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Close-to-Convex Functions Defined by Fractional Operator

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Abstract

Let S denote the class of functions $f(z) = z + a_2z^2 + ...$ analytic and univalent in the open unit disc $D = \{z \in \mathbb{C} ||z| < 1\}$. Consider the subclass and S^* of S, which are the classes of convex and starlike functions, respectively. In 1952, W. Kaplan introduced a class of analytic functions $f(z)$, called close-to-convex functions, for which there exists $\phi(z) \in \mathbb{C}$, depending on $f(z)$ with $Re(\frac{f'(z)}{\phi'(z)}) > 0$ in , and prove that every close-to-convex function is univalent. The normalized class of close-to-convex functions denoted by K. These classes are related by the proper inclusions $C \subset S^* \subset K \subset S$.
In this paper, we generalize the close-to-convex functions and denote $K(\lambda)$ the class of such functions. Various properties of this class of functions is alos studied.

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1 Introduction

Let be the family of functions $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ which are analytic in D and satisfy the conditions $p(0) = 1$, $Rep(z) > 0$ for all $z \in$.
Let S denote the class of functions $f(z)$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic and univalent in D.
We recall here the definition of the well-known classes of starlike, convex and close-to-convex functions [3], respectively,

$$S^* = \left\{ f \in S | Re\frac{zf'(z)}{f(z)} > 0, z \in D \right\}, \tag{1}$$

$$C = \left\{ f \in S | (1 + Re\frac{zf''(z)}{f'(z)}) > 0, z \in D \right\}, \tag{2}$$

$$K = \left\{ f \in S | \exists\psi \in C, Re\frac{f'(z)}{\psi(z)} > 0, z \in D \right\}. \tag{3}$$

When considering these definitions above, in general, the functions belonging to them can be represented as the functions of .
Alexander's Theorem says us "if $\phi(z)$ is convex, then $\psi(z) := z\phi'(z)$ is starlike". Hence, we can rewrite K as follows:

$$K = \left\{ f \in S | \exists\psi \in S^* \ni Re(z\frac{f'(z)}{\psi(z)}) > 0, forallz \in D \right\}. \tag{4}$$

A fairly complete treatment, wtih applications of the fractional calculus, is given in the books [6] by Oldham and Spanier, and [5] by Miller and Ross. We refer to [10] for more insight into the concept of the fractional calculus. For further details on the materials in this paper see [4].
For convenience, we shall remind some definitions of the fractional calculus (i.e, fractionla integral and fractional derivative).
The fractional integral of order λ for ana analytical function $f(z)$ in D is defined by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta, (\lambda > 0) \tag{5}$$

where the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.
The fractional derivative of order λ for an analytic function $f(z)$ in D is defined by

$$D_z^{\lambda} f(z)=\frac{d}{dz}(D_z^{-\lambda} f(z))=\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}}d\zeta,(0\leq\lambda<1), \tag{6}$$

where the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.
Under the hypothesis of the fractional derivative , the fractional derivative of order $(n+\lambda)$ for an analytic function $f(z)$ in D is defined by

$$D_z^{n+\lambda} f(z)=\frac{d^n}{dz^n}(D_z^{\lambda} f(z)),(0\leq\lambda<1,n\in N_0=\Big\{0,1,2,...\Big\}). \tag{7}$$

From the definitions of the fractional calculus, we see that

$$D_z^{-\lambda} z^k=\frac{\Gamma(k+1)}{\Gamma(k+1+\lambda)}z^{k+\lambda},(\lambda>0,k>0) \tag{8}$$

$$D_z^{\lambda} z^k=\frac{\Gamma(k+1)}{\Gamma(k+1+\lambda)}z^{k-\lambda},(0\leq\lambda<1,k>0) \tag{9}$$

and

$$D_z^{n+\lambda} z^k=\frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)}z^{k-n-\lambda},(0\leq\lambda<1,k>0,n\in N_0,k-n\neq-1,-2,...) \tag{10}$$

Therefore we see that for any real λ

$$D_z^{\lambda} z^k=\frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)}z^{k-\lambda},(k>0,k-\lambda\neq-1,-2,...) \tag{11}$$

2 Main Results

Using the rule of the fractional derivative which is mentioned in the preceding, we define the $\lambda-$ fractional operator as follows,

$$f(z)=z+a_2z^2+...+a_nz^n+...\Rightarrow D_z^{\lambda} f(z)=D_z^{\lambda}(z+a_2z^2+...+a_nz^n+...)$$

$$D^{\lambda} f(z)=\Gamma(2-\lambda)z^{\lambda}D_z^{\lambda} f(z)=z+\sum_{n=2}^{\infty}a_n\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}z^n \tag{12}$$

From the definition of $D^{\lambda} f(z)$ we have the following properties.

i.

$$D'f(z) = Df(z) = lim_{\lambda\to 1}D^{\lambda}f(z) = zf'(z);$$

ii.

$$D^{\lambda}(D^{\delta}f(z)) = D^{\delta}(D^{\lambda}f(z)) =$$

$$z + \sum_{n=2}^{\infty} a_n \frac{\Gamma(2-\lambda)\Gamma(2-\delta)(\Gamma(n+1))^2}{\Gamma(n+1-\lambda)\Gamma(n+1-\delta)} z^n;$$

iii.

$$D(D^{\delta}f(z)) = z + \sum_{n=2}^{\infty} a_n n a_n \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda} z^n = z(D^{\delta}f(z))' =$$

$$\Gamma(2-\lambda)z^{\lambda}(\lambda D_z^{\lambda} + zD_z^{\lambda+1}f(z));$$

vi.

$$\frac{D(D^{\lambda}f(z))}{D^{\lambda}f(z)} = z\frac{f'(z)}{f(z)}, for \lambda = 0,$$

$$= 1 + z\frac{f''(z)}{f'(z)}, for \lambda = 1.$$

Thus, we define the following class of functions.

Definition 2.1 *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be an element of S. Then $f(z)$ is said to be $\lambda-$ fractional close-to-convex function in D if there exists a function $g(z)$ of S^* such that*

$$Re(\frac{D(D^{\lambda}f(z))}{g}(z)) > 0$$

for all $z \in D$. The class of these functions is denoted by $K(\lambda)$.
It is obviously that $K(0) = K$.
By using the definition above and properties of $\lambda-$ fractional operator $D^{\lambda}f(z)$, we have the following properties.

i.

$$L(z) = \frac{z}{1-z} = z + z^2 + ... + z^n + ... D^{\lambda}L(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} z^n =$$

$$zF(2, 1, 2-\lambda; z).$$

Then we have,

$$Re(\frac{D(D^{\lambda}f(z))}{g(z)}) > 0 \Rightarrow Re(\frac{zf'(z) * D^{\lambda}L(z)}{g(z)}) > 0 \Rightarrow$$

$$Re(\frac{zf'(z) * zF(2,1,2-\lambda;z)}{g(z)}) > 0.$$

(a) For $\lambda = 0$,

$$Re(\frac{zf'(z) * L(z)}{g(z)}) > 0 \Rightarrow Re(\frac{zf'(z)}{g(z)}) > 0.$$

(b) For $\lambda = 1$,

$$Re(\frac{zf'(z) * L'(z)}{g(z)}) = Re(\frac{zf'(z)}{g(z)}) > 0.$$

Where $k(z)$ is a Koebe function.

ii.

$$Re(\frac{D(D^{\lambda}f(z))}{g}(z)) > 0 = Re(z\frac{f'(z)}{g(z)}(1 + z\frac{f''(z)}{f'(z)})) > 0, \lambda = 1,$$

$$= Re(z\frac{f'(z)}{g(z)}) > 0, \lambda = 0.$$

Theorem 2.2 *Let $f(z)$ be an element of $K(\lambda)$. Then*

$$\frac{r(1-r)}{(1+r)^3} \leq |D(D^{\lambda}f(z))| \leq \frac{r(1+r)}{(1-r)^3} \tag{13}$$

Proof 2.3 *Using the definition of the class $K(\lambda)$, we can write*

$$\frac{D(D^{\lambda}f(z))}{g(z)} = p(z) \Rightarrow D(D^{\lambda}f(z)) = p(z)g(z). \tag{14}$$

where $p(z) \in P$. On the other hand, we have the inequalities

$$\frac{1-r}{1+r} \leq |p(z)| \leq \frac{1+r}{1-r} \tag{15}$$

and

$$\frac{r}{(1+r)^2} \leq |g(z)| \leq \frac{r}{(1-r)^2} \tag{16}$$

from [1]*. By considering* (14)*,* (15) *and* (16)*, we obtain* (13)*.*

If $f(z)$ be an element of $K(\lambda)$. Then

$$\frac{1-r}{r(1+r)^3} \leq |f'(z)| \leq \frac{(1+r)}{r(1-r)^3}, for \lambda = 0, \tag{17}$$

$$\frac{(1-r)}{(1+r)^3} \leq |f'(z) + zf''(z)| \leq \frac{(1+r)}{(1-r)^3}, for \lambda = 1. \tag{18}$$

Theorem 2.4 *Let $f(z)$ be an element of $K(\lambda)$; then*

$$|a_n| \leq \frac{n\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)\Gamma(n+1)} z^n \tag{19}$$

We notice that this result is, indeed, sharp since the extremal function

$$f(z) = z + \sum_{n=2}^{\infty} \frac{n\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)\Gamma(n+1)} z^n \tag{20}$$

is the solution of the fractional differential equation

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(2-\lambda)} z^{-\lambda} \frac{z}{(1-z)^2}.$$

Proof 2.5 *If we use the definition of the class $K(\lambda)$, then we can write*

$$\frac{D(D^{\lambda} f(z))}{g(z)} = p(z) \Rightarrow D(D^{\lambda} f(z)) = p(z)g(z) \Rightarrow \tag{21}$$

$$z + \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda) n a_n z^n} \tag{22}$$

$$= (z + b_2 z^2 + ... + b_n z^n + ...)(1 + p_1 z + p_2 z^2 + ... + p_n z^n + ...) \Rightarrow \tag{23}$$

$$n a_n \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} = (b_n + b_{n-1} p_1 + ... + b_2 p_{n-2} + b_1 p_{n-1}) \Rightarrow \tag{24}$$

$$n\,|a_n|\, \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} \leq |b_n| + |b_{n-1}|\,|p_1| + ... + |b_1|\,|p_{n-1}| \Rightarrow \tag{25}$$

$$\leq n + (n-1)2 + (n-2)2 + ... + 2.2 + 1.2 \tag{26}$$

$$= n + 2[1 + 2 + ... + (n-1)] = n^2 \Rightarrow \tag{27}$$

$$|a_n| \leq \frac{n\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)\Gamma(n+1)} \tag{28}$$

We notice that if we take $\lambda = 0$ then we obtain $|a_n| \leq n$ which is the coefficient inequality for the close-to-convex functions, and we take $\lambda = 1$, then $|a_n| \leq 1$.

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