A Modified Reductive Perturbation Method as Applied to Nonlinear Ion-Acoustic Waves

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The basic equations describing the nonlinear ion-acoustic waves in a cold collisionless plasma, in the longwave limit, is re-examined through the use of a modified reductive perturbation method. Introducing the concept of a scale parameter and expanding the variables and the scale parameter into a power series of the smallness parameter $\epsilon$, a set of evolution equations is obtained for various order terms in the perturbation expansion. To illustrate the present derivation, a localized travelling wave solution is studied for the derived field equations and the result is compared with those of obtained by Sugimoto and Kakutani\(^3\) who introduced some slow scales, Kodama and Tanidti\(^4\) who employed the renormalization method and Malfllet and Wieers\(^6\) who employed the dressed solitary wave approach from the outset of their study.

**KEYWORDS:** reductive perturbation, ion-acoustic plasma, solitary waves

§1. **Introduction**

In plasma physics, the theory of one dimensional ion-acoustic waves is a typical topic in nonlinear waves. As is known, in a cold collisionless plasma the dynamical behavior of the ions is determined by the presence of electrons and, as a result, the ion-acoustic wave develop in the medium. Due to nonlinearity of the governing equations, in the longwave limit, one obtains the KdV equation for the lowest order term in the expansion, the solution of which may be described by solitons (Davidson\(^1\)). To study the higher order terms in the perturbation expansion, the reductive perturbation method has been introduced by use of the stretched time and space variables (Tanidti\(^2\)). However, in such an approach some secular terms appear which can be eliminated by introducing some slow scale variables (Sugimoto and Kakutani\(^3\)) or by a renormalization procedure of the velocity of the KdV soliton (Kodama and Tanidi\(^3\)). Nevertheless, this approach remains somewhat artificial, since there is no reasonable connection between the smallness parameters of the stretched variables and the one used in the perturbation expansion of the field variables. The choice of the former parameter is based on the linear wave analysis of the concerned problem and the wave number or the frequency is taken as the perturbation parameter (Washimi and Tanidti\(^5\)). On the other hand, at the lowest order, the amplitude and width of the wave are expressed in terms of the unknown perturbed velocity, which is also used as the smallness parameter. This causes some ambiguity over the correction terms. In order to remove this uncertainty, Malfllet and Wieers\(^6\) presented a dressed solitary wave approach, which is based on the assumption that the field variables admit localized travelling wave solution. Then, for the longwave limit, they expanded the field quantities into a power series of the wave number, which is assumed to be the only smallness parameter, and obtained the explicit solution for various order terms in the expansion. However, this approach can only be used when one studies progressive wave solution to the original nonlinear equations and it does not give any idea about the form of evolution equations governing various terms in the perturbation expansion.

In the present work, we introduced a modified reductive perturbation method in which the concept of scale parameter $g$ is presented. Then, expanding the field variables and the scale parameter into a power series of the smallness parameter $t$ the governing equations are obtained for various order terms in the perturbation expansion. By employing the hyperbolic tangent method, the localized travelling wave solution for the evolution equations is studied and the consistency of the present approach with those of obtained Sugimoto and Kakutani\(^3\) who introduced some slow scale variables, Kodama and Tanidti\(^4\) who employed the renormalization method presented in ref. 4 and Malfllet and Wieers\(^6\) who employed the dressed solitary wave approach. It is shown that for some specific values of the scale parameters the evolution equations of the present formulation reduce to those of renormalization method. Moreover, the use of the present formulation is quite simple as compared to the renormalization method.

§2. **Long-Wave Approximation**

The well-known dimensionless set of equations describing nonlinear ion-acoustic waves in a one dimensional collisionless plasma is (Davidson\(^1\)),

\[
\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x} (n_i u) = 0, \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial \varphi}{\partial x} = 0, \\
 \frac{\partial^2 \varphi}{\partial x^2} + n_i - \exp (\varphi) = 0, \tag{1}
\]

where $u$ is the flow velocity of ions, $n_i$ is the ion density and $\varphi$ is the electric potential. Denoting the fluctuation...
of ion density from its equilibrium value by \( n_i \), i.e., \( n_i = 1 + n \), the above equation can be rewritten as

\[
\frac{\partial n}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} (nu) = 0, \quad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x} = 0, \quad \frac{\partial^2 \varphi}{\partial x^2} + 1 + n - \exp(\varphi) = 0.
\]

(2)

Now, we introduce the following coordinates stretching

\[
\xi = c^{1/2}(x - ct), \quad \tau = c^{1/2}gt,
\]

(3)

where \( c \geq 0 \) and \( g \) are some scale parameters to be determined through the solution of the field equations. Introducing (3) into eq. (2) we obtain

\[
\begin{align*}
 ec \frac{\partial n}{\partial \tau} - c \frac{\partial n}{\partial \xi} + \frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \xi} (nu) &= 0, \\
 ec \frac{\partial u}{\partial \tau} - c \frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \xi} + \frac{\partial \varphi}{\partial \xi} &= 0, \\
 e \frac{\partial^2 \varphi}{\partial \xi^2} + 1 + n - \exp(\varphi) &= 0.
\end{align*}
\]

(4)

Throughout this work we shall assume that the field variables and the scale parameter \( g \) may be expressed as asymptotic series of \( c \) as

\[
\begin{align*}
n &= \sum_{j=1}^{\infty} c^j n_j(\xi, \tau), \\
u &= \sum_{j=1}^{\infty} c^j u_j(\xi, \tau), \\
\varphi &= \sum_{j=1}^{\infty} c^j \varphi_j(\xi, \tau), \\
g &= \sum_{j=0}^{\infty} c^j g_j.
\end{align*}
\]

(5)

where the coefficients \( n_j, u_j, \varphi_j \) and the constants \( g_j \) are to be determined from the solution of the field equations. Introducing the expansion (5) into the eqs. (2) and setting the like powers of \( c \) equal to zero we obtain the following sets of differential equations

**O(\epsilon) order equations:**

\[
\begin{align*}
-\frac{c \partial n_1}{\partial \xi} + \frac{\partial u_1}{\partial \xi} &= 0, \\
-\frac{c \partial u_1}{\partial \xi} + \frac{\partial \varphi_1}{\partial \xi} &= 0, \quad n_1 - \varphi_1 = 0.
\end{align*}
\]

(6)

**O(\epsilon^2) order equations:**

\[
\begin{align*}
\frac{\partial n_1}{\partial \tau} - c \frac{\partial n_2}{\partial \xi} + \frac{\partial u_2}{\partial \xi} + \frac{\partial (n_1 u_1)}{\partial \xi} &= 0, \\
\frac{\partial u_1}{\partial \tau} - c \frac{\partial u_2}{\partial \xi} + \frac{\partial u_1}{\partial \xi} + \frac{\partial \varphi_2}{\partial \xi} &= 0, \\
\frac{\partial^2 \varphi_1}{\partial \xi^2} + n_2 - \varphi_2 - \frac{1}{2} \varphi_1 &= 0.
\end{align*}
\]

(7)

**O(\epsilon^3) order equations:**

\[
\begin{align*}
\frac{\partial n_2}{\partial \tau} + g_1 \frac{\partial n_1}{\partial \tau} - c \frac{\partial n_3}{\partial \xi} + \frac{\partial u_3}{\partial \xi} + \frac{\partial (n_1 u_2 + n_2 u_1)}{\partial \xi} &= 0, \\
\frac{\partial u_2}{\partial \tau} + g_1 \frac{\partial u_1}{\partial \tau} - c \frac{\partial u_3}{\partial \xi} + \frac{\partial \varphi_3}{\partial \xi} + \frac{\partial (u_1 u_2)}{\partial \xi} &= 0, \\
\frac{\partial^2 \varphi_2}{\partial \xi^2} + n_3 - \frac{\varphi_1^3}{6} - \varphi_1 \varphi_2 - \varphi_3 &= 0.
\end{align*}
\]

(8)

O(\epsilon^4) order equations:

\[
\begin{align*}
&g_0 \frac{\partial n_3}{\partial \tau} + g_1 \frac{\partial n_2}{\partial \tau} + g_2 \frac{\partial n_1}{\partial \tau} - c \frac{\partial n_4}{\partial \xi} + \frac{\partial u_4}{\partial \xi} \\
&+ \frac{\partial (n_1 u_3 + n_2 u_2 + n_3 u_1)}{\partial \xi} = 0, \\
&g_0 \frac{\partial u_3}{\partial \tau} + g_1 \frac{\partial u_2}{\partial \tau} + g_2 \frac{\partial u_1}{\partial \tau} - c \frac{\partial u_4}{\partial \xi} + \frac{\partial \varphi_4}{\partial \xi} \\
&+ \frac{\partial (u_1 u_3 + u_2 u_2)}{\partial \xi} = 0, \\
&\frac{\partial^2 \varphi_3}{\partial \xi^2} + n_4 - \frac{1}{24} \varphi_1^4 - \frac{1}{2} \varphi_1^2 \varphi_2 - \frac{1}{2} \varphi_2^2 \\
&- \varphi_1 \varphi_2 - \varphi_4 = 0.
\end{align*}
\]

(9)

### 2.1 Solution of the field equations

In this subsection we shall give the localized solution to the field equations presented in (6)–(9). The solution of the set (6) gives

\[
n_1 = u_1 = \varphi_1(\xi, \tau), \quad c = 1,
\]

(10)

where \( \varphi_1(\xi, \tau) \) is an unknown function whose governing equation will be obtained later. Introducing (10) into eqs. (7) we have

\[
\begin{align*}
&g_0 \frac{\partial \varphi_1}{\partial \tau} - c \frac{\partial n_2}{\partial \xi} + \frac{\partial u_2}{\partial \xi} + 2 \varphi_1 \frac{\partial \varphi_1}{\partial \xi} = 0, \\
&g_0 \frac{\partial \varphi_1}{\partial \tau} - c \frac{\partial u_2}{\partial \xi} + \frac{\partial u_1}{\partial \xi} + \varphi_1 \frac{\partial \varphi_1}{\partial \xi} = 0,
\end{align*}
\]

(11)

\[
\begin{align*}
\frac{\partial^2 \varphi_1}{\partial \xi^2} + n_2 - \varphi_2 - \frac{1}{2} \varphi_1 &= 0.
\end{align*}
\]

Solving eqs. (11) for \( n_2 \) and \( u_2 \) in terms of \( \varphi_2 \), we have

\[
\begin{align*}
u_2 &= \varphi_2 - \frac{1}{2} \frac{\partial^2 \varphi_1}{\partial \xi^2}, \\
\varphi_2 &= \varphi_2 + \frac{1}{2} \frac{\partial^2 \varphi_1}{\partial \xi^2},
\end{align*}
\]

(12)

where \( \varphi_2(\xi, \tau) \) is another unknown function whose governing equation will be obtained later. Eliminating \( n_2, u_2 \) and \( \varphi_2 \) between the eqs. (11) we obtain the following Korteweg – de Vries equation

\[
\begin{align*}
&g_0 \frac{\partial \varphi_1}{\partial \tau} + \varphi_1 \frac{\partial \varphi_1}{\partial \xi} + \frac{1}{2} \varphi_1^3 = 0.
\end{align*}
\]

(13)

To obtain the solution to the \( \epsilon^3 \) order equations, we first introduce the eqs. (10) and (12) into (8), which results in

\[
\begin{align*}
&g_0 \frac{\partial \varphi_2}{\partial \tau} + g_1 \frac{\partial \varphi_1}{\partial \tau} - c \frac{\partial n_3}{\partial \xi} + \frac{\partial u_3}{\partial \xi} + 2 \varphi_1 \frac{\partial \varphi_1}{\partial \xi} \\
&+ \frac{\partial \varphi_1^3}{\partial \xi^2} + \frac{5}{4} \frac{\partial^2 \varphi_1}{\partial \xi^2} + \frac{1}{2} \frac{\partial^4 \varphi_1}{\partial \xi^4} = 0, \\
&g_0 \frac{\partial \varphi_2}{\partial \tau} + g_1 \frac{\partial \varphi_1}{\partial \tau} - c \frac{\partial u_3}{\partial \xi} + \frac{\partial u_1}{\partial \xi} + \frac{\partial \varphi_1}{\partial \xi} \\
&+ \frac{\partial \varphi_1^3}{\partial \xi^2} + \frac{5}{4} \frac{\partial^2 \varphi_1}{\partial \xi^2} + \frac{1}{2} \frac{\partial^4 \varphi_1}{\partial \xi^4} = 0,
\end{align*}
\]

(14)

\[
\begin{align*}
n_3 &= \varphi_3 - \frac{\partial^2 \varphi_1}{\partial \xi^2} + \varphi_1 \varphi_2 + \frac{\varphi_3^3}{6}.
\end{align*}
\]
\[ u_3 = \varphi_3 - \frac{1}{2} \frac{\partial^3 \varphi_2}{\partial \xi^3} + \frac{1}{2} \frac{\partial^3 \varphi_1}{\partial \xi^3} - \frac{3}{8} \frac{\partial^3 \varphi_1}{\partial \xi^3} - \frac{1}{8} \frac{\partial \varphi_1}{\partial \xi}, \]

(14)

where \( \varphi_3(\xi, \tau) \) is another unknown function to be determined from the solution of the problem. Eliminating \( u_3, n_3 \) and \( \varphi_3 \) between the eqs. (14) we obtain the following evolution equation

\[ g_0 \frac{\partial \varphi_2}{\partial \tau} + \frac{1}{2} \frac{\partial^3 \varphi_2}{\partial \xi^3} + \frac{\partial}{\partial \xi} (\varphi_1 \varphi_2) = S_1(\varphi_1), \]

(15)

where the function \( S_1(\varphi_1) \) is defined by

\[ S_1(\varphi_1) = -g_1 \frac{\partial \varphi_1}{\partial \tau} + \frac{1}{2} \frac{\partial^3 \varphi_1}{\partial \xi^3} - \frac{5}{8} \frac{\partial}{\partial \xi} \left( \frac{\partial \varphi_1}{\partial \xi} \right)^2 - \frac{3}{8} \frac{\partial^3 \varphi_1}{\partial \xi^3}. \]

(16)

As stated by Kodama and Taniuti, the right side of eq. (15) is the linearized KdV equation for \( \varphi_2 \) with a non-homogeneous term \( S_1(\varphi_1) \). If we set \( g_1 = 0 \) in eq. (16), the result will be exactly the same with those of obtained by Sugimoto and Kakutan(3) and Ichikawa et al.7 However, in general the present formulation is quite different from those of given in refs. 3 and 7.

For the solution of \( e^4 \) order equation we first eliminate \( u_4, n_4 \) and \( \varphi_4 \) between the eqs. (9) to have

\[ g_0 \frac{\partial}{\partial \tau} (n_3 + u_3) + g_1 \frac{\partial}{\partial \tau} (n_2 + u_2) + g_2 \frac{\partial}{\partial \tau} (n_1 + u_1) + \frac{\partial^3 \varphi_3}{\partial \xi^3} - \frac{\partial}{\partial \xi} (\varphi_1 \varphi_3) + \frac{\partial}{\partial \xi} \left( \frac{\varphi_1^2 \varphi_2 - \varphi_2^2}{2} \right) \]

\[ + n_1 u_3 + u_1 u_3 + n_3 u_1 + n_2 u_2 + \frac{1}{2} u^2_2 = 0. \]

(17)

Introducing the solutions given in (10), (12) and (14) into (17) the following evolution equation is obtained

\[ g_0 \frac{\partial \varphi_3}{\partial \tau} + \frac{1}{2} \frac{\partial^3 \varphi_3}{\partial \xi^3} + \frac{\partial}{\partial \xi} (\varphi_1 \varphi_3) = S_2(\varphi_1, \varphi_2), \]

(18)

where the function \( S_2(\varphi_1, \varphi_2) \) is defined by

\[ S_2(\varphi_1, \varphi_2) = -g_0 \frac{\partial \varphi_1}{\partial \tau} + \frac{1}{2} \frac{\partial^3 \varphi_1}{\partial \xi^3} + \frac{5}{8} \frac{\partial}{\partial \xi} \left( \frac{\partial \varphi_1}{\partial \xi} \right)^2 - \frac{3}{8} \frac{\partial^3 \varphi_1}{\partial \xi^3} + \frac{3}{4} \frac{\partial^3 \varphi_2}{\partial \xi^3} + \frac{3}{4} \frac{\partial^3 \varphi_1}{\partial \xi^3} \]

\[ + \frac{5}{8} \frac{\partial^2 \varphi_1}{\partial \xi^2} \frac{\partial^3 \varphi_1}{\partial \xi^3} - \frac{3}{4} \frac{\partial^3 \varphi_1}{\partial \xi^3} \frac{\partial^2 \varphi_1}{\partial \xi^2} - \frac{1}{4} \frac{\partial^4 \varphi_1}{\partial \xi^4} - \frac{g_0}{2} \frac{\partial \varphi_1}{\partial \tau} + g_1 \frac{\partial}{\partial \xi} \left( \varphi_1 + \varphi_2 \right) - \frac{3}{4} \frac{\partial^3 \varphi_1}{\partial \xi^3} \]

\[ - g_0 \frac{\partial}{\partial \xi} \left( \varphi_1 \varphi_2 + \frac{\varphi_1^3}{6} + \frac{3}{2} \frac{\partial^2 \varphi_2}{\partial \xi^2} + \frac{1}{2} \frac{\partial^3 \varphi_1}{\partial \xi^3} - \frac{3}{8} \frac{\partial^3 \varphi_1}{\partial \xi^3} + \frac{1}{8} \frac{\partial^4 \varphi_1}{\partial \xi^4} \right). \]

(19)

As is seen from eq. (19), this equation is the linearized KdV equation for \( \varphi_3 \) with a non-homogeneous term \( S_2(\varphi_1, \varphi_2) \). Moreover, due to existence of the scale parameters \( g_1 \) and \( g_2 \) this evolution equation is quite different from that of Kodama and Taniuti.

2.2 Solitary waves

In this sub-section we shall study the localized traveling wave solution to the field eqs. (13), (15) and (18). For that purpose we introduce

\[ \varphi_0 = \varphi_0(\eta), \quad (\alpha = 1, 2, 3), \quad \eta = k(\xi - \tau), \]

(20)

where \( k \) is a constant to be determined as a part of the solution. Introducing (20) into eq. (13) we obtain

\[ k^2 \varphi_1'' + 2 \varphi_1 \varphi_1' - 2 g_0 \varphi_1'' = 0, \]

(21)

where a prime denotes the differentiation with respect to \( \eta \). Integrating eq. (21) with respect to its argument and using the localization condition, i.e., \( \varphi_1 \to 0 \) as \( \eta \to \pm \infty \), we obtain

\[ k^2 \varphi_1'' + \varphi_1^2 - 2 g_0 \varphi_1 = 0. \]

(22)

It is a common practice to employ the hyperbolic tangent method in solving this type of wave equations (Malfliet and Weiers). For that purpose we introduce the following coordinate transformation

\[ z = \tanh \eta. \]

(23)

The finite power series solution of (22) in the variable \( z \) which satisfies the regularity conditions, \( \varphi_1 \to 0 \) as \( z \to \pm 1 \), can be expressed as

\[ \varphi_1 = a(1 - z^2), \]

(24)

where \( a \) is a constant to be determined so as to satisfy the eq. (22). Noting the differential relation \( d/d\eta = (1 - z^2) d/dz \), we have

\[ \varphi_1' = a(-2 + 8 z^2 - 6z^4). \]

(25)

Introducing (25) into (22) and setting the like powers of \( z \) equal to zero we obtain \( a = 3g_0 \) and \( k = (g_0/2)^{1/2} \). Here we note that the requirement of localized travelling wave solution made it possible to determine the scale parameter \( g_0 \) in terms of the wave amplitude. Thus, the solution of the first order equation gives

\[ \varphi_1 = 3g_0 \text{sech}^2 \eta, \quad \eta = (\frac{g_0}{2})^{1/2}(\xi - \tau). \]

(26)

This solution is exactly the same with that of obtained by Washimi and Taniuti.5

To obtain the solution for the second order term we first introduce (20) into eq. (15) which results in

\[ \frac{1}{2} k^3 \varphi_2'' - 2 k \varphi_2' + k(\varphi_1 \varphi_2)' = S_1(\varphi_1). \]

(27)

Integrating (27) with respect to \( \eta \) and using the localization condition we have

\[ \varphi_1' + 4(\frac{\varphi_1}{g_0} - 1) \varphi_2 = T_1(\varphi_1), \]

(28)

where \( T_1(\varphi_1) \) is defined by

...
where $\varphi^{(iv)}_1$ stand for the forth order derivative of $\varphi_1$ with respect to $\eta$. If we set $g_0 = 2$ the eqs. (28) and (29) will be exactly the same with that of obtained by Malfliet and Wiere, who employed the concept of dressed solitary waves, but different from that of ref. 7. However, due to existence of the scale parameter $g_1$, this evolution equation is more general than that of Ichikawa et al.\(^7\).

For the solution of eq. (28) we shall again employ the hyperbolic tangent method and introduce the following finite power series as the solution for $\varphi_2$, which satisfies the localization condition

$$\varphi_2 = (1 - z^2)(a_0 + a_1 z + a_2 z^2),$$

where $a_0, a_1$ and $a_2$ are some constants to be determined by introducing (30) into eq. (28). Noting the differential relation

$$\varphi_2'' = 2(a_2 - a_0) - 8a_1 z + (8a_0 - 20a_2)z^2 + 20a_1 z^3 + (38a_2 - 6a_0)z^4 - 12a_1 z^5 - 20a_2 z^6,$$

and introducing these expressions into (28) and setting the like powers of $z$ equal to zero we obtain $a_0 = (9/4)g_0^3$, $a_2 = -63/4g_0^3$ and $g_1 = (3/2)g_0^2$. The coefficient $a_1$ remains undetermined. Thus the solution may be expressed as follows

$$\varphi_2 = a_1 \text{sech}^2 \eta \tanh \eta + \frac{1}{4} g_0^2 \text{sech}^2 \eta (1 - 7 \tanh^2 \eta).$$

Here, as can be shown, the first term in (32) corresponds to the homogeneous solution of eq. (28). This solution is exactly the same with that of obtained by Sugimoto and Kakutani\(^3\) who introduced some slow scales, Kodama and Tanii\(^4\) who employed the renormalization method and Malfliet and Wiere\(^6\) who employed the travelling wave solution to the field equations at the outset. However, the methods presented in refs. 4, 6 and 7 can be used for progressive wave solution to the field equations. Moreover, the method presented by Ichikawa et al.\(^7\) leads to a secular term. In our case we removed the secularity by presenting the scale parameter $g_1$.

In order to obtain the solution for the third order terms, we first introduce (20) into the eqs. (18) and (19), integrate the result with respect to $\eta$ and then employ the localization condition and, finally, obtain the following equation

$$\varphi'_3 + 4\left(\frac{\varphi_1}{g_0} - 1\right)\varphi_3 = T_2(\varphi_1, \varphi_2),$$

where $T_2(\varphi_1, \varphi_2)$ is defined by

This result is again exactly the same with that of Malfliet and Wiere\(^6\). Following Malfliet and Wiere\(^6\), who employed the hyperbolic tangent method for the solution of the field equations and obtained the scaling parameter or the speed correction term $g_2$ as $g_2 = (5/2)g_0^3$. The resulting solution may be expressed as follows

$$\varphi_3 = \frac{9}{80} g_0^3 \text{sech}^2 \eta (943 \tanh^4 \eta - 663 \tanh^2 \eta + 26).$$

Here we should point out that the homogeneous solutions for both $\varphi_1$ and $\varphi_2$ are proportional to $\text{sech}^2 \eta \tanh \eta$ and were set equal to zero.

§3. Conclusions

Assuming the wave under investigation is rightgoing, a modified reductive perturbation method along with the concept of scale parameter is introduced. Expanding the field variables and the scale parameter into a power series of the smallness parameter $\epsilon$ we obtained a set of differential equations governing various order terms in the perturbation expansion. Then, solving these differential equations we obtained the evolution equations governing the first three terms in the expansion. The evolution equations for the second and third order terms in the perturbation expansion are quite different from those of Kodama and Tanii\(^4\) and Ichikawa et al.\(^7\). If one sets the scale parameters $g_1$ and $g_2$ equal to zero, the result will be the same with those of given in refs. 4 and 7. However, the disappearance of these parameters leads the solution to have some secular terms. Thus, the existence of such parameters in the evolution equation makes it possible to remove such secularities. Finally, seeking a localized travelling wave solution to the field equations we obtained the explicit expressions of the solitary waves and the scaling parameters or speed correction terms as a part of the solution. It is shown that the result of travelling wave solution to the present formulation is completely the same with those of obtained by Sugimoto and Kakutani\(^3\) who introduced some slow scales, Kodama and Tanii\(^4\) who employed the renormalization method and Malfliet and Wiere\(^6\) who assumed a localized travelling wave solution for the field variables in the beginning of their study and, thus worked with nonlinear ordinary differential equations. However, in ref. 6, the partial differential equation representing the evolution equation is not known. Therefore, one cannot study any boundary or initial value problem, except the travelling wave solution, through the use of the method in ref. 6. As to the method in ref. 4, the evolution equations in ref. 4 are different from the one presented in our work. Nevertheless, the result of travelling solution along with the re-normalization method is exactly the same with ours. But, the method presented here is more simple than that of re-normalization method.
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