankel determinants for their classes. For example, the results of [3], [5], [10], [11], [14], [17] and [19].

Recently, authors in [13] had obtained the sharp result of the Hankel determinants whose entries are the logarithmic coefficient of $f \in S$, that is,

$$H_{q,n}(F_f/2)=\begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2(q-1)} \end{vmatrix},$$

where $q, n \in N$. In their research, they dealt with finding the Hankel determinants for the classes of starlike function and convex function. Additionally, authors in [3] had obtained the second Hankel determinant of logarithmic coefficient for various subclasses of analytic function with accurate results. From there, we were motivated to find the Hankel determinant for the subclass of S especially for the class of close-to-convex functions.

In this paper, we have to deal with $H_{2,1}(F_f/2) = \gamma_1\gamma_3 - \gamma_2^2$ by using a subclass of close-to-convex function that is defined as follows,

$$Re\left\{\left(1-z^2\right) f'(z)\right\} > 0,$$

where $z \in D$. We denoted this subclass as F_1. This class have been studied by [4], [7], [14] and [15] on the estimation of logarithmic coefficients. This class has been discovered by Ozaki (As cited by [7]) as a valuable criteria of univalence. This class has a good geometrical interpretation, and it plays a major role in the geometric theory. The following lemmas will be used in order to obtain the second Hankel determinant.

Lemma 1.1 (As cited by [13]). *If $p \in \mathrm{P}$ is in the form (7) with $c_1 \geq 0$, then*

$$c_1 = 2\zeta_1, \tag{9}$$

$$c_2 = 2\zeta_1^2 + 2\left(1-\zeta_1^2\right)\zeta_2, \tag{10}$$

and

$$c_3 = 2\zeta_1^3 + 4\left(1-\zeta_1^2\right)\zeta_1\zeta_2 - 2\left(1-\zeta_1^2\right)\zeta_1\zeta_2^2 + 2\left(1-\zeta_1^2\right)\left(1-|\zeta_2|^2\right)\zeta_3 \tag{11}$$

for some $\zeta_1 \in [0,1]$ and $\zeta_2, \zeta_3 \in \bar{D}$. For $\zeta_1 \in T$, there is a unique function $p \in \mathrm{P}$ with c_1 and c_2 as in (9)-(10), namely,

$$p(z) = \frac{1+\left(\bar{\zeta}_1\zeta_2+\zeta_1\right)z+\zeta_2 z^2}{1+\left(\bar{\zeta}_1\zeta_2-\zeta_1\right)z-\zeta_2 z^2}, z \in D.$$

For $\zeta_1, \zeta_2 \in D$ and $\zeta_3 \in T$, there is a unique functionp $\in \mathrm{P}$ with c_1, c_2 and c_2 as in (9)-(11)

$$p(z) = \frac{1+\left(\bar{\zeta}_2\zeta_3+\bar{\zeta}_1\zeta_2+\zeta_1\right)z+\left(\bar{\zeta}_1\zeta_3+\zeta_1\bar{\zeta}_2\zeta_3+\zeta_2\right)z^2+\zeta_3 z^3}{1+\left(\bar{\zeta}_2\zeta_3+\bar{\zeta}_1\zeta_2-\zeta_1\right)z+\left(\bar{\zeta}_1\zeta_3-\zeta_1\bar{\zeta}_2\zeta_3-\zeta_2\right)z^2-\zeta_3 z^3}, \quad z \in D. \tag{12}$$

Lemma 1.2 (As cited by [3]). *Given the real numbers A, B, C, let*

$$Y(A,B,C) := \max\left\{\left|A+Bz+Cz^2\right|+1-|z|^2 \ : \ z \in \bar{D}\right\}.$$

I . If $AC \geq 0$, then

$$Y(A,B,C) = \begin{cases} |A|+|B|+|C|, & |B| \geq 2(1-|C|), \\ 1+|A|+\frac{B^2}{4(1-|C|)}, & |B| < 2(1-|C|). \end{cases}$$

II . If $AC < 0$, then

$$Y(A,B,C) = \begin{cases} 1-|A|+\frac{B^2}{4(1-|C|)}, & -4AC\left(\frac{1}{C^2}-1\right) \leq B^2 \wedge |B| < 2(1-|C|), \\ 1+|A|+\frac{B^2}{4(1-|C|)}, & B^2 < \min\left\{4(1+|C|)^2, -4AC\left(\frac{1}{C^2}-1\right)\right\} \\ R(A,B,C) & \textit{otherwise}, \end{cases}$$

where

$$R(A,B,C) := \begin{cases} |A|+|B|-|C|, & |C|(|B|+4|A|) \leq |AB|, \\ -|A|+|B|+|C|, & |AB| \leq |C|(|B|-4|A|), \\ (|C|+|A|)\sqrt{1-\frac{B^2}{4AC}} & \textit{otherwise}. \end{cases}$$

The following theorem will be discussed on the $H_{2,1}(F_f/2)$ for the subclass of the close-to-convex function.

Theorem 1.1. *If the function $f \in F_1$, then*

$$\left| \gamma_1\gamma_3 - \gamma_2^2 \right| \le 0.05364.$$

The inequality is sharp.

Proof. We have,

$$\left(1 - z^2\right) f'(z) = p(z) \tag{13}$$

By differentiating the equation (13) and computing the coefficients of z^2, z^3, and z^4, we get

$$a_2 = \frac{1}{2}c_1, \tag{14}$$

$$a_3 = \frac{1}{3} + \frac{1}{3}c_2, \tag{15}$$

and

$$a_4 = \frac{1}{4}c_3 + \frac{1}{4}c_1 \tag{16}$$

respectively. Note that, the logarithmic coefficients for γ_1, γ_2, and γ_3 give

$$\gamma_1 = \frac{1}{2}a_2, \tag{17}$$

$$\gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right), \tag{18}$$

and

$$\gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right) \tag{19}$$

respectively. Then, substitute the equation of (14), (15), and (16) into the logarithmic coefficients (17) - (18), yields

$$\gamma_1 = \frac{1}{4}c_1, \tag{20}$$

$$\gamma_2 = \frac{1}{6} + \frac{1}{6}c_2 - \frac{1}{16}c_1^2, \tag{21}$$

and

$$\gamma_3 = \frac{1}{48}c_1^3 + \frac{1}{48}(2 - 4c_2)\, c_1 + \frac{1}{8}c_3. \tag{22}$$

Then, by applying Lemma 1.1 into the logarithmic coefficient of the equation (20) – (22), we have

$$\gamma_1 = \frac{1}{2}\zeta_1 \tag{23}$$

$$\gamma_2 = \frac{1}{6} + \frac{1}{12}\zeta_1^2 + \frac{1}{3}\left(1 - \zeta_1^2\right)\zeta_2 \tag{24}$$

and

$$\gamma_3 = \frac{1}{12}\zeta_1^3 + \frac{1}{12}\left(1 + \left(2\zeta_2 - 3\zeta_2^2\right)\left(1 - \zeta_1^2\right)\right)\zeta_1 + \frac{1}{4}\left(1 - \zeta_1^2\right)\left(1 - |\zeta_2|^2\right)\zeta_3 \tag{25}$$

Note that, the second Hankel determinant denotes $H_{2,1}(F_f/2) = \gamma_1\gamma_3 - \gamma_2^2$. By relation of the equation (23) - (25), we have

$$\begin{aligned}\gamma_1\gamma_3 - \gamma_2^2 = {}& \tfrac{5}{144}\zeta_1^4 + \tfrac{1}{144}\left(2 + \left(4\zeta_2 - 18\zeta_2^2\right)\left(1 - \zeta_1^2\right)\right)\zeta_1^2 \\ & + \tfrac{1}{8}\left(1 - \zeta_1^2\right)\left(1 - |\zeta_2|^2\right)\zeta_1\zeta_3 - \tfrac{1}{9}\left(\left(1 - \zeta_1^2\right)\zeta_2 + \tfrac{1}{2}\right)^2\end{aligned} \tag{26}$$

A. Assuming that $\zeta_1 = 1$, then by the equation (26),

$$\left|\,\gamma_1\gamma_3 - \gamma_2^2\,\right| = \frac{1}{48}.$$

B. Assuming that $\zeta_1 = 0$, then we have the following inequality

$$\begin{aligned}\left|\,\gamma_1\gamma_3 - \gamma_2^2\,\right| &= \left| -\left(\frac{1}{6} + \frac{1}{3}\zeta_2\right)^2 \right| \\ &\le \frac{1}{9}\zeta_2^2 + \frac{1}{9}\zeta_2 + \frac{1}{36} = \psi_1\left(\zeta_2\right)\end{aligned}$$

By using elementary calculus to find the extreme point for the function $\psi_1\left(\zeta_2\right)$, we found that,

$$\max\ \psi_1\left(\,\zeta_2\,\right) = \psi_1\left(\,0.5\,\right) = \frac{1}{9} \approx 0.1111.$$

Therefore, we can say that

$$\left|\,\gamma_1\gamma_3 - \gamma_2^2\,\right| \le \frac{1}{9} \approx 0.1111.$$

C. Assuming that $\zeta_1 \in (\,0\,,\,1\,)$. Since the inequality is $|\,\zeta_3\,| \le 1$, the equation (26) will return as

$$\begin{aligned}\left|\,\gamma_1\gamma_3 - \gamma_2^2\,\right| \le\ & \tfrac{5}{144}\zeta_1^4 + \tfrac{1}{144}\left(2 + \left(4\,\zeta_2 - 18\,\zeta_2^2\right)\left(1 - \zeta_1^2\,\right)\right)\zeta_1^2 \\ & + \tfrac{1}{8}\left(1 - \zeta_1^2\right)\left(1 - |\zeta_2|^2\right)\zeta_1\ - \tfrac{1}{9}\left(\left(1 - \zeta_1^2\right)\zeta_2 + \tfrac{1}{2}\right)^2 \\ =\ & \frac{1}{8}\zeta_1\left(1 - \zeta_1^2\right)\left[\,\left|\,A + B\zeta_2 + C\zeta_2^2\,\right| + 1 - |\zeta_2|^2\right] + \frac{1}{36},\end{aligned} \tag{27}$$

where

$$A = \frac{1}{\zeta_1\left(1 - \zeta_1^2\right)}\left(\frac{1}{9}\zeta_1^2 + \frac{5}{18}\zeta_1^4\right),$$

$$B = \frac{1}{\zeta_1}\left(\frac{2}{9}\zeta_1^2 - \frac{8}{9}\right),$$

and

$$C = \frac{1}{\zeta_1}\left(-\frac{1}{9}\zeta_1^2 - \frac{8}{9}\right).$$

We use Lemma 1.2 only for case II as $AC < 0$. Before that, we know that the absolutes of A, B and C give,

$$A = \frac{1}{18}\left(\frac{4\,\zeta_1^2}{\left(1 - \zeta_1^2\right)^2} + \frac{20\,\zeta_1^4}{\left(1 - \zeta_1^2\right)^2} + \frac{25\,\zeta_1^6}{\left(1 - \zeta_1^2\right)^2}\right)^{\frac{1}{2}},$$

$$B = \frac{2}{9}\left(\zeta_1^2 - 8 + \frac{16}{\zeta_1^2}\right)^{\frac{1}{2}},$$

and

$$C = \frac{1}{9}\left(\zeta_1^2 + 16 + \frac{64}{\zeta_1^2}\right)^{\frac{1}{2}}.$$

C1. Note that the following inequality gives us

$$-4AC\left(\frac{1}{C^2} - 1\right) - B^2 \le \frac{6\,\zeta_1^6 - 636\,\zeta_1^4 - 64\,\zeta_1^2 - 512}{81\left(\zeta_1^2 + 8\right)\zeta_1^2} \le 0,$$

and it is equivalent to $6\,\zeta_1^6 - 636\,\zeta_1^4 - 64\,\zeta_1^2 - 512 \le 0$, which is false for $\zeta_1 \in (\,0\,,\,1\,)$. Same goes to the inequality $|\,B\,| < 2\,(1 - |\,C\,|)$ gives

$$\frac{2}{9}\sqrt{\frac{\zeta_1^2 - 8 + \frac{16}{\zeta_1^2}}{9}} + \frac{2}{9}\sqrt{\frac{\zeta_1^2 + 16 + \frac{64}{\zeta_1^2}}{9}} - 2 < 0,$$

which is false for $\zeta_1 \in (\,0\,,\,1\,)$.

C2. Next, we look into another case for the following inequality

$$B^2 < \min\left\{\,4\,(1 + |\,C\,|)^2\,,\ -4AC\left(\frac{1}{C^2} - 1\right)\right\}.$$

From there, we know that

$$4\,(1 + |\,C\,|)^2 = \frac{4\left(9 + \sqrt{\frac{(\,\zeta_1^2+8\,)^2}{\zeta_1^2}}\right)^2}{81} > 0, \quad \textbf{and} \quad -4AC\left(\frac{1}{C^2} - 1\right) = \frac{10\,\zeta_1^4 - 636\,\zeta_1^2 - 256}{81\,\zeta_1^2 + 648} < 0.$$

Now, we can say that

$$B^2 + 4AC\left(\frac{1}{C^2} - 1\right) = \frac{-6\,\zeta_1^6 + 636\,\zeta_1^4 + 64\,\zeta_1^2 + 512}{81\,\left(\,\zeta_1^2 + 8\,\right)\,\zeta_1^2} < 0,$$

which is false for $\zeta_1 \in (\,0\,,\,1\,)$.

C3. We found that the following inequality

$$\begin{aligned}|\,C\,|\,(|\,B\,| + 4\,|\,A\,|) - |\,AB\,| = {} & \tfrac{2}{81}\left(\tfrac{(\,\zeta_1^2+8\,)^2}{\zeta_1^2}\right)^{\frac12}\left(\tfrac{(\,\zeta_1^2-4\,)^2}{\zeta_1^2}\right)^{\frac12} + \tfrac{2}{81}\left(\tfrac{(\,\zeta_1^2+8\,)^2}{\zeta_1^2}\right)^{\frac12}\left(\tfrac{\zeta_1^2(\,5\,\zeta_1^2+2\,)^2}{(\,\zeta_1^2-1\,)^2}\right)^{\frac12} \\ & -\tfrac{1}{81}\left(\tfrac{(5\,\zeta_1^4-18\,\zeta_1^2-8\,)^2}{(\,\zeta_1^2-1\,)^2}\right)^{\frac12} \le 0\,,\end{aligned}$$

gives

$$\frac{7\zeta_1^6 + 108\,\zeta_1^4 - 32\,\zeta_1^2 + 64}{\zeta_1^2\,\left(\,\zeta_1^2 - 1\,\right)} \le 0$$

which is false for $\zeta_1 \in (\,0\,,\,1\,)$.

C4. Note that the following inequality gives

$$\begin{aligned}|AB\,| - |\,C\,|\,(|\,B\,| - 4\,|\,A\,|) = {} & \tfrac{1}{81}\left(\tfrac{(\,5\zeta_1^4-18\zeta_1^2-8\,)^2}{(\,\zeta_1^2-1\,)^2}\right)^{\frac12} - \tfrac{2}{81}\left(\tfrac{(\,\zeta_1^2+8\,)^2}{\zeta_1^2}\right)^{\frac12}\left(\tfrac{(\,\zeta_1^2-4\,)^2}{\zeta_1^2}\right)^{\frac12} \\ & +\tfrac{2}{81}\left(\tfrac{(\,\zeta_1^2+8\,)^2}{\zeta_1^2}\right)^{\frac12}\left(\tfrac{\zeta_1^2(\,5\,\zeta_1^2+2\,)^2}{(\,\zeta_1^2-1\,)^2}\right)^{\frac12} \le 0\end{aligned}$$

and it is equivalent to $13\,\zeta_1^6 + 60\,\zeta_1^4 + 96\,\zeta_1^2 - 64 \;\le\; 0$. From there, we know that the inequality has a real root, that is

$$\begin{aligned}\zeta_1 = {} & \frac{1}{(\,13\,)\,\left(\,309+(\,13\,)\,(\,565\,)^{\frac12}\,\right)^{\frac16}}\left(\,(\,13\,)^{\frac12}\,(\,2\,)^{\frac16}\left(\,(\,22\,)^{\frac13}\left(\,309 + (\,13\,)\,(\,565\,)^{\frac12}\,\right)^{\frac23}\right.\right. \\ & \left.\left. - (\,10\,)\,(\,2\,)^{\frac23}\left(309 + (\,13\,)\,(\,565\,)^{\frac12}\right)^{\frac13} - 4\right)^{\frac12}\right) \approx 0.70443\,,\end{aligned}$$

and it satisfies $\zeta_1 \in (\,0\,,\,1\,)$. Then by the equation (27) , we have

$$|\,\gamma_1\gamma_3 - \gamma_2^2\,| \le \frac{1}{8}\zeta_1\left(1 - \zeta_1^2\right)\,\left(\,-\,|\,A\,| + |\,B\,| + |\,C\,|\,\right) + \frac{1}{36}$$

$$= \psi_2(\zeta_1) + \frac{1}{36},$$

where $\psi_2(\zeta_1) = \frac{1}{144}\zeta_1\left(1-\zeta_1^2\right)\left(\frac{\zeta_1\left(5\zeta_1^2+2\right)}{\left(\zeta_1^2-1\right)} - \frac{4\left(\zeta_1^2-4\right)}{\zeta_1} - \frac{2\left(\zeta_1^2+8\right)}{\zeta_1}\right)$, and then gives

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| \leq \psi_2(0.70443) + \frac{1}{36} \approx 0.05364.$$

This concludes the proof. □

2. Conclusions

By summarising the inequality in (26), it follows from the portion A until C, the equality for the function $f \in A$ given by the equation (13), where $p \in P$ and has the form of (7) with $\zeta_1 \approx 0.70443$, $\zeta_2 = \frac{1}{9} \approx 0.1111$ and $\zeta_3 = 1$ that gives

$$p(z) = \frac{1 + (0.7827)\, z + (0.1111)\, z^2}{1 - (0.6262)\, z - (0.1111)\, z^2}, \quad z \in D.$$

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