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*Research Article*

# $n$-Tuplet Coincidence Point Theorems in Partially Ordered Probabilistic Metric Spaces

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In this paper, we investigate existence of $n$-tuplet coincidence point theorems in partially ordered probabilistic metric spaces. Also, we gave uniqueness of $n$-tuplet fixed point theorems in this space.

## 1. Introduction

Probabilistic metric spaces were introduced by Menger in his fundamental paper [1] in 1942. In Menger's theory, the notion of distance between two points $x$ and $y$ is replaced by a distribution function $F_{x,y}$. The value of $F_{x,y}$ at any point $t$ is represented by $F_{x,y}(t)$. $F_{x,y}(t)$ is interpreted as probability that the distance between $x$ and $y$ is less than $t$. Sehgal first introduced the notion of contraction mapping on a probabilistic metric space in [2, 3]. In fact the study of such spaces received an impetus with the pioneering works of Schewizer and Sklar [4]. Then several authors have studied probabilistic spaces; see [5–9].

Guo and Lakshmikantham initiated the concept of coupled fixed point [10] in 1987. After that, Bhaskar and Lakshmikantham introduced the notion of mixed monotone property and gave some coupled fixed point theorems in ordered metric spaces in 2006 [11]. In 2012, the extention of coupled fixed point theorems to tripled fixed point theorems for nonlinear mapping in partially ordered metric space was introduced by Berinde and Borcut [12]. Then, some coupled and tripled fixed point results were obtained by many authors [13–19]. In 2013, Ertürk and Karakaya gave the concept of $n$-tuplet fixed point theorems in partially ordered metric spaces [20]. Alam, Imdad, and Ali unified $n$-tuplet fixed point results in ordered metric space in 2016 [21]. Their survey article is recommended to someone who wants to have details about this theory.

Hu and Ma studied couple coincidence point theorems in partially ordered probabilistic metric spaces in [22]. Recently, Binayak S. et al. [23] gave tripled coincidence point results in partially ordered probabilistic metric spaces.

Inspired by the above studies, we introduce $n$-tuplet fixed point theorems in partially ordered probabilistic metric spaces. This paper is organized as follows. Section 2 is devoted to giving some preliminaries. In Section 3, we obtain existence of $n$-tuplet fixed point theorems. Finally, Section 4 concerns the uniqueness of fixed point. These are the extensions of coupled and tripled fixed points in partially ordered probabilistic metric spaces.

## 2. Preliminaries

*Definition 1* (see [4]). A triangular norm (shorter $\Delta -$ norm$/t-$norm) is a binary operation $\Delta$ on the interval $[0, 1]$,

$$\Delta : [0, 1] \times [0, 1] \longrightarrow [0, 1] \tag{1}$$

that satisfies the following conditions:

(a) $\Delta(a, 1) = a$, $\Delta(0, 0) = 0$,

(b) $\Delta(a, b) = \Delta(b, a)$,

(c) $\Delta(c, d) \geq \Delta(a, b)$ for $c \geq a$, $d \geq b$,

(d) For all $a, b, c \in [0, 1]$, $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$.

Principal examples of $\Delta -$ norms are

(i) $\Delta_M(a,b) = \min(a,b)$,

(ii) $\Delta_P(a,b) = a.b$,

(iii) $\Delta_L(a,b) = \max(a+b-1,0)$,

(iv) $\Delta_D(a,b) = \left\{\begin{smallmatrix}\min(a,b), & \text{if } \max(a,b)=1\\ 0, & \text{otherwise.}\end{smallmatrix}\right.$

Above $\Delta -$ norms have the following relations:

$$\Delta_D < \Delta_L < \Delta_P < \Delta_M. \tag{2}$$

*Definition 2* (see [4]). A function $F : \mathbb{R} \longrightarrow \mathbb{R}^+$ is called a distribution function if it is nondecreasing, left continuous with

$$\begin{gathered}\inf\{F(t) : t \in \mathbb{R}\} = 0\\ \text{and } \sup\{F(t) : t \in \mathbb{R}\} = 1.\end{gathered} \tag{3}$$

In addition, if $F(0) = 0$, then $F$ is called a distance distribution function. We denote the set of all distance distribution functions by $L^+$ and $H$ is a specific distance distribution function (also known as Heaviside function) defined by

$$H(t) = \begin{cases}0, & t \le 0\\ 1, & t > 0.\end{cases} \tag{4}$$

*Definition 3* (see [1]). Let $X$ be a nonempty set, $F$ be a mapping defined on $X \times X$ into $L^+$, and $\Delta$ be a $t-$norm. If the following conditions are satisfied, $(X,F,\Delta)$ is called Menger probabilistic metric spaces:

(1) $F_{x,y}(t) = H(t)$ for all $t > 0$ if and only if $x = y$;

(2) $F_{x,y}(t) = F_{y,x}(t)$, for all $t > 0$ and $x, y \in X$;

(3) $F_{x,y}(t+s) \ge \Delta(F_{x,z}(t), F_{z,y}(s)), \forall t, s \ge 0, x, y, z \in X$.

*Example 4.* Let $\Delta$ be an arbitrary $t-$norm, $X = [-1,1]$, and

$$F_{x,y}(t) = \begin{cases}e^{-|x-y|/t}, & t > 0\\ 0, & t = 0\end{cases} \tag{5}$$

for every $x, y \in X$. Then $(X,F,\Delta)$ is a Menger Probabilistic Metric (for short PM) space given in [24].

*Definition 5* (see [4, 25]). Let $(X,F,\Delta)$ be a Menger space.

(i) A sequence $(x_n)$ in $X$ is said to be convergent to a point $x \in X$ if, for every $> 0$, $\lim_{n\longrightarrow\infty} F_{x_n,x}(t) = 1$.

(ii) A sequence $(x_n)$ in $X$ is called Cauchy sequence if, for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in N$ $F_{x_n,x_m}(t) \ge 1-\varepsilon$ for each $n, m \ge n_0$.

(iii) A Menger space in which every Cauchy sequence is convergent is said to be complete.

**Lemma 6** (see [26]). *If $(X,F,\Delta)$ be a Menger space where $\Delta$ is continuous $t$-norm, then for every fixed $t > 0$, if $(x_n) \longrightarrow x$, $(y_n) \longrightarrow y$, then $\lim_{n\longrightarrow\infty} F_{x_n,y_n}(t) = F_{x,y}(t)$.*

**Lemma 7** (see [23]). *If $h : R^n \longrightarrow R$ is continuous and $(a_{\alpha,\beta})_{\alpha=1}^{\infty}, \beta = 1,2,\ldots,n$ are such that $\liminf_{\alpha\longrightarrow\infty} a_{\alpha,\gamma} = a_{\alpha,\gamma}$ for all $\gamma \neq m$ for some $m$ and $(a_{\alpha,\alpha})_{\alpha=1}^{\infty}$ is bounded, then, $\liminf_{\alpha\longrightarrow\infty} h(a_{\alpha,1}, a_{\alpha,2}, \ldots, a_{\alpha,n}) = h(a_1, a_2, \ldots, \liminf_{\alpha\longrightarrow\infty} a_{\alpha,m}, \ldots, a_n)$.*

Let $R^+ = [0,\infty)$ and $\Phi = \{\phi : \phi : R^+ \longrightarrow R^+\}$. Each $\phi \in \Phi$ satisfies the following conditions:

(1) $\phi$ is strict increasing,

(2) $\phi$ is upper semicontinuous from the right,

(3) $\forall t > 0$, $\sum_{n=0}^{\infty} \phi^n(t) < \infty$.

If $\phi \in \Phi$, $\phi(t) \le t$ for all $t > 0$.

**Lemma 8** (see [27]). *Let $(u_n)$ be sequence in a Menger space $(X,F,\Delta)$, where $\Delta$ is a minimum $t$-norm. If there exists a function $\phi \in \Phi$ $\forall t > 0$ and $n \ge 1$,*

$$F_{u_n,u_{n+1}}(\phi(t)) \ge \min\left\{F_{u_{n-1},u_n}(t), F_{u_n,u_{n+1}}(t)\right\}. \tag{6}$$

*Then $(u_n)$ is a Cauchy sequence in $X$.*

**Lemma 9** (see [27]). *Let $(X,F,\Delta)$ be a Menger space. If there exists a function $\phi \in \Phi$ $\forall t > 0$ and $x, y \in X$, $F_{x,y}(\phi(t)+0) \ge F_{x,y}(t)$, then $x = y$.*

## 3. Main Results

*Definition 10* (see [20]). Let $(X,\le)$ be a partially ordered set and $G : X^n \longrightarrow X$. $G$ is said to have mixed monotone property if $G$ is monotone nondecreasing in its odd arguments and it is monotone nonincreasing in its even argument. That is, for any $x^1, x^2, x^3, \ldots, x^n \in X$,

$$\begin{gathered}y^1, z^1 \in X,\\ y^1 \le z^1 \Longrightarrow\\ G(y^1, x^2, x^3, \ldots, x^n) \le G(z^1, x^2, x^3, \ldots, x^n)\\ y^2, z^2 \in X,\\ y^2 \le z^2 \Longrightarrow\\ G(x^1, y^2, x^3, \ldots, x^n) \ge G(x^1, z^2, x^3, \ldots, x^n)\\ \vdots\\ y^n, z^n \in X,\end{gathered}$$

$$
\begin{gathered}
y^n \leq z^n \Longrightarrow \\
G\left(x^1, x^2, x^3, \ldots, y^n\right) \leq G\left(x^1, x^2, x^3, \ldots, z^n\right) \\
\text{(if } n \text{ odd)} \\
y^n, z^n \in X, \\
y^n \leq z^n \Longrightarrow \\
G\left(x^1, y^2, x^3, \ldots, x^n\right) \geq G\left(x^1, x^2, x^3, \ldots, z^n\right) \\
\text{(if } n \text{ even)}.
\end{gathered} \tag{7}
$$

*Definition 11* (see [20]). Let $(X, \leq)$ be a partially ordered set and $G : X^n \longrightarrow X, g : X \longrightarrow X$. We say that $G$ has the mixed $g$-monotone property if $G$ is monotone $g$-nondecreasing in its odd arguments and it is monotone $g$-nonincreasing in its even argument. That is,

for any $x^1, x^2, x^3, \ldots, x^n \in X$

$$
\begin{gathered}
y^1, z^1 \in X, \\
g\left(y^1\right) \leq g\left(z^1\right) \Longrightarrow \\
G\left(y^1, x^2, x^3, \ldots, x^n\right) \leq G\left(z^1, x^2, x^3, \ldots, x^n\right) \\
y^2, z^2 \in X, \\
g\left(y^2\right) \leq g\left(z^2\right) \Longrightarrow \\
G\left(x^1, y^2, x^3, \ldots, x^n\right) \geq G\left(x^1, z^2, x^3, \ldots, x^n\right) \\
\vdots \\
y^n, z^n \in X, \\
g\left(y^n\right) \leq g\left(z^n\right) \Longrightarrow \\
G\left(x^1, x^2, x^3, \ldots, y^n\right) \leq G\left(x^1, x^2, x^3, \ldots, z^n\right) \\
\text{(if } n \text{ odd)} \\
y^n, z^n \in X, \\
g\left(y^n\right) \leq g\left(z^n\right) \Longrightarrow \\
G\left(x^1, y^2, x^3, \ldots, x^n\right) \geq G\left(x^1, x^2, x^3, \ldots, z^n\right) \\
\text{(if } n \text{ even)}.
\end{gathered} \tag{8}
$$

If $g$ is taken identity mapping Definition 11 reduces to Definition 10.

*Definition 12* (see [28]). Let $X \neq \emptyset$. An element $(x^1, x^2, x^3, \ldots, x^n) \in X^n$ is defined as a $n$–tuplet fixed point of the mapping $G : X^n \longrightarrow X$ if

$$
\begin{gathered}
G\left(x^1, x^2, x^3, \ldots, x^n\right) = x^1 \\
G\left(x^2, x^3, \ldots, x^n, x^1\right) = x^2 \\
\vdots \\
G\left(x^n, x^1, x^2, \ldots, x^{n-1}\right) = x^n.
\end{gathered} \tag{9}
$$

*Definition 13* (see [28]). Let $X \neq \emptyset$. An element $(x^1, x^2, x^3, \ldots, x^n) \in X^n$ is defined as a $n$–tuplet coincidence point of the mapping $G : X^n \longrightarrow X$ and $g : X \longrightarrow X$ if

$$
\begin{gathered}
G\left(x^1, x^2, x^3, \ldots, x^n\right) = g\left(x^1\right) \\
G\left(x^2, x^3, \ldots, x^n, x^1\right) = g\left(x^2\right) \\
\vdots \\
G\left(x^n, x^1, x^2, \ldots, x^{n-1}\right) = g\left(x^n\right).
\end{gathered} \tag{10}
$$

When $g$ is taken identity mapping Definition 13 reduces to Definition 12.

*Definition 14* (see [20]). Let $(X, \leq)$ be a partially ordered set and $G : X^n \longrightarrow X$ and $g : X \longrightarrow X$ is called commutative if for all $x^1, x^2, x^3, \ldots, x^n \in X$

$$
\begin{aligned}
& g\left(G\left(x^1, x^2, x^3, \ldots, x^n\right)\right) \\
& \quad = G\left(g\left(x^1\right), g\left(x^2\right), g\left(x^3\right), \ldots, g\left(x^n\right)\right).
\end{aligned} \tag{11}
$$

*Definition 15.* Let $(X, F, \Delta)$ be a Menger space. $G : X^n \longrightarrow X$ and $g : X \longrightarrow X$ are said to be compatible if for all $t > 0$

$$
\begin{gathered}
\lim_{k \longrightarrow \infty} F_{g(G(x_k^1, \ldots, x_k^n)), G(g(x_k^1), \ldots, g(x_k^n))}(t) = 1, \\
\lim_{k \longrightarrow \infty} F_{g(G(x_k^2, \ldots, x_k^n, x_k^1)), G(g(x_k^1), \ldots, g(x_k^n), g(x_k^1))}(t) = 1, \\
\vdots \\
\lim_{k \longrightarrow \infty} F_{g(G(x_k^n, x_k^1, \ldots, x_k^{n-1})), G(g(x_k^n), g(x_k^1), \ldots, g(x_k^{n-1}))}(t) = 1,
\end{gathered} \tag{12}
$$

whenever $(x_k^1), (x_k^2), \ldots, (x_k^n)$ sequences in $X$ such that

$$
\begin{gathered}
\lim_{k \longrightarrow \infty} G\left(x_k^1, x_k^2, \ldots, x_k^n\right) = \lim_{k \longrightarrow \infty} g\left(x_k^1\right) = x^1, \\
\lim_{k \longrightarrow \infty} G\left(x_k^2, \ldots, x_k^n, x_k^1\right) = \lim_{k \longrightarrow \infty} g\left(x_k^2\right) = x^2, \\
\vdots \\
\lim_{k \longrightarrow \infty} G\left(x_k^n, x_k^1, \ldots, x_k^{n-1}\right) = \lim_{k \longrightarrow \infty} g\left(x_k^n\right) = x^n.
\end{gathered} \tag{13}
$$

**Theorem 16.** *Let $(X, \leq)$ be a partially ordered set and $(X, F, \Delta)$ be a complete Menger space, where $\Delta$ is a minimum t-norm. Let $G : X^n \longrightarrow X, g : X \longrightarrow X$ be two mappings such that G has* *the mixed $g$-monotone property. Suppose there exist $\phi \in \Phi$ and $p \geq 0$ such that*

$$\begin{aligned} & F_{G(x^1,x^2,\ldots,x^n),G(y^1,y^2,\ldots,y^n)}(\phi(t)) + p\Big(1 \\ & \quad - \max\Big\{F_{g(x^1),G(y^1,y^2,\ldots,y^n)}(\phi(t)), \\ & \quad F_{g(y^1),G(x^1,x^2,\ldots,x^n)}(\phi(t))\Big\}\Big) \geq \min\Big\{F_{g(x^1),g(y^1)}(t), \\ & \quad F_{g(x^1),G(x^1,x^2,\ldots,x^n)}(t), F_{g(y^1),G(y^1,y^2,\ldots,y^n)}(t)\Big\} \end{aligned} \tag{14}$$

*for all $t > 0$, $x^1, x^2, \ldots, x^n, y^1, y^2, \ldots, y^n \in X$ with*

$$\begin{gathered} g(x^1) \geq g(y^1) \\ g(x^2) \leq g(y^2) \\ \vdots \\ g(x^n) \geq g(y^n) \quad (\text{if } n \text{ is odd}) \\ g(x^n) \leq g(y^n) \quad (\text{if } n \text{ is even}) \end{gathered} \tag{15}$$

*Assume that $g$ is continuous, monotonic increasing, compatible with $G$ such that $G(X^n) \subseteq g(X)$ and suppose either*

(a) *$G$ is continuous or*

(b) *$X$ has the following property:*

(i) *if nondecreasing sequence $(x_n) \longrightarrow x$, then $x_n \leq x$ for all $n \geq 0$,*

(ii) *if nondecreasing sequence $(y_n) \longrightarrow y$, then $y_n \leq y$ for all $n \geq 0$.*

*If there exist $x_0^1, x_0^2, \ldots, x_0^n \in X$ such that*

$$\begin{gathered} g(x_0^1) \leq G(x_0^1, x_0^2, \ldots, x_0^n) \\ g(x_0^2) \geq G(x_0^2, x_0^3, \ldots, x_0^n, x_0^1) \\ \vdots \\ g(x_0^n) \leq G(x_0^n, x_0^1, x_0^2, \ldots, x_0^{n-1}) \quad (\text{if } n \text{ is odd}) \\ g(x_0^n) \geq G(x_0^n, x_0^1, x_0^2, \ldots, x_0^{n-1}) \quad (\text{if } n \text{ is even}) \end{gathered} \tag{16}$$

*then there exist $x^1, x^2, \ldots, x^n \in X$ such that*

$$\begin{gathered} G(x^1, x^2, x^3, \ldots, x^n) = g(x^1), \\ G(x^2, x^3, \ldots, x^n, x^1) = g(x^2), \\ \vdots \\ G(x^n, x^1, x^2, \ldots, x^{n-1}) = g(x^n). \end{gathered} \tag{17}$$

*That is, $G$ and $g$ have a $n$-tuplet coincidence point.*

*Proof.* Let $x_0^1, x_0^2, \ldots, x_0^n \in X$ satisfy condition (16). Since $G(X^n) \subset g(X)$, we can define $(x_k^1), (x_k^2), \ldots, (x_k^n) \in X$ sequences as follows:

for $k = 1, 2, 3, \ldots$,

$$\begin{gathered} g(x_k^1) = G(x_{k-1}^1, x_{k-1}^2, \ldots, x_{k-1}^n) \\ g(x_k^2) = G(x_{k-1}^2, x_{k-1}^3, \ldots, x_{k-1}^n, x_{k-1}^1) \\ \vdots \\ g(x_k^n) = G(x_{k-1}^n, x_{k-1}^1, \ldots, x_{k-1}^{n-1}). \end{gathered} \tag{18}$$

Next step, we show that, for all $k \geq 1$,

$$\begin{gathered} g(x_{k-1}^1) \leq g(x_k^1) \\ g(x_{k-1}^2) \geq g(x_k^2) \\ \vdots \\ g(x_{k-1}^n) \leq g(x_k^n) \quad (\text{if } n \text{ is odd}) \\ g(x_{k-1}^n) \geq g(x_k^n) \quad (\text{if } n \text{ is even}). \end{gathered} \tag{19}$$

To prove this claim, we will use the inductive method for mathematics. Because of the inequalities in (16), (19) holds for $k = 1$. We have

$$\begin{gathered} g(x_0^1) \leq G(x_0^1, x_0^2, \ldots, x_0^n) = g(x_1^1), \\ g(x_0^2) \geq G(x_0^2, x_0^3, \ldots, x_0^n, x_0^1) = g(x_1^2), \\ \vdots \\ g(x_0^n) \leq G(x_0^n, x_0^1, x_0^2, \ldots, x_0^{n-1}) = g(x_1^n) \quad (\text{if } n \text{ is odd}), \\ g(x_0^n) \geq G(x_0^n, x_0^1, x_0^2, \ldots, x_0^{n-1}) = g(x_1^n) \quad (\text{if } n \text{ is even}). \end{gathered} \tag{20}$$

Let us assume that (19) is true for $k = m$; that is,

$$\begin{gathered} g(x_{m-1}^1) \leq g(x_m^1), \\ g(x_{m-1}^2) \geq g(x_m^2), \\ \vdots \\ g(x_{m-1}^n) \leq g(x_m^n) \quad (\text{if } n \text{ is odd}) \\ g(x_{m-1}^n) \geq g(x_m^n) \quad (\text{if } n \text{ is even}). \end{gathered} \tag{21}$$

Since $G$ has the mixed $g$-monotone property and from (21),

$$\begin{aligned} g(x_m^1) &= G(x_{m-1}^1, x_{m-1}^2, \ldots, x_{m-1}^n) \\ &\leq G(x_m^1, x_{m-1}^2, \ldots, x_{m-1}^n) \\ &\leq G(x_m^1, x_m^2, x_{m-1}^3, \ldots, x_{m-1}^{n-1}, x_{m-1}^n) \\ &\ \ \vdots \\ &\leq G(x_m^1, x_m^2, \ldots, x_m^{n-1}, x_m^n) = g(x_{m+1}^1). \end{aligned} \tag{22}$$

Similar way we get

$$\begin{aligned} g\left(x_m^2\right) &= G\left(x_{m-1}^2, \ldots, x_{m-1}^n, x_{m-1}^1\right) \\ &\geq G\left(x_m^2, x_{m-1}^3, \ldots, x_{m-1}^n, x_{m-1}^1\right) \\ &\geq G\left(x_m^2, x_m^3, x_{m-1}^4, \ldots, x_{m-1}^n, x_{m-1}^1\right) \\ &\vdots \\ &\geq G\left(x_m^2, x_m^3, \ldots, x_m^n, x_m^1\right) = g\left(x_{m+1}^2\right). \\ &\vdots \\ g\left(x_m^n\right) &\leq G\left(x_m^n, x_m^1, \ldots, x_m^{n-1}\right) = g\left(x_{m+1}^n\right) \quad \text{(if } n \text{ is odd)} \\ g\left(x_m^n\right) &\geq G\left(x_m^n, x_m^1, \ldots, x_m^{n-1}\right) = g\left(x_{m+1}^n\right) \quad \text{(if } n \text{ is even)}. \end{aligned} \tag{23}$$

So, inequalities in (19) are true for all $k \geq 1$.

$$\begin{aligned} &\cdots \geq g\left(x_k^1\right) \geq g\left(x_{k-1}^1\right) \geq \cdots \geq g\left(x_1^1\right) \geq g\left(x_0^1\right) \\ &\cdots \leq g\left(x_k^2\right) \leq g\left(x_{k-1}^2\right) \leq \cdots \leq g\left(x_1^2\right) \leq g\left(x_0^2\right) \\ &\vdots \\ &\cdots \geq g\left(x_k^n\right) \geq g\left(x_{k-1}^n\right) \geq \cdots \geq g\left(x_1^n\right) \geq g\left(x_0^n\right) \quad \text{(if } n \text{ is odd)} \\ &\cdots \leq g\left(x_k^n\right) \leq g\left(x_{k-1}^n\right) \leq \cdots \leq g\left(x_1^n\right) \leq g\left(x_0^n\right) \quad \text{(if } n \text{ is even)}. \end{aligned} \tag{24}$$

Now, we will show that $(g(x_k^1)), (g(x_k^2)), \ldots, (g(x_k^n))$ are Cauchy sequences.

For all $t > 0, n \geq 1$,

$$\begin{aligned} &F_{g(x_k^1),g(x_{k+1}^1)}(\phi(t)) \\ &\quad = F_{G(x_{k-1}^1,x_{k-1}^2,\ldots,x_{k-1}^n),G(x_k^1,x_k^2,\ldots,x_k^n)}(\phi(t)) \\ &\geq \min\left\{F_{g(x_{k-1}^1),g(x_k^1)}(t), F_{g(x_{k-1}^1),G(x_{k-1}^1,x_{k-1}^2,\ldots,x_{k-1}^n)}(t),\right. \\ &\quad \left.F_{g(x_k^1),G(x_k^1,x_k^2,\ldots,x_k^n)}(t)\right\} \\ &- p\left(1 - \max\left\{F_{g(x_{k-1}^1),G(x_k^1,x_k^2,\ldots,x_k^n)}(\phi(t)),\right.\right. \\ &\quad \left.\left.F_{g(x_k^1),G(x_{k-1}^1,x_{k-1}^2,\ldots,x_{k-1}^n)}(\phi(t))\right\}\right) \\ &= \min\left\{F_{g(x_{k-1}^1),g(x_k^1)}(t), F_{g(x_{k-1}^1),g(x_k^1)}(t),\right. \\ &\quad \left.F_{g(x_k^1),g(x_{k+1}^1)}(t)\right\} \\ &- p\left(1 - \max\left\{F_{g(x_{k-1}^1),g(x_{k+1}^1)}(\phi(t)),\right.\right. \\ &\quad \left.\left.F_{g(x_k^1),g(x_k^1)}(\phi(t))\right\}\right) \\ &F_{g(x_k^1),g(x_{k+1}^1)}(\phi(t)) \geq \min\left\{F_{g(x_{k-1}^1),g(x_k^1)}(t),\right. \\ &\quad \left.F_{g(x_k^1),g(x_{k+1}^1)}(t)\right\} - p(1-1) \\ &F_{g(x_k^1),g(x_{k+1}^1)}(\phi(t)) \geq \min\left\{F_{g(x_{k-1}^1),g(x_k^1)}(t),\right. \\ &\quad \left.F_{g(x_k^1),g(x_{k+1}^1)}(t)\right\}. \end{aligned} \tag{25}$$

By Lemma 8, we obtain that $(g(x_k^1))$ is a Cauchy sequence.

$$\begin{aligned} &F_{g(x_k^2),g(x_{k+1}^2)}(\phi(t)) \\ &\quad = F_{G(x_{k-1}^2,x_{k-1}^3,\ldots,x_{k-1}^n,x_{k-1}^1),G(x_k^2,x_k^3,\ldots,x_k^n,x_k^1)}(\phi(t)) \\ &\geq \min\left\{F_{g(x_{k-1}^2),g(x_k^2)}(t), F_{g(x_{k-1}^2),G(x_{k-1}^2,x_{k-1}^3,\ldots,x_{k-1}^n,x_{k-1}^1)}(t),\right. \\ &\quad \left.F_{g(x_k^2),G(x_k^2,x_k^3,\ldots,x_k^n,x_k^1)}(t)\right\} \\ &- p\left(1 - \max\left\{F_{g(x_{k-1}^2),G(x_k^2,x_k^3,\ldots,x_k^n,x_k^1)}(\phi(t)),\right.\right. \\ &\quad \left.\left.F_{g(x_k^2),G(x_{k-1}^2,x_{k-1}^3,\ldots,x_{k-1}^n,x_{k-1}^1)}(\phi(t))\right\}\right) \\ &= \min\left\{F_{g(x_{k-1}^2),g(x_k^2)}(t), F_{g(x_{k-1}^2),g(x_k^2)}(t),\right. \\ &\quad \left.F_{g(x_k^2),g(x_{k+1}^2)}(t)\right\} \\ &- p\left(1 - \max\left\{F_{g(x_{k-1}^2),g(x_{k+1}^2)}(\phi(t)),\right.\right. \\ &\quad \left.\left.F_{g(x_k^2),g(x_k^2)}(\phi(t))\right\}\right) \\ &F_{g(x_k^2),g(x_{k+1}^2)}(\phi(t)) \geq \min\left\{F_{g(x_{k-1}^2),g(x_k^2)}(t),\right. \\ &\quad \left.F_{g(x_k^2),g(x_{k+1}^2)}(t)\right\} - p(1-1) \\ &F_{g(x_k^2),g(x_{k+1}^2)}(\phi(t)) \geq \min\left\{F_{g(x_{k-1}^2),g(x_k^2)}(t),\right. \\ &\quad \left.F_{g(x_k^2),g(x_{k+1}^2)}(t)\right\}. \end{aligned} \tag{26}$$

We obtain that $(g(x_k^2))$ is a Cauchy sequence.

Using same way we conclude that $(g(x_k^3)), \ldots, (g(x_k^n))$ are also Cauchy sequences. Since $X$ is complete metric space, there exist $x^1, x^2, \ldots, x^n \in X$ such that

$$\begin{aligned} \lim_{k\longrightarrow\infty} g\left(x_k^1\right) &= x^1, \\ \lim_{k\longrightarrow\infty} g\left(x_k^2\right) &= x^2, \\ &\vdots \\ \lim_{k\longrightarrow\infty} g\left(x_k^n\right) &= x^n. \end{aligned} \tag{27}$$

Since $g$ is continuous, we can write

$$\begin{aligned} \lim_{k\longrightarrow\infty} g\left(g\left(x_k^1\right)\right) &= g\left(x^1\right), \\ \lim_{k\longrightarrow\infty} g\left(g\left(x_k^2\right)\right) &= g\left(x^2\right), \\ &\vdots \\ \lim_{k\longrightarrow\infty} g\left(g\left(x_k^n\right)\right) &= g\left(x^n\right). \end{aligned} \tag{28}$$

As $(G, g)$ is compatible and $g$ is continuous,

$$\begin{aligned}
g(x^1) &= \lim_{k\to\infty} g(g(x^1_{k+1})) \\
&= \lim_{k\to\infty} g(G(x^1_k, x^2_k, \dots, x^n_k)) \\
&= \lim_{k\to\infty} G(g(x^1_k), g(x^2_k), \dots, g(x^n_k)), \\
g(x^2) &= \lim_{k\to\infty} g(g(x^2_{k+1})) \\
&= \lim_{k\to\infty} g(G(x^2_k, \dots, x^n_k, x^1_k)) \\
&= \lim_{k\to\infty} G(g(x^2_k), \dots, g(x^n_k), g(x^1_k)), \\
&\vdots \\
g(x^n) &= \lim_{k\to\infty} g(g(x^n_{k+1})) \\
&= \lim_{k\to\infty} g(G(x^n_k, x^1_k, \dots, x^{n-1}_k)) \\
&= \lim_{k\to\infty} G(g(x^n_k), g(x^1_k), \dots, g(x^{n-1}_k)).
\end{aligned} \tag{29}$$

We will indicate that

$$\begin{aligned}
G(x^1, x^2, \dots, x^n) &= g(x^1), \\
G(x^2, x^3, \dots, x^n, x^1) &= g(x^2), \\
&\vdots \\
G(x^n, x^1, \dots, x^{n-1}) &= g(x^n).
\end{aligned} \tag{30}$$

Suppose that (a) holds. From (27), (28), and (29),

$$\begin{aligned}
g(x^1) &= \lim_{k\to\infty} G(g(x^1_k), g(x^2_k), \dots, g(x^n_k)) \\
&= G\left(\lim_{k\to\infty} g(x^1_k), \lim_{k\to\infty} g(x^2_k), \dots, \lim_{k\to\infty} g(x^n_k)\right) \\
&= G(x^1, x^2, \dots, x^n) \\
g(x^2) &= \lim_{k\to\infty} G(g(x^2_k), g(x^3_k), \dots, g(x^n_k), g(x^1_k)) \\
&= G\left(\lim_{k\to\infty} g(x^2_k), \lim_{k\to\infty} g(x^3_k), \dots, \lim_{k\to\infty} g(x^n_k), \lim_{k\to\infty} g(x^1_k)\right) \\
&= G(x^2, x^3, \dots, x^n, x^1) \\
&\vdots \\
g(x^n) &= \lim_{k\to\infty} G(g(x^n_k), g(x^1_k), \dots, g(x^{n-1}_k)) \\
&= G\left(\lim_{k\to\infty} g(x^n_k), \lim_{k\to\infty} g(x^1_k), \dots, \lim_{k\to\infty} g(x^{n-1}_k)\right) \\
&= G(x^n, x^1, \dots, x^{n-1}).
\end{aligned} \tag{31}$$

Now assume that (b) holds. From (19) and (29), we have for all $k$

$$\begin{aligned}
&g(x^1_k) \le x^1 \\
&g(x^2_k) \ge x^2 \\
&\vdots \\
&g(x^n_k) \le x^n \quad (\text{if } n \text{ is odd}) \\
&g(x^n_k) \ge x^n \quad (\text{if } n \text{ is even}).
\end{aligned} \tag{32}$$

For all $t > 0, 0 < \lambda < 1$, we have

$$\begin{aligned}
&F_{g(x^1),G(x^1,x^2,\dots,x^n)}(\phi(t)) \\
&\ge \Delta\{F_{g(x^1),g(g(x^1_{k+1}))}(\phi(t) - \phi(\lambda t)), \\
&F_{g(g(x^1_{k+1})),G(x^1,x^2,\dots,x^n)}(\phi(\lambda t))\}.
\end{aligned} \tag{33}$$

If $g(x^1) = \lim_{k\to\infty} g(g(x^1_{k+1})), \lim_{k\to\infty} F_{g(x^1),g(g(x^1_{k+1}))}(t) = 1$.

When we apply limit to both parts of above inequality, we get

$$\begin{aligned}
&F_{g(x^1),G(x^1,x^2,\dots,x^n)}(\phi(t)) \ge \lim_{k\to\infty} \inf \Delta\{F_{g(x^1),g(g(x^1_{k+1}))}(\phi(t) - \phi(\lambda t)), F_{g(g(x^1_{k+1})),G(x^1,x^2,\dots,x^n)}(\phi(\lambda t))\} \\
&= \Delta\left\{\lim_{k\to\infty} F_{g(x^1),g(g(x^1_{k+1}))}(\phi(t) - \phi(\lambda t)), \lim_{k\to\infty} \inf F_{g(g(x^1_{k+1})),G(x^1,x^2,\dots,x^n)}(\phi(\lambda t))\right\} = \min\Big\{1, \\
&\lim_{k\to\infty} \inf F_{G(g(x^1_k),g(x^2_k),\dots,g(x^1_k)),G(x^1,x^2,\dots,x^n)}(\phi(\lambda t))\Big\} = \lim_{k\to\infty} \inf F_{G(g(x^1_k),g(x^2_k),\dots,g(x^1_k)),G(x^1,x^2,\dots,x^n)}(\phi(\lambda t)) \\
&\ge \lim_{k\to\infty} \inf \Big[\min\{F_{g(g(x^1_k)),g(x^1)}(\lambda t), F_{g(g(x^1_k)),G(g(x^1_k),g(x^2_k),\dots,g(x^n_k))}(\lambda t), F_{g(x^1),G(x^1,x^2,\dots,x^n)}(\lambda t)\} \\
&- p\left(1 - \max\{F_{g(g(x^1_k)),G(x^1,x^2,\dots,x^n)}(\phi(\lambda t)), F_{g(x^1),G(g(x^1_k),g(x^2_k),\dots,g(x^n_k))}(\phi(\lambda t))\}\right)\Big] = \min\Big\{\lim_{k\to\infty} F_{g(g(x^1_k)),g(x^1)}(\lambda t), \\
&\lim_{k\to\infty} \inf F_{g(g(x^1_k)),G(g(x^1_k),g(x^2_k),\dots,g(x^n_k))}(\lambda t), F_{g(x^1),G(x^1,x^2,\dots,x^n)}(\lambda t)\Big\} - p\Big(1 - \max\Big\{\lim_{k\to\infty} F_{g(g(x^1_k)),G(x^1,x^2,\dots,x^n)}(\phi(\lambda t)),
\end{aligned}$$

$$\lim_{k\longrightarrow\infty} F_{g(x^1),G(g(x_k^1),g(x_k^2),\ldots,g(x_k^n))}(\phi(\lambda t))\}) = \min\{F_{g(x^1),g(x^1)}(\lambda t), F_{g(x^1),g(x^1)}(\lambda t), F_{g(x^1),G(x^1,x^2,\ldots,x^n)}(\lambda t)\} - p(1 - \max\{F_{g(x^1),G(x^1,x^2,\ldots,x^n)}(\phi(\lambda t)), F_{g(x^1),g(x^1)}(\phi(\lambda t))\}) \geq \min\{1, F_{g(x^1),G(x^1,x^2,\ldots,x^n)}(\lambda t)\} - p(1-1) \geq F_{g(x^1),G(x^1,x^2,\ldots,x^n)}(\lambda t). \tag{34}$$

Because $\lambda$ is arbitrary $0 < \lambda < 1$, by taking $\lambda \longrightarrow 1$, and left continuous property of distribution function $F$, we have

$$F_{g(x^1),G(x^1,x^2,\ldots,x^n)}(\phi(t)) \geq F_{g(x^1),G(x^1,x^2,\ldots,x^n)}(t). \tag{35}$$

$\phi$ is increasing and also is monotone increasing. Accordingly,

$$\phi(t) + 0 \geq \phi(t) \Longrightarrow$$

$$F_{g(x^1),G(x^1,x^2,\ldots,x^n)}(\phi(t)+0) \geq F_{g(x^1),G(x^1,x^2,\ldots,x^n)}(\phi(t)) \geq F_{g(x^1),G(x^1,x^2,\ldots,x^n)}(t). \tag{36}$$

From Lemma 9, $g(x^1) = G(x^1, x^2, \ldots, x^n)$.

By the same way, we can find

$$\begin{gathered} g(x^2) = G(x^2, \ldots, x^n, x^1), \\ \vdots \\ g(x^n) = G(x^n, x^1, \ldots, x^{n-1}). \end{gathered} \tag{37}$$

Consequently, $g$ and $G$ have $n$-tuplet coincidence point in $X$. □

Taking $p = 0$ in Theorem 16, we obtain Corollary 17.

**Corollary 17.** *Let $(X, \leq)$ be a partially ordered set and $(X, F, \Delta)$ be a complete Menger space, where $\Delta$ is minimum t-norm. Let $G : X^n \longrightarrow X, g : X \longrightarrow X$ be two mappings such that $G$ has the mixed $g$-monotone property. Let $\phi \in \Phi$ exist such that*

$$F_{G(x^1,x^2,\ldots,x^n),G(y^1,y^2,\ldots,y^n)}(\phi(t)) \geq \min\{F_{g(x^1),g(y^1)}(t), F_{g(x^1),G(x^1,x^2,\ldots,x^n)}(t), F_{g(y^1),G(y^1,y^2,\ldots,y^n)}(t)\} \tag{38}$$

*for all $t > 0$, $x^1, x^2, \ldots, x^n, y^1, y^2, \ldots, y^n \in X$ with*

$$\begin{gathered} g(x^1) \geq g(y^1) \\ g(x^2) \leq g(y^2) \\ \vdots \\ g(x^n) \geq g(y^n) \quad (\text{if } n \text{ is odd}) \\ g(x^n) \leq g(y^n) \quad (\text{if } n \text{ is even}). \end{gathered} \tag{39}$$

*Also $g$ is continuous, monotonic increasing, compatible with $G$ and $G(X^n) \subseteq g(X)$. And suppose either*

(a) *$G$ is continuous or*

(b) *$X$ has the following properties:*

  (i) *if a nondecreasing sequence $(x_n) \longrightarrow x$, then $x_n \leq x$ $\forall n \geq 0$,*

  (ii) *if a nonincreasing sequence $(y_n) \longrightarrow y$, then $y_n \leq y$ $\forall n \geq 0$.*

*If there are $x_0^1, x_0^2, \ldots, x_0^n \in X$ such that*

$$\begin{gathered} g(x_0^1) \leq G(x_0^1, x_0^2, \ldots, x_0^n) \\ g(x_0^2) \geq G(x_0^2, x_0^3, \ldots, x_0^n, x_0^1) \\ \vdots \\ g(x_0^n) \leq G(x_0^n, x_0^1, \ldots, x_0^{n-1}) \quad (\text{if } n \text{ is odd}) \\ g(x_0^n) \geq G(x_0^n, x_0^1, \ldots, x_0^{n-1}) \quad (\text{if } n \text{ is even}), \end{gathered} \tag{40}$$

*then $g$ and $G$ have $n$-tuplet coincidence point in $X$. That is, there exist $x^1, x^2, \ldots, x^n \in X$ such that*

$$\begin{gathered} G(x^1, x^2, x^3, \ldots, x^n) = g(x^1) \\ G(x^2, x^3, \ldots, x^n, x^1) = g(x^2) \\ \vdots \\ G(x^n, x^1, x^2, \ldots, x^{n-1}) = g(x^n). \end{gathered} \tag{41}$$

When we take $\phi(t) = ct$, $c \in (0, 1)$ in Theorem 16, we obtain Corollary 18.

**Corollary 18.** *Let $(X, \leq)$ be a partially ordered set and $(X, F, \Delta)$ be a complete Menger space, where $\Delta$ is minimum t-norm. Let $G : X^n \longrightarrow X, g : X \longrightarrow X$ be two mappings such that $G$ has the mixed $g$-monotone property. Let $\phi \in \Phi$ and $p \geq 0$ exist such that*

$$F_{G(x^1,x^2,\ldots,x^n),G(y^1,y^2,\ldots,y^n)}(ct) + p(1 - \max\{F_{g(x^1),G(y^1,y^2,\ldots,y^n)}(ct), F_{g(y^1),G(x^1,x^2,\ldots,x^n)}(ct)\}) \geq \min\{F_{g(x^1),g(y^1)}(t), F_{g(x^1),G(x^1,x^2,\ldots,x^n)}(t), F_{g(y^1),G(y^1,y^2,\ldots,y^n)}(t)\} \tag{42}$$

*for all* $t > 0$, $x^1, x^2, \ldots, x^n, y^1, y^2, \ldots, y^n \in X$ *with*

$$\begin{gathered} g\left(x^1\right) \geq g\left(y^1\right) \\ g\left(x^2\right) \leq g\left(y^2\right) \\ \vdots \\ g\left(x^n\right) \geq g\left(y^n\right) \quad (\text{if } n \text{ is odd}) \\ g\left(x^n\right) \leq g\left(y^n\right) \quad (\text{if } n \text{ is even}). \end{gathered} \tag{43}$$

*Also* $g$ *is continuous, monotonic increasing, compatible with* $G$ *and* $G(X^n) \subseteq g(X)$. *And suppose either*

(a) $G$ *is continuous or*

(b) $X$ *has the following properties:*

(i) *if a nondecreasing sequence* $(x_n) \longrightarrow x$, *then* $x_n \leq x$ $\forall n \geq 0$,

(ii) *if a nonincreasing sequence* $(y_n) \longrightarrow y$, *then* $y_n \leq y$ $\forall n \geq 0$.

*If there are* $x_0^1, x_0^2, \ldots, x_0^n \in X$ *such that*

$$\begin{gathered} g\left(x_0^1\right) \leq G\left(x_0^1, x_0^2, \ldots, x_0^n\right) \\ g\left(x_0^2\right) \geq G\left(x_0^2, x_0^3, \ldots, x_0^n, x_0^1\right) \\ \vdots \\ g\left(x_0^n\right) \leq G\left(x_0^n, x_0^1, \ldots, x_0^{n-1}\right) \quad (\text{if } n \text{ is odd}) \\ g\left(x_0^n\right) \geq G\left(x_0^n, x_0^1, \ldots, x_0^{n-1}\right) \quad (\text{if } n \text{ is even}), \end{gathered} \tag{44}$$

*then* $g$ *and* $G$ *have* $n$*-tuplet coincidence point in* X. *That is, there exist* $x^1, x^2, \ldots, x^n \in X$ *such that*

$$\begin{gathered} G\left(x^1, x^2, x^3, \ldots, x^n\right) = g\left(x^1\right) \\ G\left(x^2, x^3, \ldots, x^n, x^1\right) = g\left(x^2\right) \\ \vdots \\ G\left(x^n, x^1, x^2, \ldots, x^{n-1}\right) = g\left(x^n\right). \end{gathered} \tag{45}$$

## 4. Uniqueness of $n$-Tuplet Fixed Point

For all $(x^1, x^2, \ldots, x^n), (x^{1*}, x^{2*}, \ldots, x^{n*}) \in X^n$,

$$\begin{gathered} \left(x^1, x^2, \ldots, x^n\right) \leq \left(x^{1*}, x^{2*}, \ldots, x^{n*}\right) \Longleftrightarrow \\ x^1 \leq x^{1*}, \\ x^2 \geq x^{2*}, \\ \vdots \\ x^n \leq x^{n*} \quad (\text{if } n \text{ is odd}), \\ x^n \geq x^{n*} \quad (\text{if } n \text{ is even}). \end{gathered} \tag{46}$$

$(x^1, x^2, \ldots, x^n)$ is equal to $(x^{1*}, x^{2*}, \ldots, x^{n*})$ iff $x^1 = x^{1*}, x^2 = x^{2*}, \ldots, x^n = x^{n*}$.

**Theorem 19.** *In addition to the hypotheses of Theorem 16, suppose that for all* $(x^1, x^2, \ldots, x^n), (x^{1*}, x^{2*}, \ldots, x^{n*}) \in X^n$ *there exist* $(y^1, y^2, \ldots, y^n) \in X^n$ *such that*

$$\begin{gathered} \left(G\left(y^1, y^2, \ldots, y^n\right), G\left(y^2, \ldots, y^n, y^1\right), \ldots, \right. \\ \left. G\left(y^n, y^1, \cdots, y^{n-1}\right)\right) \end{gathered} \tag{47}$$

*is comparable to*

$$\begin{gathered} \left(G\left(x^1, x^2, \ldots, x^n\right), G\left(x^2, \ldots, x^n, x^1\right), \ldots, \right. \\ \left. G\left(x^n, x^1, \ldots, x^{n-1}\right)\right) \end{gathered} \tag{48}$$

*and*

$$\begin{gathered} \left(G\left(x^{1*}, x^{2*}, \ldots, x^{n*}\right), G\left(x^{2*}, \ldots, x^{n*}, x^{1*}\right), \ldots, \right. \\ \left. G\left(x^{n*}, x^{1*}, \ldots, x^{(n-1)*}\right)\right). \end{gathered} \tag{49}$$

*Then* $g$ *and* $G$ *have a unique* $n$*-tuplet common fixed point; that is, there exist* $(x^1, x^2, \ldots, x^n) \in X^n$ *such that*

$$\begin{gathered} x^1 = g\left(x^1\right) = G\left(x^1, x^2, x^3, \ldots, x^n\right), \\ x^2 = g\left(x^2\right) = G\left(x^2, x^3, \ldots, x^n, x^1\right), \\ \vdots \\ x^n = g\left(x^n\right) = G\left(x^n, x^1, x^2, \ldots, x^{n-1}\right). \end{gathered} \tag{50}$$

*Proof.* The set of $n$-tuplet coincidence points is nonempty due to Theorem 16. We shall show that if $(x^1, x^2, \ldots, x^n)$ and $(x^{1*}, x^{2*}, \ldots, x^{n*})$ are $n$-tuplet coincidence points, that is, if

$$\begin{gathered} g\left(x^1\right) = G\left(x^1, x^2, \ldots, x^n\right), \\ g\left(x^2\right) = G\left(x^2, \ldots, x^n, x^1\right), \\ \vdots \\ g\left(x^n\right) = G\left(x^n, x^1, \ldots, x^{n-1}\right) \end{gathered} \tag{51}$$

and

$$\begin{gathered} g\left(x^{1*}\right) = G\left(x^{1*}, x^{2*}, \ldots, x^{n*}\right), \\ g\left(x^{2*}\right) = G\left(x^{2*}, \ldots, x^{n*}, x^{1*}\right), \\ \vdots \\ g\left(x^{n*}\right) = G\left(x^{n*}, x^{1*}, \ldots, x^{(n-1)*}\right), \end{gathered} \tag{52}$$

then

$$g\left(x^{1}\right)=g\left(x^{1*}\right),\\ g\left(x^{2}\right)=g\left(x^{2*}\right),\\ \vdots\\ g\left(x^{n}\right)=g\left(x^{n*}\right). \tag{53}$$

By assumption there is $(y^1, y^2, \ldots, y^n) \in X^n$ such that

$$\left(G\left(y^{1},y^{2},\ldots,y^{n}\right),G\left(y^{2},\ldots,y^{n},y^{1}\right),\ldots,\right.\\ \left.G\left(y^{n},y^{1},\ldots,y^{n-1}\right)\right) \tag{54}$$

is comparable to

$$\left(G\left(x^{1},x^{2},\ldots,x^{n}\right),G\left(x^{2},\ldots,x^{n},x^{1}\right),\ldots,\right.\\ \left.G\left(x^{n},x^{1},\ldots,x^{n-1}\right)\right) \tag{55}$$

and

$$\left(G\left(x^{1*},x^{2*},\ldots,x^{n*}\right),G\left(x^{2*},\ldots,x^{n*},x^{1*}\right),\ldots,\right.\\ \left.G\left(x^{n*},x^{1*},\ldots,x^{(n-1)*}\right)\right). \tag{56}$$

We can determine sequences $(g(y_k^1)), (g(y_k^2)), \ldots, (g(y_k^n))$ such that $y^1 = y_0^1, y^2 = y_0^2, \ldots, y^n = y_0^n$ and

$$g\left(y_{k}^{1}\right)=G\left(y_{k-1}^{1},y_{k-1}^{2},\ldots,y_{k-1}^{n}\right),\\ g\left(y_{k}^{2}\right)=G\left(y_{k-1}^{2},\ldots,y_{k-1}^{n},y_{k-1}^{1}\right),\\ \vdots\\ g\left(y_{k}^{n}\right)=G\left(y_{k-1}^{n},y_{k-1}^{1},\ldots,y_{k-1}^{n-1}\right). \tag{57}$$

From comparability of (56) and (55) with (54), we suppose that

$$\left(g\left(x^{1}\right),g\left(x^{2}\right),\ldots,g\left(x^{n}\right)\right)\\ \geq\left(g\left(y^{1}\right),g\left(y^{2}\right),\ldots,g\left(y^{n}\right)\right)\\ =\left(g\left(y_{0}^{1}\right),g\left(y_{0}^{2}\right),\ldots,g\left(y_{0}^{n}\right)\right). \tag{58}$$

Using (24), we have

$$\left(g\left(x^{1}\right),g\left(x^{2}\right),\ldots,g\left(x^{n}\right)\right)\\ \geq\left(g\left(y_{k}^{1}\right),g\left(y_{k}^{2}\right),\ldots,g\left(y_{k}^{n}\right)\right)\quad\text{for all } k. \tag{59}$$

Also from (46), we get

$$g\left(x^{1}\right)\geq g\left(y_{k}^{1}\right),\\ g\left(x^{2}\right)\leq g\left(y_{k}^{2}\right),\\ \vdots\\ g\left(x^{n}\right)\geq g\left(y_{k}^{n}\right)\quad(\text{if } n \text{ is odd}),\\ g\left(x^{n}\right)\leq g\left(y_{k}^{n}\right)\quad(\text{if } n \text{ is even}).$$

$$F_{g(x^1),g(y_{k+1}^1)}\left(\phi\left(t\right)\right)=F_{G(x^1,x^2,\ldots,x^n),G(y_k^1,y_k^2,\ldots,y_k^n)}\left(\phi\left(t\right)\right)\\ \geq\min\left\{F_{g(x^1),g(y_k^1)}\left(t\right),F_{g(x^1),G(x^1,x^2,\ldots,x^n)}\left(t\right),\right.\\ \left.F_{g(y_k^1),G(y_k^1,y_k^2,\ldots,y_k^n)}\left(t\right)\right\}-p\left(1\right.\\ -\max\left\{F_{g(x^1),G(y_k^1,y_k^2,\ldots,y_k^n)}\left(\phi\left(t\right)\right),\right.\\ \left.\left.F_{g(y_k^1),G(x^1,x^2,\ldots,x^n)}\left(\phi\left(t\right)\right)\right\}\right)=\min\left\{F_{g(x^1),g(y_k^1)}\left(t\right),\right.\\ \left.F_{g(x^1),g(x^1)}\left(t\right),F_{g(y_k^1),g(y_{k+1}^1)}\left(t\right)\right\}-p\left(1\right.\\ \left.-\max\left\{F_{g(x^1),g(y_{k+1}^1)}\left(\phi\left(t\right)\right),F_{g(y_k^1),g(x^1)}\left(\phi\left(t\right)\right)\right\}\right)\\ \geq\min\left\{F_{g(x^1),g(y_k^1)}\left(t\right),F_{g(y_k^1),g(y_{k+1}^1)}\left(t\right)\right\},$$

$$F_{g(x^2),g(y_{k+1}^2)}\left(\phi\left(t\right)\right)\\ =F_{G(x^2,x^3,\ldots,x^n,x^1),G(y_k^2,y_k^3,\ldots,y_k^n,y_k^1)}\left(\phi\left(t\right)\right)\\ \geq\min\left\{F_{g(x^2),g(y_k^2)}\left(t\right),F_{g(x^2),G(x^2,x^3,\ldots,x^n,x^1)}\left(t\right),\right.\\ \left.F_{g(y_k^2),G(y_k^2,y_k^3,\ldots,y_k^n,y_k^1)}\left(t\right)\right\}-p\left(1\right.\\ -\max\left\{F_{g(x^2),G(y_k^2,\ldots,y_k^n,y_k^1)}\left(\phi\left(t\right)\right),\right.\\ \left.\left.F_{g(y_k^2),G(x^2,\ldots,x^n,x^1)}\left(\phi\left(t\right)\right)\right\}\right)=\min\left\{F_{g(x^2),g(y_k^2)}\left(t\right),\right.\\ \left.F_{g(x^2),g(x^2)}\left(t\right),F_{g(y_k^2),g(y_{k+1}^2)}\left(t\right)\right\}-p\left(1\right.\\ \left.-\max\left\{F_{g(x^2),g(y_{k+1}^2)}\left(\phi\left(t\right)\right),F_{g(y_k^2),g(x^2)}\left(\phi\left(t\right)\right)\right\}\right)\\ \geq\min\left\{F_{g(x^2),g(y_k^2)}\left(t\right),F_{g(y_k^2),g(y_{k+1}^2)}\left(t\right)\right\},\\ \vdots$$

$$F_{g(x^n),g(y_{k+1}^n)}\left(\phi\left(t\right)\right)\\ =F_{G(x^n,x^1,\ldots,x^{n-1}),G(y_k^n,y_k^1,\ldots,y_k^{n-1})}\left(\phi\left(t\right)\right)\\ \geq\min\left\{F_{g(x^n),g(y_k^n)}\left(t\right),F_{g(x^n),G(x^n,x^1,\ldots,x^{n-1})}\left(t\right),\right.\\ \left.F_{g(y_k^n),G(y_k^n,y_k^1,\ldots,y_k^{n-1})}\left(t\right)\right\}-p\left(1\right.\\ -\max\left\{F_{g(x^n),G(y_k^n,y_k^1,\ldots,y_k^{n-1})}\left(\phi\left(t\right)\right),\right.$$

$$F_{g(y_k^n),G(x^n,x^1,\ldots,x^{n-1})}(\phi(t))\}\Big) = \min\Big\{F_{g(x^n),g(y_k^n)}(t), F_{g(x^n),g(x^n)}(t), F_{g(y_k^n),g(y_{k+1}^n)}(t)\Big\} - p\Big(1 - \max\Big\{F_{g(x^n),g(y_{k+1}^n)}(\phi(t)), F_{g(y_k^n),g(x^n)}(\phi(t))\Big\}\Big) \geq \min\Big\{F_{g(x^n),g(y_k^n)}(t), F_{g(y_k^n),g(y_{k+1}^n)}(t)\Big\} \quad (60)$$

for each $k \geq 1$. If we take lower limit when $k \longrightarrow \infty$ from Lemmas 8 and 9, we have

$$\lim_{k\longrightarrow\infty} g(y_{k+1}^1) = g(x^1), \quad \lim_{k\longrightarrow\infty} g(y_{k+1}^2) = g(x^2), \quad \vdots \quad \lim_{k\longrightarrow\infty} g(y_{k+1}^n) = g(x^n). \quad (61)$$

Likewise, one can show that

$$\lim_{k\longrightarrow\infty} g(y_{k+1}^1) = g(x^{1*}), \quad \lim_{k\longrightarrow\infty} g(y_{k+1}^2) = g(x^{2*}), \quad \vdots \quad \lim_{k\longrightarrow\infty} g(y_{k+1}^n) = g(x^{n*}). \quad (62)$$

Using (61), (62), and triangle inequality property,

$$F_{g(x^1),g(x^{1*})}(t) \geq \Delta\Big\{F_{g(x^1),g(y_{k+1}^1)}\Big(\frac{t}{2}\Big), F_{g(y_{k+1}^1),g(x^{1*})}\Big(\frac{t}{2}\Big)\Big\} \longrightarrow 1 \quad (k \longrightarrow \infty),$$
$$F_{g(x^2),g(x^{2*})}(t) \geq \Delta\Big\{F_{g(x^2),g(y_{k+1}^2)}\Big(\frac{t}{2}\Big), F_{g(y_{k+1}^2),g(x^{2*})}\Big(\frac{t}{2}\Big)\Big\} \longrightarrow 1 \quad (k \longrightarrow \infty),$$
$$\vdots$$
$$F_{g(x^n),g(x^{n*})}(t) \geq \Delta\Big\{F_{g(x^n),g(y_{k+1}^n)}\Big(\frac{t}{2}\Big), F_{g(y_{k+1}^n),g(x^{n*})}\Big(\frac{t}{2}\Big)\Big\} \longrightarrow 1 \quad (k \longrightarrow \infty). \quad (63)$$

So, we have $g(x^1) = g(x^{1*}), g(x^2) = g(x^{2*}), \ldots, g(x^n) = g(x^{n*})$. Thus, we showed (53).

Using commutativity of $G$ and $g$,

$$g(g(x^1)) = g(G(x^1, x^2, \ldots, x^n)) = G(g(x^1), g(x^2), \ldots, g(x^n)),$$
$$g(g(x^2)) = g(G(x^2, \ldots, x^n, x^1)) = G(g(x^2), \ldots, g(x^n), g(x^1)),$$
$$\vdots$$
$$g(g(x^n)) = g(G(x^n, x^1, \ldots, x^{n-1})) = G(g(x^n), g(x^1), \ldots, g(x^{n-1})). \quad (64)$$

Indicate $g(x^1) = z^1, g(x^2) = z^2, \ldots, g(x^n) = z^n$. From (64),

$$g(z^1) = G(z^1, z^2, \ldots, z^n), \quad g(z^2) = G(z^2, \ldots, z^n, z^1), \quad \vdots \quad g(z^n) = G(z^n, z^1, \ldots, z^{n-1}). \quad (65)$$

Therefore, $(z^1, z^2, \ldots, z^n)$ is a $n$-tuplet coincidence point. Then from the assumption with $x^{1*} = z^1, \ldots, x^{n*} = z^n$ it follows $g(z^1) = g(x^1), g(z^2) = g(x^2), \ldots, g(z^n) = g(x^n)$; that is,

$$g(z^1) = z^1, \quad g(z^2) = z^2, \quad \vdots \quad g(z^n) = z^n. \quad (66)$$

By (65) and (66),

$$z^1 = g(z^1) = G(z^1, z^2, \ldots, z^n), \quad z^2 = g(z^2) = G(z^2, \ldots, z^n, z^1), \quad \vdots \quad z^n = g(z^n) = G(z^n, z^1, \ldots, z^{n-1}). \quad (67)$$

Hence, $(z^1, z^2, \ldots, z^n)$ is a $n$-tuplet common fixed point of $G$ and $g$. To prove the uniqueness, assume that $(q^1, q^2, \ldots, q^n)$

is another $n$-tuplet common fixed point. Then by assumption we have

$$\begin{aligned} q^1 &= g(q^1) = g(z^1), \\ q^2 &= g(q^2) = g(z^2), \\ &\vdots \\ q^n &= g(q^n) = g(z^n). \end{aligned} \tag{68}$$

□

## Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

## Disclosure

A part of the results of this paper were presented in International Conference on Operators in Morrey-Type Spaces and Applications, OMSTA 2017 [29].

## Conflicts of Interest

The authors have no conflicts of interest regarding this paper.

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