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POSITIVE SOLUTIONS FOR A SUM-TYPE SINGULAR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION WITH m-POINT BOUNDARY CONDITIONS

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We study the existence and uniqueness of positive solutions for a sum-type singular fractional integro-differential equation with m-point boundary condition. Also, we provide an example to illustrate our main result.

Keywords: Fixed point, Fractional derivative, Riemann-Liouville integral, Singular integro-differential equation.

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1. Introduction and Preliminaries

Many researchers have been investigated distinct fractional differential equations and inclusions which have been applied in modeling of different problems in some sciences. There are many papers on the existence of solutions (or positive solution) for some singular fractional differential equations (see for example, [1]-[3], [10], [12], [14], [17], [18] and [19]). In 2013, the existence and uniqueness of positive solutions for the fractional differential equation $-D^{\alpha}u(t) = f(t, u(t)) + g(t, u(t))$ with the boundary conditions $u(0) = u'(0) = u''(0) = u''(1) = 0$ or $u(0) = u'(0) = u''(0) = 0$, $u''(1) = \beta u''(\eta)$ investigated, where $0 < t < 1$, $3 < \alpha \leq 4$ and D^{α} is the Riemann-Liouville fractional derivative ([21]). By using idea of the work and some another published works such [4], [8], [9] and [16], we investigate the singular fractional integro-differential equation

$$\begin{gathered} D^{\alpha}u(t) + f(t, u(t), \int_0^t \gamma(t,s)h(t,s,u(s), D^{\beta_1}u(s), \dots, D^{\beta_L}u(s))ds, \\ D^{\mu_1}u(t), \dots, D^{\mu_N}u(t)) \\ +g(t, u(t), \int_0^t \gamma(t,s)h(t,s,u(s), D^{\beta_1}u(s), \dots, D^{\beta_L}u(s))ds, \\ D^{\mu_1}u(t), \dots, D^{\mu_N}u(t)) = 0 \quad (1.1) \end{gathered}$$

with m-point boundary conditions $D^{\mu_i}u(0) = D^{\beta_j}u(0) = 0$ for $1 \leq i \leq N$ and $1 \leq j \leq L$, $D^{\mu_N+1}u(0) = \cdots = D^{\mu_N+3}u(0) = 0$ and $D^{\mu_N+3}u(1) = \sum_{j=1}^{m-2} a_j D^{\mu_N+3}u(\xi_j)$, where $0 < t < 1$, $n \geq 5$, $n-1 < \alpha \leq n$, $0 < \mu_1 < \cdots < \mu_N$, $0 < \beta_1 < \cdots < \beta_L \leq \mu_N$, $4 < \alpha - \mu_N \leq 5$, $a_j \in (0,\infty)$, $0 < \xi_1 < \cdots < \xi_{m-2} < 1$, $\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4} < 1$, $\gamma : [0,1] \times [0,1] \to [0,\infty)$, $h : [0,1] \times [0,1] \times \mathbb{R}^{L+1} \to [0,\infty)$ and $f, g : (0,1] \times \mathbb{R}^{N+2} \to [0,\infty)$ are continuous mappings, D is the Riemann-Liouville fractional derivative, $\lim_{t\to 0^+} f(t, ., ., \dots, .) = +\infty$

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and $\lim_{t\to 0^+} g(t,.,.,\dots,.) = +\infty$, that is, f, g are singular at $t = 0$. As you know, the Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $f : [0,\infty) \to \mathbb{R}$ is defined by $I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds$ for $t > 0$ provided the integral exists ([7], [13] and [15]). Also, the Riemann-Liouville fractional derivative of order $\alpha > 0$ for a continuous function $f : [0,\infty) \to \mathbb{R}$ is defined by $D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s)\, ds$ for $t > 0$, where n is the smallest integer greater than or equal α ([7], [13] and [15]).

Lemma 1.1. *([7], [13]) If $u \in C(0,1) \cap L^1(0,1)$ with $D^\alpha u \in C(0,1) \cap L^1(0,1)$, then*

$$I^\alpha D^\alpha u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_1, \dots, c_n$ are some real numbers and n is the smallest integer greater than or equal α.

Lemma 1.2. *([7], [13]) If $u \in L^1[0,1]$ and $\rho > \sigma > 0$, then $I^\rho I^\sigma u(t) = I^{\rho+\sigma} u(t)$, $D^\sigma I^\rho u(t) = I^{\rho-\sigma} u(t)$ and $D^\sigma I^\sigma u(t) = u(t)$. If $\rho > 0$ and $\nu > -1$, then $D^\rho t^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\nu-\rho+1)} t^{\nu-\rho}$. Also, $I^\alpha u \in C[0,1]$ for all $\alpha > 0$ and $u \in C[0,1]$.*

By using Lemma 1.2, one can easily conclude next result.

Lemma 1.3. *Let $v \in C([0,1])$ and $u(t) = I^{\mu_N} v(t)$. Then (1.1) reduces to the problem*

$$D^{\alpha-\mu_N} v(t) + f(t, I^{\mu_N} v(t), \int_0^t \gamma(t,s) h(t,s,I^{\mu_N} v(s), I^{\mu_N-\beta_1} v(s), \dots, I^{\mu_N-\beta_L} v(s)) ds,$$

$$I^{\mu_N-\mu_1} v(t), I^{\mu_N-\mu_2} v(t), \dots, I^{\mu_N-\mu_{N-1}} v(t), v(t))$$

$$+ g(t, I^{\mu_N} v(t), \int_0^t \gamma(t,s) h(t,s,I^{\mu_N} v(s), I^{\mu_N-\beta_1} v(s), \dots, I^{\mu_N-\beta_L} v(s)) ds,$$

$$I^{\mu_N-\mu_1} v(t), I^{\mu_N-\mu_2} v(t), \dots, I^{\mu_N-\mu_{N-1}} v(t), v(t)) = 0 \quad (1.2)$$

with boundary conditions $v'''(1) = \sum_{j=1}^{m-2} a_j v'''(\xi_j)$ and $v(0) = v'(0) = v''(0) = v'''(0) = 0$, where $0 < t < 1$. Moreover if $v \in C([0,1])$ is a positive solution of the problem (1.2), then $u(t) = I^{\mu_N} v(t)$ is a positive solution for the problem (1.1).

Lemma 1.4. *Let $4 < \alpha - \mu_N \le 5$ and $\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4} \ne 1$. If $y \in C(0,1]$, then the problem $D^{\alpha-\mu_N} w(t) + y(t) = 0$ with boundary conditions $w(0) = w'(0) = w''(0) = w'''(0) = 0$ and $w'''(1) = \sum_{j=1}^{m-2} a_j w'''(\xi_j)$ has the unique solution*

$$w(t) = \int_0^1 G(t,s) y(s) ds$$

$$+ \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \int_0^1 H(\xi_j, s) y(s) ds,$$

where $G(t,s) = \frac{1}{\Gamma(\alpha-\mu_N)} t^{\alpha-\mu_N-1}(1-s)^{\alpha-\mu_N-4} - (t-s)^{\alpha-\mu_N-1}$ whenever $0 \le s \le t \le 1$, $G(t,s) = \frac{1}{\Gamma(\alpha-\mu_N)} t^{\alpha-\mu_N-1}(1-s)^{\alpha-\mu_N-4}$ whenever $0 \le t \le s \le 1$, $H(t,s) = \frac{\partial^3 G(t,s)}{\partial t^3} = \frac{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)}{\Gamma(\alpha-\mu_N)} (t^{\alpha-\mu_N-4}(1-s)^{\alpha-\mu_N-4} - (t-s)^{\alpha-\mu_N-4})$ whenever $0 \le s \le t \le 1$ and $H(t,s) = \frac{\partial^3 G(t,s)}{\partial t^3} = \frac{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)}{\Gamma(\alpha-\mu_N)} t^{\alpha-\mu_N-4}(1-s)^{\alpha-\mu_N-4}$ whenever $0 \le t \le s \le 1$.

Proof. By using Lemma 1.1, the solution of this problem is

$$w(t) = -I^{\alpha-\mu_N} y(t) + c_1 t^{\alpha-\mu_N-1} + c_2 t^{\alpha-\mu_N-2} + c_3 t^{\alpha-\mu_N-3} + c_4 t^{\alpha-\mu_N-4} + c_5 t^{\alpha-\mu_N-5},$$

where $c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}$ are some constants. By using the boundary conditions, we get $c_2 = c_3 = c_4 = c_5 = 0$ and

$$c_1 = \frac{1}{\Gamma(\alpha-\mu_N)(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})}$$

$$\times \Big[\int_0^1 (1-s)^{\alpha-\mu_N-4} y(s) ds - \sum_{j=1}^{m-2} a_j \int_0^{\xi_j} (\xi_j - s)^{\alpha-\mu_N-4} y(s) ds \Big].$$

Thus, the unique solution of this problem is

$$\begin{aligned}
w(t) &= -\int_0^t \frac{(t-s)^{\alpha-\mu_N-1}}{\Gamma(\alpha-\mu_N)} y(s) ds + \frac{t^{\alpha-\mu_N-1}}{\Gamma(\alpha-\mu_N)(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \\
&\quad \times \Big[\int_0^1 (1-s)^{\alpha-\mu_N-4} y(s) ds - \sum_{j=1}^{m-2} a_j \int_0^{\xi_j} (\xi_j - s)^{\alpha-\mu_N-4} y(s) ds \Big] \\
&= -\int_0^t \frac{(t-s)^{\alpha-\mu_N-1}}{\Gamma(\alpha-\mu_N)} y(s) ds + \Big(\frac{t^{\alpha-\mu_N-1}}{\Gamma(\alpha-\mu_N)} \\
&\quad + \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4}}{\Gamma(\alpha-\mu_N)(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \Big) \int_0^1 (1-s)^{\alpha-\mu_N-4} y(s) ds \\
&\quad - \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j}{\Gamma(\alpha-\mu_N)(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \int_0^{\xi_j} (\xi_j - s)^{\alpha-\mu_N-4} y(s) ds \\
&= \frac{1}{\Gamma(\alpha-\mu_N)} \int_0^t \Big(t^{\alpha-\mu_N-1}(1-s)^{\alpha-\mu_N-4} - (t-s)^{\alpha-\mu_N-1} \Big) y(s) ds \\
&\quad + \frac{1}{\Gamma(\alpha-\mu_N)} \int_t^1 t^{\alpha-\mu_N-1}(1-s)^{\alpha-\mu_N-4} y(s) ds \\
&\quad + \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j}{\Gamma(\alpha-\mu_N)(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \\
&\quad \times \Big[\int_0^1 \xi_j^{\alpha-\mu_N-4}(1-s)^{\alpha-\mu_N-4} y(s) ds - \int_0^{\xi_j} (\xi_j - s)^{\alpha-\mu_N-4} y(s) ds \Big] \\
&= \int_0^1 G(t,s) y(s) ds \\
&\quad + \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \\
&\quad \times \int_0^1 H(\xi_j, s) y(s) ds.
\end{aligned}$$

This completes the proof. □

One can check that G is a continuous function on $[0,1] \times [0,1]$, $G(t,s) \geq 0$ and $H(t,s) \geq 0$ for all $t, s \in [0,1]$ and $G(t,s) > 0$ for all $t, s \in (0,1)$. Also,

$$\frac{1}{\Gamma(\alpha-\mu_N)} s(s^2 - 3s + 3)(1-s)^{\alpha-\mu_N-4} t^{\alpha-\mu_N-1} \leq G(t,s)$$

$$\leq \frac{1}{\Gamma(\alpha-\mu_N)}(1-s)^{\alpha-\mu_N-4}t^{\alpha-\mu_N-1}$$

for all $t, s \in [0,1]$. Also, $\sup_{0\leq t\leq 1}\int_0^1 G(t,s)s^{-\sigma}ds = \frac{1}{\Gamma(\alpha-\mu_N)}(\beta(1-\sigma,\alpha-\mu_N-3) - \beta(1-\sigma,\alpha-\mu_N))$ and

$$\int_0^1 H(\eta,s)s^{-\sigma}ds =$$

$$\frac{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)}{\Gamma(\alpha-\mu_N)}(\eta^{\alpha-\mu_N-4}-\eta^{\alpha-\mu_N-\sigma-3})\beta(1-\sigma,\alpha-\mu_N-3)$$

for all $0<\sigma<1$ and $0<\eta<1$. This conclude that

$$K := \sup_{0\leq t\leq 1}\int_0^1 \Bigg[G(t,s) + \frac{t^{\alpha-\mu_N-1}\sum_{j=1}^{m-2}a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2}a_j\xi_j^{\alpha-\mu_N-4})}$$

$$\times H(\xi_j,s)\Bigg]s^{-\sigma}ds$$

$$= \frac{1}{\Gamma(\alpha-\mu_N)}\Bigg[\Bigg(1+\frac{\sum_{j=1}^{m-2}a_j(\xi_j^{\alpha-\mu_N-4}-\xi_j^{\alpha-\mu_N-\sigma-3})}{1-\sum_{j=1}^{m-2}a_j\xi_j^{\alpha-\mu_N-4}}\Bigg)\beta(1-\sigma,\alpha-\mu_N-3)-\beta(1-\sigma,\alpha-\mu_N)\Bigg].$$

By using some calculations, one can prove next key result.

Theorem 1.1. *Let $0<\sigma<1$, $4<\alpha-\mu_N\leq 5$, $\sum_{j=1}^{m-2}a_j\xi_j^{\alpha-\mu_N-4}\neq 1$ and $F:(0,1]\to\mathbb{R}$ be a continuous function with $\lim_{t\to 0^+}F(t)=\infty$. Suppose that $t^\sigma F(t)$ is a continuous function on $[0,1]$. Then the functions defined by $L(t)=\int_0^1 G(t,s)F(s)ds$ and $W(t) = \frac{t^{\alpha-\mu_N-1}\sum_{j=1}^{m-2}a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2}a_j\xi_j^{\alpha-\mu_N-4})}\int_0^1 H(\xi_j,s)F(s)ds$ are continuous on $[0,1]$.*

Let $(X,\|.\|)$ be a real Banach space which has a partially order by using a cone $P\subset X$. A nonempty closed convex set $P\subset X$ is a cone whenever $x\in P$ and $\lambda\geq 0$ implies $\lambda x\in P$ and $P\cap(-P)=\{0\}$ ([5] and [20]). A cone P is called solid whenever interior of P is nonempty. Each cone P defines the order $\leq$ on X by $x\leq y$ if and only if $y-x\in P$ ([5] and [20]). A cone P is called normal if there exists a constant $N>0$ such that $\theta\leq x\leq y$ implies $\|x\|\leq N\|y\|$ ([5] and [20]). In this case, least number N is called the normal constant of P ([5] and [20]). Define $x\sim y$ whenever there exist $\lambda>0$ and $\mu>0$ such that $\lambda x\leq y\leq \mu x$. Then, $\sim$ is an equivalence relation on X ([5] and [20]). For each $k\geq\theta$ with $k\neq\theta$, define $P_k=\{x\in X:\ x\sim k\}$. One can check that $P_k\subset P$ for all $k\in P$ ([5] and [20]).

Theorem 1.2. *([6]) Let (X,d) be a complete metric space, $\leq$ an order on X, $T:X\to X$ an increasing map and $x_n\leq x$ for all n whenever $\{x_n\}$ is an increasing sequence in X with $x_n\to x$. Suppose that there exists a continuous and increasing function $\psi:[0,\infty)\to[0,\infty)$ such that ψ is positive on $(0,\infty)$, $\psi(0)=0$ and $d(T(x),T(y))\leq d(x,y)-\psi(d(x,y))$ for all $x\geq y$. If there exists $x_0\in X$ with $x_0\leq Tx_0$, then T has a fixed point. If for each $x,y\in X$, there exists $z\in X$ which is comparable to x and y, then T has a unique fixed point.*

Let X be a real Banach space, P a cone in X and $0\leq\gamma<1$ a real number. An operator $A:P\to P$ is said to be γ-concave whenever $A(tx)\geq t^\gamma Ax$ for all $t\in(0,1)$ and $x\in P$ ([20]). Also, $A:P\to P$ is called homogeneous whenever $A(\lambda x)=\lambda Ax$ for all $\lambda>0$ and $x\in P$ ([20]). Finally, $A:P\to P$ is said to be sub-homogeneous whenever $A(tx)\geq tAx$ for all $t\in(0,1)$ and $x\in P$ ([20]). In 2011, Zhai and Anderson proved next result.

Theorem 1.3. *([20]) Let P be a normal cone in a real Banach space X, $A:P\to P$ an increasing γ-concave map and $B:P\to P$ an increasing sub-homogeneous operator. Assume that there is $h>\theta$ such that $Ah\in P_h$ and $Bh\in P_h$. Also, there exists $\delta_0>0$ such that*

$Ax \geq \delta_0 Bx$ for all $x \in P$. Then the operator equation $Ax + Bx = x$ has a unique solution x^* in P_k. Moreover, the sequence $y_n = Ay_{n-1} + By_{n-1}$ $(n \geq 1)$ with initial value $y_0 \in P_k$ converges to x^*.

Note that last result holds whenever B is a null operator. In this paper, we use the Banach space $X = C([0,1])$ with the partial order $x \leq y$ if and only if $x(t) \leq y(t)$ for all $t \in [0,1]$ and $x, y \in C([0,1])$. It has been proved $(C([0,1]), \leq)$ has this property that $x_n \leq x$ for all n whenever $\{x_n\}$ is an increasing sequence in $C([0,1])$ with $x_n \to x$ ([11]). Moreover, $\max\{x, y\} \in C([0,1])$ for all $x, y \in C([0,1])$, that is, for each $x, y \in C([0,1])$ there exists $z \in C([0,1])$ which is comparable to x and y. We consider the normal cone $P = \{x \in C([0,1]) : x(t) \geq 0 \text{ for all } t \in [0,1]\}$ with normal constant 1.

2. **Main Results**

Now, we are ready to state and prove our main result.

Theorem 2.1. *Suppose that $f, g : (0,1] \times \mathbb{R}^{N+2} \to [0,\infty)$ are continuous mappings with $f(t,.,.,\ldots,.), g(t,.,.,\ldots,.) \to +\infty$ as $t \to 0^+$, $0 < \sigma < 1$, the maps $t^\sigma f(t, x_1, x_2, \ldots, x_{N+2})$ and $t^\sigma g(t, x_1, x_2, \ldots, x_{N+2})$ are continuous on $[0,1] \times \mathbb{R}^{N+2}$, $t^\sigma f(t,.,.,\ldots,.)$, $t^\sigma g(t,.,.,\ldots,.)$ and $h(t,s,.,\ldots,.)$ are increasing with respect to their components on $[0,\infty)$ for each fixed t and s in $[0,1]$ and also $t^\sigma g(t,0,0,\ldots,0) \not\equiv 0$. Also, assume that*

$$t^\sigma g(t, \lambda x_1, \lambda x_2, \ldots, \lambda x_{N+2}) \geq \lambda t^\sigma g(t, x_1, x_2, \ldots, x_{N+2}),$$

$h(t, s, \lambda y_1, \lambda y_2, \ldots, \lambda y_{L+1}) \geq \lambda h(t, s, y_1, y_2, \ldots, y_{L+1})$ and there exists a constant $\gamma \in [0,1)$ such that $t^\sigma f(t, \lambda x_1, \lambda x_2, \ldots, \lambda x_{N+2}) \geq \lambda^\gamma t^\sigma f(t, x_1, x_2, \ldots, x_{N+2})$ for all $t, s \in [0,1]$, $\lambda \in (0,1)$ and $x_i, y_j \in [0,\infty)$ $(1 \leq i \leq N+2,\ 1 \leq j \leq L+1)$. If there exists $\delta_0 > 0$ such that $t^\sigma f(t, x_1, x_2, \ldots, x_{N+2}) \geq \delta_0 t^\sigma g(t, x_1, x_2, \ldots, x_{N+2})$, for all $t \in [0,1]$ and $x_i \in [0,\infty)$ $(1 \leq i \leq N+2)$, then the problem (1.2) has a unique solution $u^* \in P_k$, where $k(t) = t^{\alpha - \mu_N - 1}$ for all $t \in [0,1]$. Moreover, the sequence

$$\begin{aligned} u_{n+1}(t) &= \int_0^1 \Bigg[G(t,s) \\ &\quad + \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \\ &\quad \times H(\xi_j, s) \Bigg] \times \Big[\widetilde{f}(s, u_n(s)) + \widetilde{g}(s, u_n(s)) \Big] ds \end{aligned}$$

converges to u^* for each initial value $u_0 \in P_k$.

Proof. Define the operators $A, B : P \to X$ by

$$Au(t) = \int_0^1 \Bigg[G(t,s) + \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \times H(\xi_j, s) \Bigg] \widetilde{f}(s, u(s)) ds$$

and

$$Bu(t) = \int_0^1 \Bigg[G(t,s) + \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \times H(\xi_j, s) \Bigg] \widetilde{g}(s, u(s)) ds$$

for all $t \in [0,1]$. It is easy to check that u is a solution for the problem (1.2) if and only if $u = Au + Bu$. From the assumptions and Theorem 1.1, we know that the operators A and

B maps P into P. We show that A and B are increasing operators. Let $u \geq v$. Then, we have

$$Au(t) = \int_0^1 \Bigg[G(t,s) + \frac{t^{\alpha-\mu_N-1}\sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\mu_N-4})} H(\xi_j,s)\Bigg] \times \widetilde{f}(s,u(s))ds$$

$$= \int_0^1 \Bigg[G(t,s) + \frac{t^{\alpha-\mu_N-1}\sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\mu_N-4})} H(\xi_j,s)\Bigg] \times s^{-\sigma}s^{\sigma}\widetilde{f}(s,u(s))ds$$

$$\geq \int_0^1 \Bigg[G(t,s) + \frac{t^{\alpha-\mu_N-1}\sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\mu_N-4})} H(\xi_j,s)\Bigg] \times s^{-\sigma}s^{\sigma}\widetilde{f}(s,v(s))ds$$

$$= \int_0^1 \Bigg[G(t,s) + \frac{t^{\alpha-\mu_N-1}\sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\mu_N-4})} H(\xi_j,s)\Bigg] \times \widetilde{f}(s,v(s))ds = Av(t)$$

for all $t \in [0,1]$. Hence, $Au \geq Av$. Similarly, we can show that $Bu \geq Bv$. Now, we show that A is γ-concave and B is a sub-homogeneous operator. Let $\lambda \in (0,1)$ and $u \in P$. Then,

$$A(\lambda u)(t) = \int_0^1 \Bigg[G(t,s) + \frac{t^{\alpha-\mu_N-1}\sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\mu_N-4})} \times H(\xi_j,s)\Bigg] \times \widetilde{f}(s,\lambda u(s))ds$$

$$= \int_0^1 \Bigg[G(t,s) + \frac{t^{\alpha-\mu_N-1}\sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\mu_N-4})} \times H(\xi_j,s)\Bigg] \times s^{-\sigma}s^{\sigma}\widetilde{f}(s,\lambda u(s))ds$$

$$\geq \lambda^{\gamma}\Bigg(\int_0^1 \Bigg[G(t,s) + \frac{t^{\alpha-\mu_N-1}\sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\mu_N-4})} \times H(\xi_j,s)\Bigg] \times s^{-\sigma}s^{\sigma}\widetilde{f}(s,u(s))ds\Bigg)$$

$$= \lambda^{\gamma}\Bigg(\int_0^1 \Bigg[G(t,s) + \frac{t^{\alpha-\mu_N-1}\sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\mu_N-4})} \times H(\xi_j,s)\Bigg] \times \widetilde{f}(s,u(s))ds\Bigg) = \lambda^{\gamma}Au(t)$$

for all $t \in [0,1]$. Hence, $A(\lambda u) \geq \lambda^{\gamma}Au$ for all $\lambda \in (0,1)$ and $u \in P$ and so the A is a γ-concave operator. By using similar calculations, we can show that the operator B is sub-homogeneous. Now, we show that $Ak, Bk \in P_k$. Note that,

$$Ak(t) = \int_0^1 \Bigg[G(t,s) + \frac{t^{\alpha-\mu_N-1}\sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\mu_N-4})}$$

$$\times H(\xi_j, s)\Big] \times \widetilde{f}(s, k(s))ds$$

$$= \int_0^1 \Big[G(t,s) + \frac{t^{\alpha-\mu_N-1}\sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\mu_N-4})}$$

$$\times H(\xi_j, s)\Big] \times s^{-\sigma}s^{\sigma}\widetilde{f}(s, k(s))ds$$

$$\leq \frac{1}{\Gamma(\alpha-\mu_N)}k(t)\int_0^1 (1-s)^{\alpha-\mu_N-4}s^{-\sigma}s^{\sigma}\widetilde{f}(s,1)ds$$

$$+\frac{k(t)\sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\mu_N-4})}$$

$$\times \int_0^1 H(\xi_j, s)s^{-\sigma}s^{\sigma}\widetilde{f}(s,1)ds := l_2k(t)$$

and

$$Ak(t) = \int_0^1 \Big[G(t,s) + \frac{t^{\alpha-\mu_N-1}\sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\mu_N-4})}$$

$$\times H(\xi_j, s)\Big] \times \widetilde{f}(s, k(s))ds$$

$$= \int_0^1 \Big[G(t,s) + \frac{t^{\alpha-\mu_N-1}\sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\mu_N-4})}$$

$$\times H(\xi_j, s)\Big] \times s^{-\sigma}s^{\sigma}\widetilde{f}(s, k(s))ds$$

$$\geq \frac{1}{\Gamma(\alpha-\mu_N)}k(t)\int_0^1 s(s^2-3s+3)(1-s)^{\alpha-\mu_N-4}s^{-\sigma}s^{\sigma}\widetilde{f}(s,0)ds$$

$$+\frac{k(t)\sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\mu_N-4})}$$

$$\times \int_0^1 H(\xi_j, s)s^{-\sigma}s^{\sigma}\widetilde{f}(s,0)ds := l_1k(t)$$

for all $t \in [0,1]$. By using the assumptions, we get

$$s^{\sigma}\widetilde{f}(s,1) \geq s^{\sigma}\widetilde{f}(s,0) \geq s^{\sigma}f(s,0,0,\dots,0) \geq \delta_0 s^{\sigma}g(s,0,0,\dots,0) \geq 0.$$

Since $s^{\sigma}g(s,0,0,\dots,0) \not\equiv 0$,

$$\int_0^1 s^{\sigma}\widetilde{f}(s,1)ds \geq \int_0^1 s^{\sigma}\widetilde{f}(s,0)ds \geq \delta_0\int_0^1 s^{\sigma}g(s,0,0,\dots,0)ds > 0$$

and so $l_1 > 0$ and $l_2 > 0$. Thus, $l_1k(t) \leq Ak(t) \leq l_2k(t)$ for all $t \in [0,1]$ and so $Ak \in P_k$. Similarly, we can show that $Bk \in P_k$. Now, let $u \in P$. Then, we have

$$Au(t) = \int_0^1 \Big[G(t,s) + \frac{t^{\alpha-\mu_N-1}\sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\mu_N-4})}$$

$$\times H(\xi_j, s)\Big] \times \widetilde{f}(s, u(s))ds$$

$$= \int_0^1 \Big[G(t,s) + \frac{t^{\alpha-\mu_N-1}\sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-\mu_N-4})}$$

$$\left.\times H(\xi_j, s)\right] \times s^{-\sigma} s^{\sigma} \widetilde{f}(s, u(s))ds$$

$$\geq \delta_0 \int_0^1 \left[G(t,s) + \frac{t^{\alpha-\mu_N-1}\sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \right.$$

$$\left.\times H(\xi_j, s)\right] \times s^{-\sigma} s^{\sigma} \widetilde{g}(s, u(s))ds$$

$$= \delta_0 \int_0^1 \left[G(t,s) + \frac{t^{\alpha-\mu_N-1}\sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \right.$$

$$\left.\times H(\xi_j, s)\right] \times \widetilde{g}(s, u(s))ds = \delta_0 Bu(t).$$

Hence, $Au \geq \delta_0 Bu$ for all $u \in P$. Now by using Theorem 1.3, the operator equation $Au + Bu = u$ has a unique solution $u^* \in P_k$. Moreover, the sequence $u_n = Au_{n-1} + Bu_{n-1}$ for $n \geq 1$ with initial value $u_0 \in P_k$ converges to u^*. This implies that the problem (1.2) has the unique positive solution $u^* \in P_k$. Thus, $I^{\mu_N} u^*$ is a unique positive solution for the problem (1.1). □

Here, We list another similar result by using different conditions and Theorem 1.2 which we omit its proof.

Theorem 2.2. *Suppose that $g = 0$, $\beta_0 = 0$, $0 < \sigma < 1$, $f : (0,1] \times \mathbb{R}^{N+2} \to [0,\infty)$ is continuous, $\lim_{t\to 0^+} f(t,.,.,\ldots,.) = +\infty$ and the map $t^\sigma f(t, y_1, y_2, \ldots, y_{N+2})$ is continuous on $[0,1] \times \mathbb{R}^{N+2}$. Assume there exist positive constants $\vartheta_1, \ldots, \vartheta_L$ such that*

$$0 \leq h(t,s,x_0,x_1,\ldots,x_L) - h(t,s,y_0,y_1,\ldots,y_L) \leq \sum_{j=0}^{L} \vartheta_j (x_j - y_j)$$

for all $t, s \in [0,1]$ and $x_j, y_j \in [0,\infty)$ with $x_j \geq y_j$ ($0 \leq j \leq L$). Then the problem (1.1) has a unique positive solution if there exist $p_1, \ldots, p_{N+2} > 0$ such that $K\left(\sum_{i=1}^{N+2} p_i\right) \leq 1$ and there exist $0 < \theta_2 \leq \left(\gamma_0 \sum_{j=0}^{L} \vartheta_j \frac{1}{\Gamma(\mu_N - \beta_j + 1)}\right)^{-1}$, $0 < \theta_1 \leq \Gamma(\mu_N + 1)$, $0 < \theta_{i+2} \leq \Gamma(\mu_N - \mu_i + 1)$ for $1 \leq i \leq N-1$ and $0 < \theta_{N+2} \leq 1$ such that $0 \leq t^\sigma \Big(f(t, u_1, u_2, \ldots, u_{N+2}) - f(t, v_1, v_2, \ldots, v_{N+2}) \Big) \leq \sum_{i=1}^{N+2} p_i \phi(\theta_i(u_i - v_i))$ for all $t \in [0,1]$ and $u_i, v_i \in [0,\infty)$ with $u_i \geq v_i$ ($1 \leq i \leq N+2$), where $\phi : [0,\infty) \to [0,\infty)$ is a nondecreasing continuous map such that $\psi : [0,\infty) \to [0,\infty)$ is nondecreasing, $\psi(0) = 0$ and ψ is positive on $(0,\infty)$. Here, $\psi(t) = t - \phi(t)$.

Now, we give the following example to illustrate our main result.

Example 2.1. *Consider the singular boundary value problem*

$$D^{\frac{36}{5}} u(t) = \frac{1}{\sqrt{t}} \left[|u(t)|^{\frac{1}{2}} + p(t) \left(\frac{|\varphi u(t) + u(t)|}{1 + |\varphi u(t) + u(t)|} \right)^{\frac{1}{3}} + q(t) \left(\arctan |D^{\frac{1}{4}} u(t)| \right)^{\frac{1}{4}} \right.$$

$$+ \left(|D^{\frac{10}{9}} u(t)|^3 + |D^{\frac{31}{10}} u(t)| \right)^{\frac{1}{12}} + \arctan(|u(t) + D^{\frac{1}{4}} u(t)|) + r(t) \frac{|\varphi u(t)|}{1 + |\varphi u(t)|} + s(t)$$

$$\left. \times \frac{|D^{\frac{1}{4}} u(t) + D^{\frac{10}{9}} u(t)|}{1 + |D^{\frac{1}{4}} u(t) + D^{\frac{10}{9}} u(t)|} + \ln \left(1 + \left(|D^{\frac{10}{9}} u(t)|^3 + |D^{\frac{31}{10}} u(t)| \right)^{\frac{1}{12}} \right) + a(t) + \frac{\pi}{2} + b \right] \quad (2.1)$$

with boundary conditions $D^{\frac{1}{4}}u(0) = D^{\frac{10}{9}}u(0) = D^{\frac{31}{10}}u(0) = D^{\frac{41}{10}}u(0) = D^{\frac{51}{10}}u(0) = D^{\frac{61}{10}}u(0) = D^{\frac{1}{7}}u(0) = D^{\frac{4}{3}}u(0) = D^{\frac{21}{10}}u(0) = 0$ *and* $D^{\frac{61}{10}}u(1) = \frac{\pi}{4}D^{\frac{61}{10}}u(\frac{1}{13}) + \frac{1}{17}D^{\frac{61}{10}}u(\frac{1}{7}) + \frac{e^{-3}}{2}D^{\frac{61}{10}}u(\frac{3}{4}) + \frac{3}{19}D^{\frac{61}{10}}u(\frac{5}{6}) + \frac{4}{21}D^{\frac{61}{10}}u(\frac{7}{8})$*, where* $b > 0$ *is a constant,* $a, p, q, r, s : [0,1] \to [0,\infty)$ *are continuous,* $(\varphi u)(t) = \int_0^t \frac{(t-s)^5}{\sqrt{1+s^2}}\left[\frac{e^{-t^3}\sin^2 s}{\sqrt{1+t^4}} + \frac{(3+\cos(s^3))}{1+s^2} \times \left(\ln(1+|u(s)|) + |u(s)|^{\alpha_1}|D^{\frac{1}{7}}u(s)|^{\alpha_2} |D^{\frac{4}{3}}u(s)|^{\alpha_3}|D^{\frac{21}{10}}u(s)|^{\alpha_4} + \left(b_1|u(s)|^p + b_2|D^{\frac{1}{7}}u(s)|^p + b_3|D^{\frac{4}{3}}u(s)|^p + b_4|D^{\frac{21}{10}}u(s)|^p\right)^{\frac{1}{p}}\right)\right]ds$, α_i, $b_i \geq 0$ $(1 \leq i \leq 4)$, $\sum_{i=1}^4 \alpha_i \leq 1$, $p > 0$, $\alpha = \frac{36}{5}$, $N = 3$, $\mu_1 = \frac{1}{4}$, $\mu_2 = \frac{10}{9}$, $\mu_3 = \frac{31}{10}$, $L = 3$, $\beta_1 = \frac{1}{7}$, $\beta_2 = \frac{4}{3}$, $\beta_3 = \frac{21}{10}$, $a_1 = \frac{\pi}{4}$, $a_2 = \frac{1}{17}$, $a_3 = \frac{e^{-3}}{2}$, $a_4 = \frac{3}{19}$, $a_5 = \frac{4}{21}$, $\xi_1 = \frac{1}{13}$, $\xi_2 = \frac{1}{7}$, $\xi_3 = \frac{3}{4}$, $\xi_4 = \frac{5}{6}$, $\xi_5 = \frac{7}{8}$. *Put* $f(t,x_1,x_2,x_3,x_4,x_5) = \frac{1}{\sqrt{t}}\left[|x_1|^{\frac{1}{2}} + p(t)\left(\frac{|x_2+x_1|}{1+|x_2+x_1|}\right)^{\frac{1}{3}} + q(t)\left(\arctan|x_3|\right)^{\frac{1}{4}} + \left(|x_4|^3 + |x_5|\right)^{\frac{1}{12}} + a(t) + \frac{\pi}{2} + c\right]$,

$$g(t,x_1,x_2,x_3,x_4,x_5) = \frac{1}{\sqrt{t}}\left[\arctan(|x_1+x_3|) + r(t)\frac{|x_2|}{1+|x_2|}\right.$$
$$\left. + s(t)\frac{|x_3+x_4|}{1+|x_3+x_4|} + \ln\left(1 + \left(|x_4|^3 + |x_5|\right)^{\frac{1}{12}}\right) + b - c\right]$$

and $h(t,s,y_1,y_2,y_3,y_4) = \frac{e^{-t^3}\sin^2 s}{\sqrt{1+t^4}} + \frac{(3+\cos(s^3))}{1+s^2}\left(\ln(1+|y_1|) + |y_1|^{\alpha_1}|y_2|^{\alpha_2}|y_3|^{\alpha_3}|y_4|^{\alpha_4} + (b_1|y_1|^p + b_2|y_2|^p + b_3|y_3|^p + b_4|y_4|^p)^{\frac{1}{p}}\right)$ *for all* $x_i, y_j \in \mathbb{R}$ $(1 \leq i \leq 5, 1 \leq j \leq 4)$ *and* $t, s \in [0,1]$. *Let* $\sigma = \frac{1}{2}$, $\gamma = \frac{1}{2}$, $r_{\max} = \max\{r(t) : t \in [0,1]\}$, $s_{\max} = \max\{s(t) : t \in [0,1]\}$ *and* $0 < c \leq \frac{b}{2}$. *It is easy to check that the maps* $t^\sigma f(t,.,.,\ldots,.)$, $t^\sigma g(t,.,.,\ldots,.)$ *and* $h(t,s,.,.,\ldots,.)$ *are increasing with respect to their components on* $[0,\infty)$ *for all* $t,s \in [0,1]$ *and* $t^\sigma g(t,0,0,\ldots,0) = b - c > 0$. *Also, we have*

$$t^\sigma g(t,\lambda x_1,\lambda x_2,\lambda x_3,\lambda x_4,\lambda x_5) = \arctan(\lambda(x_1+x_3)) + r(t)\frac{\lambda x_2}{1+\lambda x_2} + s(t)\frac{\lambda(x_3+x_4)}{1+\lambda(x_3+x_4)}$$
$$+ \ln\left(1 + \left(\lambda^3 x_4^3 + \lambda x_5\right)^{\frac{1}{12}}\right) + b - c$$
$$\geq \lambda\left(\arctan(x_1+x_3) + r(t)\frac{x_2}{1+x_2} + s(t)\frac{x_3+x_4}{1+x_3+x_4} + \ln\left(1 + \left(x_4^3 + x_5\right)^{\frac{1}{12}}\right) + b - c\right)$$
$$= \lambda t^\sigma g(t,x_1,x_2,x_3,x_4,x_5),$$

$$t^\sigma f(t,\lambda x_1,\lambda x_2,\lambda x_3,\lambda x_4,\lambda x_5) = \lambda^{\frac{1}{2}}x_1^{\frac{1}{2}} + p(t)\left(\frac{\lambda(x_2+x_1)}{1+\lambda(x_2+x_1)}\right)^{\frac{1}{3}} + q(t)\left(\arctan(\lambda x_3)\right)^{\frac{1}{4}}$$
$$+ \left(\lambda^3 x_4^3 + \lambda x_5\right)^{\frac{1}{12}} + a(t) + \frac{\pi}{2} + c$$
$$\geq \lambda^{\frac{1}{2}}\left(x_1^{\frac{1}{2}} + p(t)\left(\frac{x_2+x_1}{1+x_2+x_1}\right)^{\frac{1}{3}} + q(t)\left(\arctan(x_3)\right)^{\frac{1}{4}} + \left(x_4^3 + x_5\right)^{\frac{1}{12}} + a(t) + \frac{\pi}{2} + c\right)$$

$= \lambda^\gamma t^\sigma f(t,x_1,x_2,x_3,x_4,x_5)$ *and*

$$h(t,s,\lambda y_1,\lambda y_2,\lambda y_3,\lambda y_4) = \frac{e^{-t^3}\sin^2 s}{\sqrt{1+t^4}} + \frac{(3+\cos(s^3))}{1+s^2}\left(\ln(1+\lambda y_1)\right)$$
$$\left. + \lambda^{(\sum_{i=1}^4 \alpha_i)} y_1^{\alpha_1}y_2^{\alpha_2}y_3^{\alpha_3}y_4^{\alpha_4} + \lambda\left(b_1 y_1^p + b_2 y_2^p + b_3 y_3^p + b_4 y_4^p\right)^{\frac{1}{p}}\right) \geq \lambda\left(\frac{e^{-t^3}\sin^2 s}{\sqrt{1+t^4}}\right.$$
$$+ \frac{(3+\cos(s^3))}{1+s^2}\left(\ln(1+y_1) + y_1^{\alpha_1}y_2^{\alpha_2}y_3^{\alpha_3}y_4^{\alpha_4} + \left(b_1 y_1^p + b_2 y_2^p + b_3 y_3^p + b_4 y_4^p\right)^{\frac{1}{p}}\right)$$

$= \lambda h(t,s,y_1,y_2,y_3,y_4)$ *for all* $t,s \in [0,1]$, $\lambda \in (0,1)$ *and* $x_i, y_j \in [0,\infty)$ $(1 \le i \le 5, 1 \le j \le 4)$. *If* δ_0 *belongs to* $(0, \frac{c}{r_{\max}+s_{\max}+b-c}]$, *then we get*

$$t^\sigma f(t,x_1,x_2,x_3,x_4,x_5) = x_1^{\frac{1}{2}} + p(t)\left(\frac{x_2+x_1}{1+x_2+x_1}\right)^{\frac{1}{3}}$$

$$+q(t)\left(\arctan(x_3)\right)^{\frac{1}{4}} + \left(x_4^3+x_5\right)^{\frac{1}{12}} + a(t) + \frac{\pi}{2} + c \ge \frac{\pi}{2} + (x_4^3+x_5)^{\frac{1}{12}} + c$$

$$\ge \arctan(x_1+x_3) + \ln(1+(x_4^3+x_5)^{\frac{1}{12}}) + \frac{c}{r_{\max}+s_{\max}+b-c} \times (r_{\max}+s_{\max}+b-c)$$

$$\ge \delta_0\Bigg(\arctan(x_1+x_3) + r(t)\frac{x_2}{1+x_2} + s(t)\frac{x_3+x_4}{1+x_3+x_4} + \ln\left(1+\left(x_4^3+x_5\right)^{\frac{1}{12}}\right) + b - c\Bigg)$$

$$= \delta_0 t^\sigma g(t,x_1,x_2,x_3,x_4,x_5).$$

Now by using Theorem 2.1, we get the problem (2.1) *has a unique positive solution.*

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