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# OPTIMAL MANAGEMENT OF A PREY-PREDATOR SYSTEM IN A POLLUTED ENVIRONMENT WITH EFFORT SHARED BETWEEN POLLUTION REDUCTION AND HARVESTING

S. D. ZAWKA[1*], SRINIVASU P. D. N.[2], §

Abstract. This paper is concerned with the optimal management of a prey-predator system in a polluted environment with effort allocated between pollution reduction and prey harvesting. The growth rates of species and, consequently, the economic gain from the resources are impacted by the pollutants released from external sources (such as industrial wastes). We assume that both the prey and predator populations have economic value. While revenue due to the prey comes from it's harvesting, it is from tourism that the revenue due to the predator is realised. The aim is to determine the optimal allocation of the total effort capacity between harvesting and pollution reduction to maximize the revenue. First, we study the qualitative behavior of the dynamical system and its dependence on the control parameter viz., effort, followed by an application of Pontryagin's maximum principle to solve an optimal harvesting problem. The results indicate that the system has three possible equilibrium solutions whose existence and stability heavily depend on the level of effort allocation made between harvesting and pollution reduction. It underlines that the effort allocation plays vital role in determining the eventual state of the system viz., permanence or extinction of the species. The result also shows that the optimal harvesting policy under pollution reduction efforts (as compared to one under no pollution reduction) calls for employing a lower level of optimal effort, resulting in higher resource stock levels and increased income. On the other hand, when the stock benefit of the predator rises, the optimal harvest effort falls, leading to increased stock levels in both the species.

Keywords: Prey-predator, pollution, optimal harvesting, stock benefit, pollution-reduction.

AMS Subject Classification: 92D25, 34D23, 37N25, 34H05.

## 1. Introduction

Environmental pollution has become one of the key influencing variables that should be taken into consideration when managing renewable resources. Pollutants released from external sources (such as industrial wastes, sewage, agricultural runoff etc.) are well

[1] Department of Mathematics, Arba Minch University, Arba Minch, Ethiopia.
e-mail: simondereke@gmail.com; ORCID: https://orcid.org/ 0000-0002-8814-5516.
[*] Corresponding author.
[2] Department of Mathematics, Andhra University, Visakhapatnam-530003, Andhra Pradesh, India.
e-mail: pdnsrini@gmail.com; ORCID: https://orcid.org/0000-0002-4679-0453

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known to threaten the survival of exploited ecologically-interdependent populations (such as prey and predator populations). The simultaneous effects of pollution, exploitation, and interdependence make such stocks more vulnerable to depletion, cause a significant decline in the yield, and may end up with extinction of the species. Therefore, optimal management of such resource stocks with pollution control seems to be crucial to ensure the persistence and productivity of the ecosystem.

Voluminous literature is devoted to presenting the effect of pollutants on a single species as well as ecologically-interdependent populations [10, 12, 13, 14, 15, 16, 17, 19, 28, 29]. A considerable number of studies present the simultaneous effects of exploitation and pollutants on the prey-predator system [8, 9, 18, 20, 22, 23, 34]. In particular, the work presented in Das et.al [8] deals with the prey-predator dynamics in the presence of pollutants (released from some external sources such as industrial wastes) and nonselective harvesting.

Despite the threats to the survival of the species and the negative effect on the economic benefits from the resource, studies on the optimal management of prey-predator in a polluted environment rarely considered pollution control as an alternative mechanism to enhance the productivity and hence the economic benefit from such stocks. The authors in [21, 25, 30, 31, 35, 32] discussed the dynamics of exploited single and competing species with pollution control activities. While the studies in [25, 30, 31] deal with the optimal exploitation of single-species populations in a polluted environment with various pollution control mechanisms, the most recent contribution in [35] focuses on the coexistence of harvested two-species competition systems in a polluted environment with pollution control. To the best of our knowledge, no other literature seems to exist that deals with the optimal management of a harvested prey-predator system with effort allocated towards pollution control.

This study examines optimal management of a harvested prey-predator system in a polluted environment. Here, we consider bifurcating the overall effort capacity so that a portion of it is utilised for pollution reduction and the remaining is utilised for harvesting and study the consequences of such bifurcated allocation of efforts on the system dynamics. Ultimately, we derive an optimal division of the effort capacity that maximises revenue to the sole owner. In this work, we assume that only prey is harvested and that the predator's stock has monetary worth in its native habitat. In other words, the owner of the resource has monitory gains from harvesting of the prey population as well as from the presence of the predator. The latter indicates the income derived from the predator's stock in its natural environment, such as through tourism [25, 32]. In the light of aforementioned factors, we investigate the existence of distinct equilibria and dependence of their stability on division of the effort. This is followed by solving an optimal harvesting problem to maximize the net economic revenue from optimally allocating effort towards pollution reduction and prey harvesting. The considered optimization problem takes into consideration the stock benefit to the sole owner from the presence of the predator.

The paper is organized as follows. A coupled prey-predator equations that characterize their dynamics in the presence of prey harvesting and pollution reduction activity followed by formulation of an appropriate optimal harvesting problem are presented in the section 2. The analysis of steady-state equilibria, including their dependency on effort as a control parameter and maximum sustainable yield, are addressed in Section 3. The considered optimal harvest problem is solved in section 4, followed by a presentation of numerical simulations in section 5, and concluding remarks in section 6.

Table 1. Parameters associated with the model and their descriptions.

| Symbol | Description |
|---|---|
| $r$ | Intrinsic growth rate of prey. |
| $K$ | Environmental carrying capacity of prey |
| $x_0$ | Initial biomass of prey |
| $y_0$ | Initial biomass of predator |
| $b$ | The rate at which the prey is consumed by predator per unit time |
| $\alpha_1$ | The coefficient of toxicity to the prey |
| $\alpha_2$ | The coefficient of toxicity to the predator |
| $\beta$ | Conversion factor |
| $c$ | The birth rate of predator in the presence of prey population |
| $\eta$ | Natural death rate of predator |
| $q$ | Catchability coefficient |
| $\alpha$ | Conversion factor |
| $E^{max}$ | Total effort capacity |
| $\tau$ | Price per unit harvest |
| $\delta$ | Discount factor |
| $\rho$ | Maximum stock benefit |
| $\sigma$ | Half saturation constant |

## 2. Mathematical Problem

Consider a prey-predator system being managed by a sole owner in a polluted environment. The growth rates of both prey and predator species are impacted by external pollutants (such industrial wastes). Das et.al [8] studied a mathematical model of the prey-predator system in the presence of environmental pollution and nonselective harvesting, given by

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - bxy - \alpha_1 x^3 - q_1 Ex, \; x(0) = x_0 > 0$$
$$\dot{y} = cxy - \alpha_2 y^2 - q_2 Ey - \eta y, \; y(0) = y_0 > 0,$$

where $x(t)$ and $y(t)$ represent the prey and predator populations at time $t$, respectively.

In the present study, it is assumed that only the prey population is harvested, whereas the predator population contributes to the revenue through its presence in its natural environment. A shark, for example, can be considered as a predator and smaller fish as its prey. A recent study found that sharks are an integral part of international tourism and are more valuable in the oceans (via tourism) than on the market (via harvesting) [33]. Therefore, it seems reasonable to associate harvesting with prey alone and linking predator with with its tourism value. In order to improve the rates of growth of the prey and predator and to increase the revenue from the harvesting activity as well as from the stock of the predator through tourism, the sole owner, who is experiencing a decline in the revenue from the resource due to the presence of pollution, wishes to allocate a portion of the total effort capacity towards pollution reduction rather than using the entire effort for harvesting alone.

Let $E^{max}$ (which is measured in monetary units) denote the total effort capacity which can be utilized for harvesting and reduction of pollution per unit time $t$. Suppose $E$ and $E^{max} - E$ represent the effort allocation between prey harvesting and pollution reduction, respectively. Following [8], the dynamics of the prey-predator system under the aforesaid allocation of effort towards prey harvesting ($E$) and pollution reduction ($E^{max} - E$) can be represented as

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - bxy - \alpha_1 x^3 + \alpha_1\beta(E^{max} - E)x^3 - q\alpha Ex,\ x(0) = x_0 > 0$$
$$\dot{y} = cxy - \alpha_2 y^2 + \alpha_2\beta(E^{max} - E)y^2 - \eta y,\ y(0) = y_0 > 0.$$

In the above equations, the terms $\alpha_1\beta(E^{max} - E)x^3$ and $\alpha_2\beta(E^{max} - E)y^2$, respectively represent the improvement in the growth rates of the prey and predator as a result of allocating $(E^{max} - E)$ effort capacity towards pollution reduction. It is assumed that $0 < \beta(E^{max} - E) < 1$ for positive constant $\beta$, and hence

$$0 < [1 - \beta(E^{max} - E)]\alpha_1 \leq \alpha_1 \text{ and } 0 < [1 - \beta(\mathrm{E^{max}} - \mathrm{E})]\alpha_2 \leq \alpha_2.$$

$\alpha E$ represents the harvest effort (measured in units of vessels where $\alpha$ is the conversion factor). For biological reasons, all parameters and constants in the considered model are assumed to be positive, and their description is given in Table 1.

It is worth noting a few points regarding the decay terms ($-\alpha_1 x^3$ and $-\alpha_2 y^2$) that are present in the mathematical model under consideration. The presence of pollutants in the environment is assumed to affect the growth rates of both prey and predator. The prey is directly impacted, while the predator which consumes the infected prey is indirectly impacted. As a result, the effects on the two species will differ, demanding careful consideration of the appropriate decay terms in the model. Numerous authors have represented the growth responses of prey and predator towards pollution in various ways. According to [2, 7, 8, 22, 27], a cubic term ($-ux^3$) on the prey and a quadratic term ($-vy^2$) on predator were taken into consideration as done in this article. The authors in [23, 34] considered a quadratic term ($-ux^2$) on the prey and a linear term ($-vy$) on the predator. Some studies consider a linear term on both the prey and predator [11]. The primary justification for considering the said cubic term in the prey equation and the square term in the predator equation is as follows. The pollution in the environment has direct and strong influence on the growth rate of prey, whereas it influences the predator's growth rate in an indirect manner. In other words, the impact is more severe on the prey than on the predator with severity being represented by the order of the considered term. This becomes evident from the computation of the first and second variations of the decay terms with respect to the respective population density. In the case of prey, they are $\frac{d(\alpha_1 x^3)}{dx} = 3\alpha_1 x^2 > 0$ and $\frac{d^2(\alpha_1 x^3)}{dx^2} = 6\alpha_1 x > 0$, whereas for the predator, they are $\frac{d(\alpha_2 y^2)}{dy} = 2\alpha_2 y > 0$, $\frac{d^2(\alpha_2 y^2)}{dy^2} = 2\alpha_2 > 0$. The order of these variations indicates the strength of the impact of pollution with changes in population density.

Now, consider the benefit to the owner from the resource. The owner has twofold benefit from the resource: the harvesting benefit from the prey and the stock benefit from the predator. The latter one stands for the income from the resource stock (predator) in its natural place such as through tourism [25, 32]. Although the predator is not harvested, prey harvesting indirectly affects the predator. Thus, it is crucial to ensure a reasonable level of predator population in the environment to reap the stock benefit besides the prey harvesting benefit. Therefore, to get the maximum possible revenue from the resource, we need to determine the optimal allocation of the available effort capacity between prey harvesting and pollution reduction as the latter helps in improving population levels of the prey and the predator. First, let us consider the harvesting benefit of the prey population. The gross harvesting benefit (denoted by $B(x, E)$) is given by

$$B(x, E) = \tau q\alpha Ex,$$

where $\tau$ is the price per unit catch. Next, the stock benefit to the owner due to presence of the predator (denoted by $S(y)$) is given by

$$S(y) = \frac{\rho y}{\sigma + y},$$

where $\rho > 0$ and $\sigma > 0$ represent the maximum stock benefit and half-saturation constant, respectively. Clearly, the function $S$ has the following properties:

$$S \in \mathcal{C}^2, \ S' > 0, \ S'' < 0.$$

Hence, the instantaneous net revenue from harvesting and the stock benefit (denoted by $R(x, y, E)$) becomes

$$R(x, y, E) = B(x, E) + S(y) - E^{max}.$$

Therefore, the present value of the total net revenues (denoted by $PV$) is

$$PV = \int_0^\infty e^{-\delta t} R(x, y, E)dt.$$

The aim is to determine the proper allocation of the total effort capacity ($E^{max}$) between harvesting ($E$) and pollution reduction ($E^{max} - E$) so that the net total revenue gets maximized subject to the given dynamic constraints on the stock. Formally expressed, the problem is as follows:

$$\max_{E \in [0, E^{max}]} \int_0^\infty e^{-\delta t} \left( \tau \alpha q E x + \frac{\rho y}{\sigma + y} - E^{max} \right) dt \tag{1a}$$

subject to (1b)

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - bxy - \alpha_1 x^3 + \alpha_1 \beta (E^{max} - E)x^3 - q\alpha E x, \ x(0) = x_0 > 0 \tag{1c}$$

$$\dot{y} = cxy - \alpha_2 y^2 + \alpha_2 \beta (E^{max} - E)y^2 - \eta y, \ y(0) = y_0 > 0 \tag{1d}$$

$$x(\infty), \ y(\infty) \text{ free,} \tag{1e}$$

where $\delta$ is the discount factor. The above problem is an optimal control problem in two state variables ($x$, $y$) and one control variable ($E$). Solving problem (1) is to finding out the optimal allocation of the total effort capacity between harvesting and pollution reduction such that the integral in (1a) is as large as possible.

If no action is taken to reduce pollution, the aforementioned optimal control problem takes the following form:

$$\max_{E \in [0, E^{max}]} \int_0^\infty e^{-\delta t} \left( (\tau \alpha q x - 1)E + \frac{\rho y}{\sigma + y} \right) dt \tag{2a}$$

subject to (2b)

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - bxy - \alpha_1 x^3 - q\alpha E x, \ x(0) > 0 \tag{2c}$$

$$\dot{y} = cxy - \alpha_2 y^2 - \eta y, \ y(0) > 0 \tag{2d}$$

$$x(\infty), \ y(\infty) \text{ free.} \tag{2e}$$

This is figuring out the optimal harvesting effort ($E_o$) so that the integral in (2a) is at its maximum given the constraints.

Before proceeding forward to solve the above optimal harvest problem, we wish to initially study the dynamical behavior of the system (1c)-(1d) and its dependence on the control parameter $E$. The following lemma discusses the existence, positivity and boundedness of solutions for the system under consideration. The proof is quite simple, and hence omitted (ref. [24]).

**Lemma 2.1.** *The initial value problem* (1c)-(1d) *admits a unique solution* $(x(t), y(t))$ *which is positive for all* $t \geq 0$*. Moreover, the solutions are uniformly bounded.*

## 3. Steady-state Analysis

In this section we shall study the existence and stability behaviour of equilibrium solutions of (1c)-(1d). To reduce the number of parameters and simplify the analysis we non-dimensionalise the system (1c)-(1d). The transformations:

$$X = \frac{x}{K},\ Y = \frac{b}{r}y,\ T = rt \tag{3}$$

reduce the system (1c)-(1d) to the equivalent form

$$\begin{aligned} \frac{dX}{dT} &= X\big(1 - X - Y - \overline{\phi}X^2 - \overline{q}E\big) \\ \frac{dY}{dT} &= Y\big(\overline{c}X - \overline{\psi}Y - \overline{\eta}\big), \end{aligned} \tag{4}$$

where

$$\overline{\phi} = \frac{\overline{\alpha}_1 K^2}{r},\ \overline{\psi} = \frac{\overline{\alpha}_2}{b},\ \overline{q} = \frac{q\alpha}{r},\ \overline{c} = \frac{cK}{r},\ \overline{\eta} = \frac{\eta}{r}$$

with

$$\overline{\alpha}_1 = \alpha_1[1 - \beta(E^{max} - E)],\ \overline{\alpha}_2 = \alpha_2[1 - \beta(E^{max} - E)]. \tag{5}$$

Clearly, the isoclines of (4) are

$$Y = 1 - X - \overline{\phi}X^2 - \overline{q}E,\ \text{and Y} = \frac{\overline{\text{c}}\text{X} - \overline{\eta}}{\overline{\psi}}$$

and the feasible equilibrium solutions of (4) are given by $(0, 0)$ - the trivial equilibrium point, $(\overline{X}, 0)$ - the axial equilibrium point, $(X^*, Y^*)$ - the interior equilibrium point, where

$$\begin{aligned} \overline{X} = \overline{X}(E) &= \frac{-1 + \sqrt{1 + 4\overline{\phi}(1 - \overline{q}E)}}{2\overline{\phi}} \\ X^* = X^*(E) &= \frac{-(1 + \frac{\overline{c}}{\overline{\psi}}) + \sqrt{(1 + \frac{\overline{c}}{\overline{\psi}})^2 + 4\overline{\phi}(1 - \overline{q}E + \frac{\overline{\eta}}{\overline{\psi}})}}{2\overline{\phi}} \\ Y^* = Y^*(E) &= \frac{\overline{c}X^*(E) - \overline{\eta}}{\overline{\psi}}. \end{aligned}$$

From the above expressions we observe that signs of $1 - \overline{q}E, 1 - \overline{q}E + \frac{\overline{\eta}}{\overline{\psi}}$ and $\overline{c}X^*(E) - \overline{\eta}$ play crucial role in determining the existence of foresaid equilibrium solutions. Let $\bar{E}$ and $E^*$, respectively, stand for solutions of

$$1 - \overline{q}E = 0 \tag{6}$$

and

$$\overline{c}X^*(E) - \overline{\eta} = 0. \tag{7}$$

It can be observed that $\overline{c}\overline{X}(E) - \overline{\eta} < 0$ for $E \in (E^*, \bar{E})$, $\overline{X} = X^*$ whenever $Y^* = 0$ i.e., when $E = E^*$. We have the following result regarding existence and local stability of various equilibrium solutions for the system (4).

**Theorem 3.1** (**Existence and Local Stability**).
*(a) The system* (4) *always admits the trivial equilibrium* $(0,0)$ *and it is asymptotically stable as long as* $E > \bar{E}$.
*(b) The axial equilibrium point* $(\overline{X},0)$ *comes into existence when* $E < \bar{E}$. *Emergence of axial equilibrium point destabilises the trivial equilibrium point and* $(\overline{X},0)$ *is asymptotically stable as long as* $E \in (E^*, \bar{E})$.
*(c) The interior equilibrium point* $(X^*, Y^*)$ *comes into existence when* $E \in [0, E^*)$. *Emergence of interior equilibrium point destabilises the axial equilibrium and the interior equilibrium point is always asymptotically stable.*

*Proof.* The Jacobian matrix associated with the system (4) is

$$J(X,Y) = \begin{bmatrix} 1 - 2X - Y - 3\overline{\phi}X^2 - \overline{q}E & -X \\ \overline{c}Y & \overline{c}X - 2\overline{\psi}Y - \overline{\eta} \end{bmatrix}.$$

$J(X,Y)$ evaluated at $(0,0)$ gives

$$J(0,0) = \begin{bmatrix} 1 - \overline{q}E & 0 \\ 0 & -\overline{\eta} \end{bmatrix}.$$

Clearly, the trivial equilibrium is locally asymptotically stable whenever $1 - \overline{q}E < 0$, and hence $(a)$.
$J(X,Y)$ evaluated at the axial equilirbium $(\overline{X},0)$ gives

$$J(\overline{X},0) = \begin{bmatrix} -(\overline{X} + 2\overline{\phi X}^2) & -\overline{X} \\ 0 & \overline{c}\overline{X} - \overline{\eta} \end{bmatrix}.$$

Clearly, $(\overline{X},0)$ is locally asymptotically stable since $\overline{c}\overline{X} - \overline{\eta} < 0$ and hence $(b)$.
$J(X,Y)$ evaluated at the interior equilibrium $(X^*, Y^*)$ gives

$$J(X^*,Y^*) = \begin{bmatrix} -(X^* + 2\overline{\phi}X^{*2}) & -X^* \\ \overline{c}Y^* & -\overline{\psi}Y^* \end{bmatrix}.$$

The trace and determinant of matrix $J(X^*,Y^*)$ are given by

$$\text{trace}J(X^*,Y^*) = -(X^* + 2\overline{\phi}X^{*2}) - \overline{\psi}Y^*$$
$$\det J(X^*,X^*) = \left(X^* + 2\overline{\phi}X^{*2}\right)\overline{\psi}Y^* + \overline{c}X^*Y^*.$$

Clearly, $\text{trace}J(X^*,Y^*) < 0$ and $\det J(X^*,Y^*) > 0$. Consequently, both the eigenvalues of $J(X^*,Y^*)$ are negative, and hence $(c)$. □

The following result establishes the global stability behaviour of the equilibria.

**Theorem 3.2** (**Global Stability**).
*(a) The interior equilibrium of the system* (4) *is globally stable whenever it is locally stable.*
*(b) The axial equilibrium of the system* (4) *is globally stable whenever it is locally stable.*
*(c) The trivial equilibrium of the system* (4) *is globally stable whenever it is locally stable.*

*Proof.* We know that, whenever the system (4) admits interior equilibrium then it is unique and locally asymptotically stable. To establish its global asymptotic stability, it suffices to verify the Bendixson – Dulac's criterion which ensures that there is no closed orbit in the positive quadrant of the $XY$-plane. To verify this criterion, let us consider the functions $F(X,Y)$, $G(X,Y)$, and $B(X,Y)$ as follows:

$$F(X,Y) = X(1 - X - Y - \overline{\phi}X^2 - \overline{q}E)$$
$$G(X,Y) = Y(\overline{c}X - \overline{\psi}Y - \overline{\eta})$$

and

$$B(X,Y) = \frac{1}{XY}.$$

For the above choice for $B$ we have,

$$\nabla(BF, BG) = -\frac{1}{Y} - 2\frac{\overline{\phi}X}{Y} - \frac{\overline{\psi}}{X} < 0$$

for all $(X,Y)$ inside the positive quadrant of the $XY$-plane. Hence no closed orbit exists inside the positive quadrant of the $XY$-plane. This proves $(a)$. The proofs for items (b) and (c) are similar, and hence the theorem. □

Parallel results for the original system (1c)-(1d) can be obtained using inverse mapping for the transformations defined in (3), thus moving back to the original variables $x, y, t$. Hence forth we shall discuss about the system (1c)-(1d).

3.1. **Maximum Sustainable Yield.** Let $(x^*(E), y^*(E))$ be the unique interior equilibrium of the system (1c)-(1d) for each $E$ in $[0, E^*)$. For the given effort allocation $(E, E^{max} - E)$, the sustainable yield $(SY)$ is given by

$$SY(E) = q\alpha E x^*(E). \tag{8}$$

We observe that increasing the harvest effort $(E)$ in (8) may not improve the yield in general. The reason is, an increase in the harvest effort causes a decrease in the depollution effort $(E^{max} - E)$. This may cause a reduction in the resource stock, and hence possibly the yield. Thus, it is quite important to determine the proper effort allocation between harvesting and pollution reduction in order to maximize the yield.

The maximum sustainable yield (MSY) is nothing but the maximum of the function $SY(E)$ on a closed and bounded interval $[0, E^{max}]$. Since $SY(E)$ is continuous on $[0, E^{max}]$, it certainly attains its maximum at $E_{\text{MSY}}$ giving rise to MSY. Here, the maximum may occur either at the boundary (i.e., $E_{\text{MSY}} = E^{max}$) or in the interior $0 < E_{\text{MSY}} < E^{max}$. In the latter case, $E_{\text{MSY}}$ must satisfy the equation $\frac{dSY}{dE} = 0$ i.e.,

$$x^*(E) + \frac{dx^*(E)}{dE} = 0. \tag{9}$$

## 4. Optimal harvest Strategy

In this section we shall consider the optimal harvest problem formulated in section 1 and solve it by using Pontryagyn's Maximum Principle and hence derive optimal harvest strategies for the sole owner to maximize his revenues. Recall the optimal harvest problem (1):

$$\max_{E \in [0, E^{max}]} \int_0^\infty e^{-\delta t} \left( \tau \alpha q E x + \frac{\rho y}{\sigma + y} - E^{max} \right) dt$$

subject to

$$\dot{x} = rx\big(1 - \frac{x}{K}\big) - bxy - \overline{\alpha}_1 x^3 - q\alpha E x, \;\; x(0) = x_0 > 0$$
$$\dot{y} = cxy - \overline{\alpha}_2 y^2 - \eta y, \;\; y(0) = y_0 > 0$$
$$x(\infty), \;\; y(\infty) \text{ free},$$

where $\overline{\alpha}_1$ and $\overline{\alpha}_2$ are given in (5). By the maximum principle [6, 26], the Hamiltonian $H(x, y, E, \mu_1, \mu_2, t)$ associated with the problem (1) is

$$H(x, y, E, \mu_1, \mu_2, t) = e^{-\delta t}\left(\tau\alpha q E x + \frac{\rho y}{\sigma + y} - E^{max}\right) + \mu_1\left[rx\left(1 - \frac{x}{K}\right) - bxy - \overline{\alpha}_1 x^3 - q\alpha E x\right] + \mu_2\left[cxy - \overline{\alpha}_2 y^2 - \eta y\right],$$

and the associated adjoint differential equations are

$$\dot{\mu_1} = -\, e^{-\delta t}(\tau q E) - \mu_1\left[r(1 - \frac{2x}{K}) - by - 3\overline{\alpha}_1 x^2 - q\alpha E\right] - \mu_2 cy,$$
$$\dot{\mu_2} = -\, e^{-\delta t}\left(\frac{\sigma\rho}{(\sigma + y)^2}\right) + \mu_1 bx - \mu_2\left[cx - \eta - 2\overline{\alpha}_2 y\right],$$

where $\mu_1$ and $\mu_2$ are the adjoint variables. Because of the presence of the term $e^{-\delta t}$ we need the following transformation:

$$\lambda_i(t) = \mu_i(t) e^{\delta t},\ i = 1, 2 \text{ and } \mathcal{H} = He^{\delta t},$$

where $\mathcal{H}$ is known as the current value Hamiltonian defined by

$$\mathcal{H}(x, y, E, \lambda_1, \lambda_2) = \left[\tau q\alpha E x + \frac{\rho y}{\sigma + y} - E^{max}\right] + \lambda_1\left[rx\left(1 - \frac{x}{K}\right) - bxy - \overline{\alpha}_1 x^3 - q\alpha E x\right] + \lambda_2\left[cxy - \overline{\alpha}_2 y^2 - \eta y\right]. \tag{10}$$

$\lambda_1$, $\lambda_2$ are known as the current value adjoint variables satisfying the following differential equations:

$$\dot{\lambda}_1 = \delta\lambda_1 - \frac{\partial\mathcal{H}}{\partial x} \text{ and } \dot{\lambda}_2 = \delta\lambda_2 - \frac{\partial\mathcal{H}}{\partial \text{y}}$$

i.e.,

$$\dot{\lambda}_1 = \delta\lambda_1 - \tau q\alpha E - \lambda_1\left[r(1 - \frac{2x}{K}) - by - 3\overline{\alpha}_1 x^2 - q\alpha E\right] - \lambda_2 cy \tag{11a}$$
$$\dot{\lambda}_2 = \delta\lambda_2 - \frac{\sigma\rho}{(\sigma + y)^2} + \lambda_1 bx - \lambda_2\left[cx - \eta - 2\overline{\alpha}_2 y\right]. \tag{11b}$$

Since (1) is linear in the control variable $E$, the optimal control is a combination of bang-bang and singular controls [5]. First, we wish to determine the optimal steady state (singular) solution. Now, differentiating the current value Hamiltonian (10) (partially) with respect to $E$ gives

$$\mathcal{H}_E = \tau q\alpha x - \lambda_1(\alpha_1\beta x^3 + q\alpha x) - \lambda_2\alpha_2\beta y^2,$$

and hence the switching function ($s(t)$) is given by

$$s(t) = \tau\alpha q x - \lambda_1(\alpha_1\beta x^3 + q\alpha x) - \lambda_2\alpha_2\beta y^2. \tag{12}$$

In the case of singular solution, we have $s(t) = 0$ i.e.,

$$\tau\alpha q x - \lambda_1(\alpha_1\beta x^3 + q\alpha x) - \lambda_2\alpha_2\beta y^2 = 0. \tag{13}$$

Now, the unique interior steady state of the four dimensional dynamical systems (1c), (1d), (11a) and (11b) is

$$\begin{aligned} x^*(E) &= \frac{-\left(\frac{r}{K}+\frac{bc}{\overline{\alpha}_2}\right)+\sqrt{\left(\frac{r}{K}+\frac{bc}{\overline{\alpha}_2}\right)^2+4\overline{\alpha}_1(r-q\alpha E+\frac{b\eta}{\overline{\alpha}_2})}}{2\overline{\alpha}_1} \\ y^*(E) &= \frac{cx^*-\eta}{\overline{\alpha}_2} \\ \lambda_1^*(E) &= \frac{\alpha\tau qE(\delta+\overline{\alpha}_2y^*)+\frac{c\sigma\rho y^*}{(\sigma+y^*)^2}}{(\delta+\frac{r}{K}x^*+2\overline{\alpha}_1(x^*)^2)(\delta+\overline{\alpha}_2y^*)+bcx^*y^*} \\ \lambda_2^*(E) &= \frac{1}{cy^*}\left[(\delta+\frac{r}{K}x^*+2\overline{\alpha}_1(x^*)^2)\lambda_1^*-\tau q\alpha E\right]. \end{aligned} \tag{14}$$

Substituting $(x^*(E), y^*(E), \lambda_1^*(E), \lambda_2^*(E))$ into (13) gives the following equation (involving $E$ as the only variable)

$$\tau\alpha q x^* - \lambda_1^* x^*[\alpha_1\beta(x^*)^2+q\alpha] - \alpha_2\beta\lambda_2^*(y^*)^2 = 0. \tag{15}$$

Then, a solution $\widehat{E}$ of (15) satisfying $0 < \widehat{E} < E^{max}$ (together with part (c) of Theorem 3.1) is singular control to the given problem. If $\widehat{E}$ is unique, it becomes an optimal singular control, otherwise the optimum is one with the largest integral value along with the associated optimal solution $\big(x^*(\widehat{E}), y^*(\widehat{E})\big)$ [3, 4].

With the optimal solution $\big(x^*(\widehat{E}), y^*(\widehat{E})\big)$ identified, it remains to reach this solution optimally from the given initial state $(x_0, y_0)$. Since the problem under consideration is linear in the control variable, the solution $(x^*(\widehat{E}), y^*(\widehat{E}))$ can be reached by a bang-bang control [26]. If we denote the bang-bang control by $\widetilde{E}(t)$, then

$$\widetilde{E} = \begin{cases} 0, \text{ for } s(t) < 0 \\ E^{max}, \text{ for } s(t) > 0, \end{cases} \tag{16}$$

where $s(t)$ is the switching function given in (12). Let $T$ be the time taken to reach the optimal solution from the given initial state. Then, the optimal harvest strategy (denoted by $E_o(t)$) becomes

$$E_o(t) = \begin{cases} \widetilde{E}, \text{ for } 0 \le t < T \\ \widehat{E}, \text{ for } t \ge T. \end{cases} \tag{17}$$

If $\big(\widetilde{x}(t), \widetilde{y}(t)\big)$ denotes the optimal approach path from the initial state $(x_0, y_0)$ to the optimal solution $(x^*(\widehat{E}), y^*(\widehat{E}))$, and

$$(\widetilde{x}(t), \widetilde{y}(t)) = \begin{cases} (x_m(t), y_m(t)), \text{ for } E = 0 \\ (x_M(t), y_M(t)), \text{ for } E = E^{max}, \end{cases} \tag{18}$$

then the optimal stock path (denoted by $(x_o(t), y_o(t))$) is

$$(x_o(t), y_o(t)) = \begin{cases} (\widetilde{x}(t), \widetilde{y}(t)) \text{ for } 0 \le t \le T \\ (x(\widehat{E}), y(\widehat{E})) \text{ for } t \ge T. \end{cases} \tag{19}$$

Here $(x_m(t), y_m(t))$ and $(x_M(t), y_M(t))$ are the trajectories of dynamical system (1c)-(1d) for $E = 0$ and $E = E^{max}$, respectively, and their respective initial conditions.

Note that, if the optimal singular control $(\widehat{E})$ is employed right from the given initial state $(x_0, y_0)$, by the global stability behavior of $(x^*(\widehat{E}), y^*(\widehat{E}))$, the corresponding stock path (also known as suboptimal path) approaches the optimal solution asymptotically.

Table 2. Values of parameters and constants

| Symbol | Parameter Values | Unit |
|---|---|---|
| $r$ ($\delta$ ) | 0.85 (0.02) | 1/year |
| $K$ ($\sigma$) | $4.5 \times 10^4$ ($1 \times 10^4$) | ton |
| $x_0$ ($y_0$) | 1000 (200) | ton |
| $b$ ($c$) | $1 \times 10^{-5}$ ($8 \times 10^{-6}$) | 1/ton/year |
| $\alpha_1$ | $1 \times 10^{-9}$ | 1/ton$^2$/year |
| $\alpha_2$ | $6 \times 10^{-5}$ | 1/ton/year |
| $\beta$ | $5 \times 10^{-7}$ | 1/vessel |
| $\eta$ | $7 \times 10^{-2}$ | 1/year |
| $q$ | $5 \times 10^{-4}$ | 1/vessel/year |
| $\alpha$ | $1 \times 10^{-3}$ | vessel/US$ |
| $E^{max}$ | $1.8 \times 10^6$ | US$ |
| $\tau$ | $1 \times 10^4$ | US$/ton |
| $\rho$ | $5 \times 10^5$ | US$/year |

## 5. Numerical Simulations

This section uses numerical simulations to illustrate the study's important findings. The values given to parameters correspond to the potential values that can be found in a fishery [1, 5].

With the set of parameter values in Table 2, we have from (7), $E^* = 1.2578 \times 10^6$. Thus, for each $E \in [0, E^*)$ the system (1c)-(1d) has a unique interior equilibrium $(x^*(E), y^*(E))$ which is globally asymptotically stable. The equilibrium points are shown in Figure 1. The figure further highlights the maximum sustainable yield (MSY) and the corresponding harvest effort $E_{\text{MSY}} = 9.0289 \times 10^5$ (US$). The associated steady state is $(14361, 1357)$ (tons). In the figure one can also see the maximum(minimum) equilibrium stock levels for both prey and predator. Also, from (6) we have, $\bar{E} = \frac{1}{q} = 1.7 \times 10^6$. Thus, system (1c)-(1d) has a unique axial equilibrium $(\overline{x}(E), 0)$ for each $E \in [0, \bar{E})$ which is globally asymptotically stable only when $E \in (E^*, \bar{E})$. The stable axial equilibrium corresponding to $E = 1.5 \times 10^6$ is shown in Frame (b) of Figure 2. If we consider $E = 1.75 \times 10^6 > \bar{E}$, the trivial equilibrium $(0, 0)$ is globally asymptotically stable (see Frame (c) of Figure 2).

Consider the optimal harvesting problem given in (1). For the given set of parameter values in Table 2, and using (15), the optimal singular harvest effort is $\widehat{E} = 9.0732 \times 10^5$ (US$) ($E^{max} - \widehat{E}$ goes for pollution reduction) and the associated optimal steady state is $(14290, 1334)$ (tons). The stability of this equilibrium can be seen in Frame (a) of Figure 3. The corresponding sustainable yield ($SY(\widehat{E})$) is 6483 (tons). Note that this yield is less than MSY (6483.2) as it was expected. The optimal control, which is a combination of bang-bang and singular controls is given by

$$E_o(t) = \begin{cases} 0, \ \text{for } 0 \leq t < 13.7 \\ E^{max}, \ \text{for } 13.7 \leq t < 14.69 \\ \widehat{E}, \ \text{for } t \geq 14.69 \end{cases}$$

Graphs of the optimal control and the associated stock path are present in Figure 3. The figure also presents the suboptimal path. While the optimal approach path reaches the singular solution in 14.69 (years), it takes 150 (years) for the suboptimal path to enter a neighborhood with sufficiently small radius from the given initial state (see Frame (b) of Figure 3). Using (1a), the present value of the total net revenues on the time period

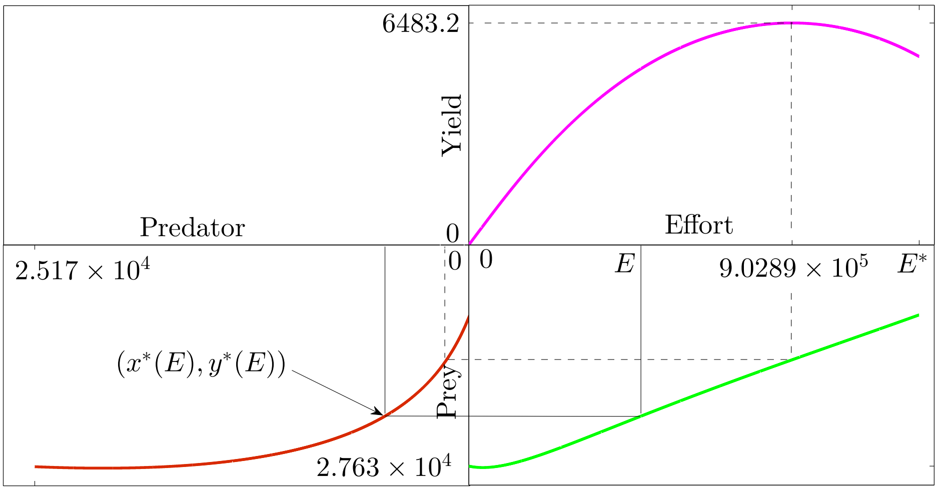



FIGURE 1. This four-quadrant figure depicts the interior equilibrium $(x^*(E), y^*(E))$ and the corresponding sustainable yield $SY(E)$ for each effort $E \in [0, E^*]$. For the given effort $E \in [0, E^*]$, the yield can be seen in quadrant I and the associated steady state in Quadrant III (via quadrant IV). We observe that MSY occurs at some critical effort level $(0 < E_{\text{MSY}} < E^{max})$. With increased harvesting effort $E$, the prey and predator stocks both decline. Particularly, the largest(smallest) prey stock occurs when $E = 0 (E = E^*)$. The largest (smallest) predator stock level is also visible. Remember that we only examined $0 \leq E \leq E^*$ for which the interior equilibrium existed while creating this graph.

$[0, 100]$ (years) is

$$PV = \int_0^{13.7} e^{-\delta t}\left(\frac{\rho y_m}{\sigma + y_m} - E^{max}\right)dt + \int_{13.7}^{14.69} e^{-\delta t}\left((\tau\alpha q x_M - 1)E^{max} + \frac{\rho y_M}{\sigma + y_M}\right)dt$$
$$+ \int_{14.69}^{100} e^{-\delta t}\left(\tau\alpha q\widehat{E}x^*(\widehat{E}) + \frac{\rho y^*(\widehat{E})}{\sigma + y^*(\widehat{E})} - E^{max}\right)dt = 2.533 \times 10^9 \text{ (in US\$)}$$

with the associated pollution reduction cost (PRC) given by,

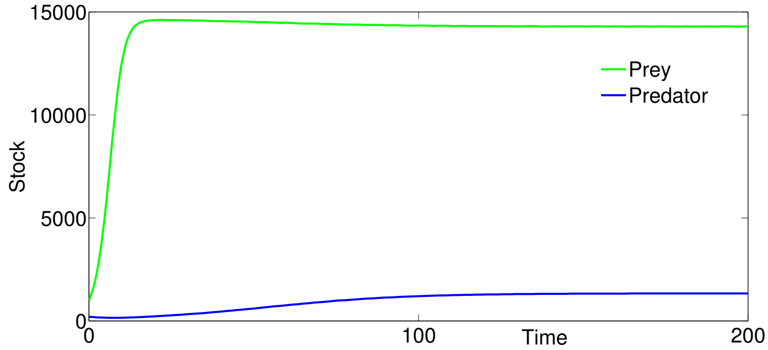
$$\text{PRC} = \int_0^{13.7} e^{-\delta t}(E^{max})dt + \int_{13.7}^{14.69} e^{-\delta t}(0)dt + \int_{14.69}^{100} e^{-\delta t}(E^{max} - \widehat{E})dt$$
$$= 4.895 \times 10^7 \text{ (in US\$)}.$$

The revenue along suboptimal solution is computed as follows. If $(x_{so}(t), y_{so}(t))$ denotes the suboptimal path, the present value for the first 100 years is given by

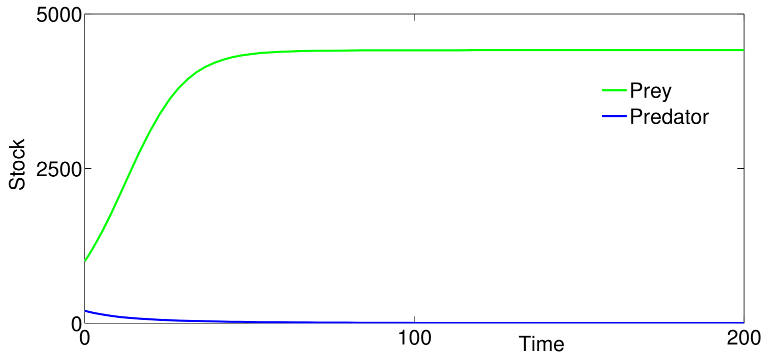
$$PV = \int_0^{100} e^{-\delta t}\left(\tau\alpha q\widehat{E}x_{so} + \frac{\rho y_{so}}{\sigma + y_{so}} - E^{max}\right)dt = 2.399 \times 10^9 \text{ US\$}.$$

We observe that the present value along the optimal solution is greater than that of the suboptimal solution in the given period, and this will be true for all future times. The present value curves for both optimal and suboptimal solutions are shown in Figure 4.

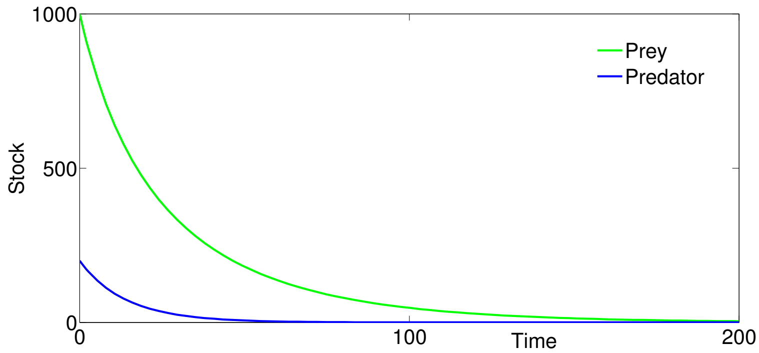
To see how the profit from the stock influences the optimal harvesting policy, let's look at the parameter values in Table 2 with the exception of $\rho$. The optimal singular effort $\widehat{E}$ and the associated stock levels for different value of $\rho$ are determined and are given in




(a) The interior equilibrium is stable for $E = \widehat{E}$



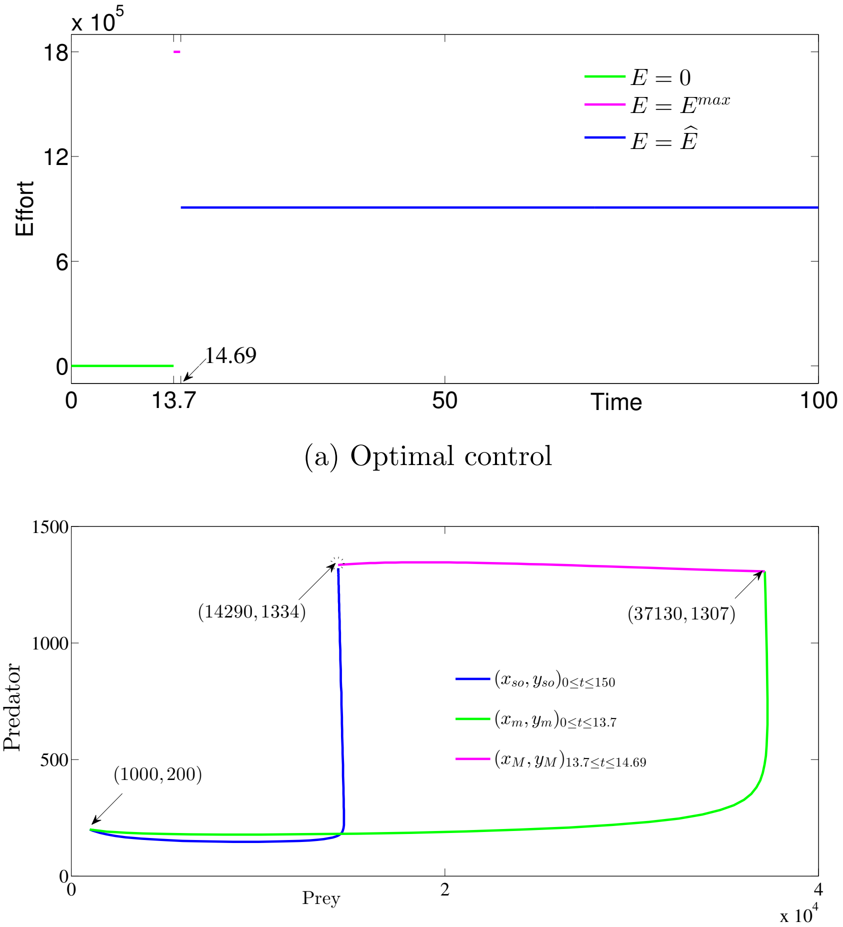

(b) The axial equilibrium is stable for $E = 1.5 \times 10^6$




(c) The trivial equilibrium is stable for $E = 1.75 \times 10^6$

FIGURE 2. The figure depicts the stability of the distinct equilibrium points of the system (1). Frames (a), (b) and (c) stand for the interior, axial and trivial equilibrium points, respectively (with all the parameter values in Table 2 are unchanged except for $E$).

Table 3. According to the table, we can see that as stock benefits rise, harvest efforts fall, leading to higher stock levels of both prey and predator. It emphasizes the need to harvest less when the benefit due to the stock (predator) is greater, which ensures availability of sufficient food to the predator.




(a) Optimal control

(b) Optimal approach path

FIGURE 3. This figure depicts the optimal control and associated optimal approach path. The optimal control ($E_o$) follows $E = 0$ for $0 \leq t < 13.7$, $E = E^{max}$ for $13.7 \leq t < 14.69$, and then $E = \widehat{E}$ for $t \geq 14.69$ (see Frame (a)). The optimal approach path (starting from the initial state $(1000, 200)$) first takes the trajectory $(x_m(t), y_m(t))$ for $0 \leq t \leq 13.7$, and then $(x_M(t), y_M(t))$ for $13.7 \leq t \leq 14.69$ to reach the optimal solution $(14290, 1334)$ (see Frame (b)). The suboptimal path $(x_{so}(t), y_{so}(t))$ takes $t = 150$ (years) to enter a neighborhood of $(14290, 1334)$ with sufficiently small radius.

Recall the optimal control problem (2), where pollution reduction was not taken into account. In this instance, the optimal solution can be constructed using the same methodology as that of (1). For the time period $[0, 100]$, discounted net income is given by

$$PV = \int_0^{12.75} e^{-\delta t}\left(\frac{\rho y_m}{\sigma + y_m}\right) dt + \int_{12.75}^{13.81} e^{-\delta t}\left((\tau\alpha q x_M - 1)E^{max} + \frac{\rho y_M}{\sigma + y_M}\right) dt$$
$$+ \int_{13.81}^{100} e^{-\delta t}\left([\tau\alpha q x^*(\widehat{E}) - 1]\widehat{E} + \frac{\rho y^*(\widehat{E})}{\sigma + y^*(\widehat{E})}\right) dt = 2.288 \times 10^9 \text{ (in US\$)},$$

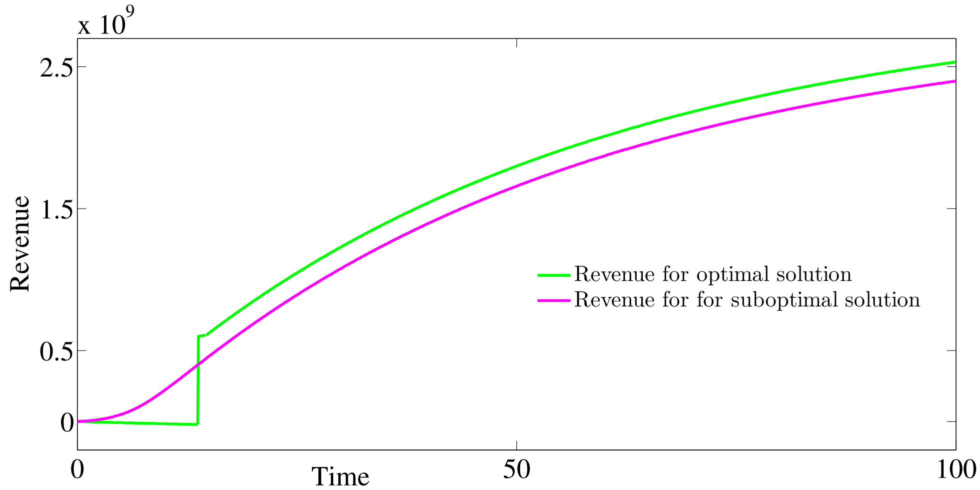



FIGURE 4. The figure shows the revenue curves for both the optimal and suboptimal solutions to (1). We notice that for a limited length of time (only), the revenue from the suboptimal solution predominates during the initial phase. However, after the two curves cross, the situation is reversed, and the revenue from the optimal solution takes over in all subsequent periods.

TABLE 3. The table presents the optimal harvest efforts and the associated optimal steady states for different values of $\rho$ (with all the other parameters in Table 2 are unchanged).

| SNO | $\rho$ | $\widehat{E}$ | $x^*(\widehat{E})$ | $y^*(\widehat{E})$ |
|---|---|---|---|---|
| 1 | $5 \times 10^4$ | $9.0815 \times 10^5$ | $1.4277 \times 10^4$ | 1330 |
| 2 | $5 \times 10^5$ | $9.0732 \times 10^5$ | $1.4290 \times 10^4$ | 1334 |
| 3 | $5 \times 10^6$ | $8.9887 \times 10^5$ | $1.4425 \times 10^4$ | 1377 |

where the optimal harvest strategy is

$$E_o(t) = \begin{cases} 0, \text{ for } 0 \leq t < 12.75 \\ E^{max}, \text{ for } 12.75 \leq t < 13.81 \\ \widehat{E} = 9.9664 \times 10^5, \text{ for } t \geq 13.81. \end{cases}$$

Clearly, sharing the total effort capacity in an optimal manner between harvesting and pollution reduction enhances net revenue. Figure 5 depicts the revenue curves for the two alternative scenarios, and Table 4 provides a summary of the optimal values for the two alternative scenarios. Note that the yield and revenue are higher than they would be in the case when there is no pollution reduction, despite the fact that less optimal harvesting effort is spent. It is because of the increased resource stock levels brought about by investments in pollution abatement.

## 6. CONCLUSION

In this work, we investigated the effects of stock benefits and pollution control efforts on a prey-predator system in a polluted environment. The growth rates of both species are impacted by pollutants from outside sources, which reduces the resource's value. In order

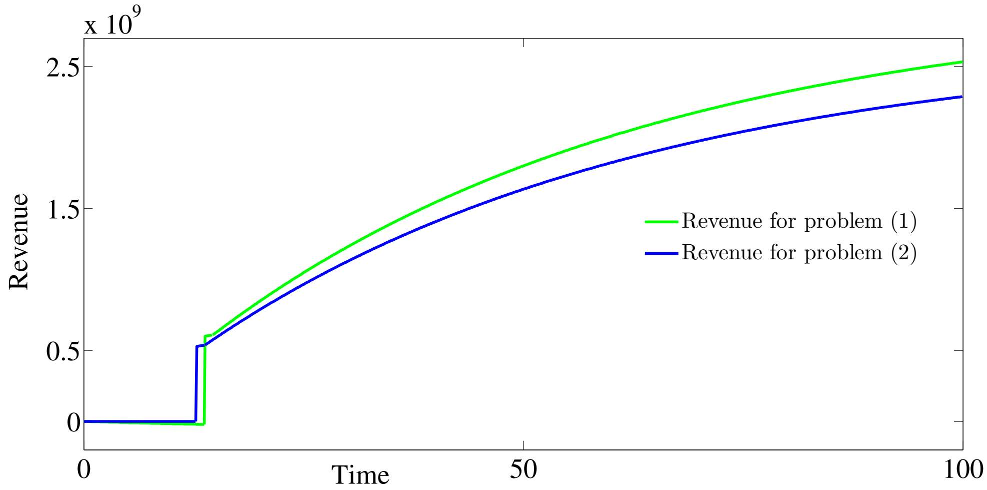



FIGURE 5. This figure presents the revenue curves for the optimal harvest problems (1) and (2). Observe that the revenue associated with (2) is superior during the initial period for a finite amount of time, and that once the curves cross, the situation is reversed, with the income associated with (1) dominating in all subsequent times.

TABLE 4. The table presents the optimal harvest effort ($\widehat{E}$), the associated optimal steady state ($x^*(\widehat{E}), y^*(\widehat{E})$), optimal sustainable yield ($h(\widehat{E})$), and the instantanuous net revenue $R(\widehat{E})$ for problems (1) and (2), with the numerical values in Table 2.

| SNO | $\widehat{E}$ | $x^*(\widehat{E})$ | $y^*(\widehat{E})$ | $h(\widehat{E})$ | $R(\widehat{E})$ |
|---|---|---|---|---|---|
| Problem (1) | $9.0732 \times 10^5$ | $1.429 \times 10^3$ | $1.334 \times 10^3$ | $6.483 \times 10^3$ | $6.3089 \times 10^7$ |
| Problem (2) | $9.9664 \times 10^5$ | $1.147 \times 10^3$ | 362 | $5.714 \times 10^3$ | $5.6159 \times 10^7$ |

to increase the productivity of the resource, we have thought about assigning a portion of the overall effort capacity to reducing pollution. We have investigated the existence and stability of the equilibrium solutions. The system has three equilibrium solutions, which we have identified as the trivial, axial, and interior equilibrium points. Interior and axial equilibria are stable for each harvesting effort below some critical effort levels $E^*$ and $\bar{E}$, respectively. Trivial equilibrium is stable for each harvest effort above $\bar{E}$, and it loses stability at the birth of the axial equilibrium point. The stocks of prey and predator decrease with the harvest effort ($E$) and increase with the depollution effort ($E^{max} - E$). At a certain critical effort level, the sustainable yield reaches its maximum (where a part of the effort capacity goes for pollution reduction).

In order to maximize the benefits of harvesting prey and the stock benefit of the predator(in its natural environment), we also investigated the optimal harvest problem on an infinite horizon and constructed the optimal harvest policy. When the stock benefit is more alluring, it is preferable to harvest less, according to the optimal strategy. We have addressed a comparison of the optimal values for two different scenarios by considering the optimal harvest problem in the absence of any efforts towards pollution reduction. We found that the optimal harvest strategy with pollution control calls for reduced harvesting effort, resulting in higher stock levels of both prey and predator. The sustainable yield and revenue are higher than they would be without pollution reduction, despite the lower harvest effort. It is a result of significantly increased resource stocks as a result of investments in pollution control. Therefore, taking into account the effort allocation towards pollution reduction may result in a clean environment with a higher resource stock and better revenue.

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**Simon Derkee Zawka** received his M.Sc. degree in 2010 from Addis Ababa University, Ethiopia, and his Ph.D. in 2018 from Andhra University, India. Currently, he is serving as assistant professor of Mathematics and chair of the department of Mathematics at Arba Minch University. His research interests are Dynamical Systems, mathematical biology and optimal control.

**Srinivasu P. D. N.** received his Master's degree in 1989 and Ph.D degree in 1992 from Sri Sathya Sai Institute of Higher Learning, Prasanthinilayam, India. At present, he is serving as a professor of Mathematics at Andhra University, Visakhapatnam, India. His current research interests are Dynamical Systems, Mathematical Biology, and Mathematical Bio-economics.