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VARIATION FORMULAS OF SOLUTION FOR A CONTROLLED FUNCTIONAL-DIFFERENTIAL EQUATION CONSIDERING DELAY PERTURBATION

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Abstract. Variation formulas of solution are proved for a controlled non-linear functional-differential equation with constant delay and the continuous initial condition. In this paper, the essential novelty is the effect of delay perturbation in the variation formulas.The continuity of the initial condition means that the values of the initial function and the trajectory always coincide at the initial moment.

Keywords: Controlled delay functional-differential equation; variation formula of solution; effect of delay perturbation; continuous initial condition.

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1. Introduction

Linear representation of the main part of the increment of a solution of an equation with respect to perturbations is called the variation formula of solution (variation formula).The variation formula allows one to construct an approximate solution of the perturbed equation in an analytical form on the one hand, and in the theory of optimal control plays the basic role in proving the necessary conditions of optimality [1-3,6,7,10], on the other. Variation formulas for various classes of functional-differential equations without perturbation of delay are given in [2-5,7,8,10,11]. Here we are interested in the variation formulas for the controlled delay functional-differential equation

$$\dot{x}(t) = f(t, x(t), x(t-\tau_0), u_0(t))$$

with the continuous initial condition

$$x(t) = \varphi_0(t), t \in [t_{00} - \tau_0, t_{00}]$$

under perturbations of initial moment t_{00}, delay parameter τ_0, initial function $\varphi_0(t)$ and control function $u_0(t)$. In this paper, the essential novelty is the effect of perturbation of delay τ_0 in the variation formulas (see **c2**). The variation formula in a neighborhood of

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the right end of the main interval (so called local variation formula) for the differential equation

$$\dot{x}(t) = f_0(t, x(t), x(t - \tau_0)) \tag{1.1}$$

with the initial condition

$$x(t) = \varphi_0(t), t \in [t_{00} - \tau_0, t_{00}), x(t_{00}) = x_{00},$$

when perturbation of initial data, delay and right-hand side of the equation (1.1) occurs is proved in [12]. It is important to note that the variation formula which is proved in the present work doesn't follows from the formula proved in [12] (see **c6**).

2. Notation and auxiliary assertions

Let R_x^n be the n-dimensional vector space of points $x = (x^1, ..., x^n)^T$, where T means transpose; suppose that $O \subset R_x^n$ and $V \subset R_u^r$ are open sets. Let the n-dimensional function $f(t, x, y, u)$ satisfies the following conditions: for almost all $t \in J = [a, b]$, the function $f(t, \cdot) : O^2 \times V \to R_x^n$ is continuously differentiable; for any $(x, y, u) \in O^2 \times V$, the functions $f(t, x, y, u), f_x(\cdot), f_y(\cdot), f_u(\cdot)$ are measurable on J; for arbitrary compacts $K \subset O, U \subset V$ there exists a function $m_{K,U}(\cdot) \in L(J, [0, \infty))$, such that for any $(x, y, u) \in K^2 \times U$ and for almost all $t \in J$ the following inequality is fulfilled

$$\mid f(t, x, y, u) \mid + \mid f_x(\cdot) \mid + \mid f_y(\cdot) \mid + \mid f_u(\cdot) \mid \leq m_{K,U}(t).$$

Further, let $0 < \tau_1 < \tau_2$ be given numbers; E_φ be the space of continuous functions $\varphi : J_1 \to R_x^n$, where $J_1 = [\hat{\tau}, b], \hat{\tau} = a - \tau_2; \Phi = \{\varphi \in E_\varphi : \varphi(t) \in O, t \in J_1\}$ be a set of initial functions; let E_u be the space of bounded measurable functions $u : J \to R_u^r$ and $\Omega = \{u \in E_u : clu(J) \subset V\}$ be a set of control functions, where $u(J) = \{u(t) : t \in J\}$ and $clu(J)$ is the closer of the set $u(J)$.

To each element $\mu = (t_0, \tau, \varphi, u) \in \Lambda = (a, b) \times (\tau_1, \tau_2) \times \Phi \times \Omega$ we assign the controlled delay functional-differential equation

$$\dot{x}(t) = f(t, x(t), x(t - \tau), u(t)) \tag{2.1}$$

with the initial condition

$$x(t) = \varphi(t), t \in [\hat{\tau}, t_0]. \tag{2.2}$$

The condition (2.2) is said to be continuous initial condition since always $x(t_0) = \varphi(t_0)$.

Definition 2.1. Let $\mu = (t_0, \tau, \varphi, u) \in \Lambda$. A function $x(t) = x(t; \mu) \in O, t \in [\hat{\tau}, t_1], t_1 \in (t_0, b)$, is called a solution of equation (2.1) with the initial condition (2.2) or a solution corresponding to μ and defined on the interval $[\hat{\tau}, t_1]$ if it satisfies condition (2.2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (2.1) almost everywhere on $[t_0, t_1]$.

Let $\mu_0 = (t_{00}, \tau_0, \varphi_0, u_0) \in \Lambda$ be a fixed element. In the space $E_\mu = R_{t_0}^1 \times R_\tau^1 \times E_\varphi \times E_u$ we introduce the set of variations:

$$V = \{\delta\mu = (\delta t_0, \delta\tau, \delta\varphi, \delta u) \in E_\mu - \mu_0 : \mid \delta t_0 \mid \leq \alpha, \mid \delta\tau \mid \leq \alpha,$$
$$\delta\varphi = \sum_{i=1}^{k} \lambda_i \delta\varphi_i, \delta u = \sum_{i=1}^{k} \lambda_i \delta u_i, \mid \lambda_i \mid \leq \alpha, i = \overline{1, k}\}, \tag{2.3}$$

where $\delta\varphi_i \in E_\varphi - \varphi_0, \delta u_i \in E_u - u_0, i = \overline{1, k}$ are fixed functions ; $\alpha > 0$ is a fixed number.

Lemma 2.1. *Let $x_0(t)$ be the solution corresponding to $\mu_0 = (t_{00}, \tau_0, \varphi_0, u_0) \in \Lambda$ and defined on $[\hat{\tau}, t_{10}], t_{10} \in (t_{00}, b)$ and let $K_0 \subset O$ and $U_0 \subset V$ be compact sets containing neighborhoods of sets $\varphi_0(J_1) \cup x_0([t_{00}, t_{10}])$ and $clu_0(J)$, respectively. Then there exist numbers $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that, for any $(\varepsilon, \delta\mu) \in [0, \varepsilon_1] \times V$, we have $\mu_0 + \varepsilon\delta\mu \in \Lambda$. In addition, a solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\hat{\tau}, t_{10} + \delta_1] \subset J_1$ corresponds to this element. Moreover,*

$$\begin{cases} x(t; \mu_0 + \varepsilon\delta\mu) \in K_0, t \in [\hat{\tau}, t_{10} + \delta_1], \\ u_0(t) + \varepsilon\delta u(t) \in U_0, t \in J. \end{cases} \tag{2.4}$$

This lemma is a result of Theorem 5.3 in [9, p.111].

Remark 2.1. Due to the uniqueness, the solution $x(t; \mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\hat{\tau}, t_{10} + \delta_1]$. Therefore, in the sequel the solution $x_0(t)$ is assumed to be defined on the interval $[\hat{\tau}, t_{10} + \delta_1]$.

Lemma 2.1 allows one to define the increment of the solution $x_0(t) = x(t; \mu_0)$:

$$\begin{cases} \Delta x(t) = \Delta x(t; \varepsilon\delta\mu) = x(t; \mu_0 + \varepsilon\delta\mu) - x_0(t), \\ (t, \varepsilon, \delta\mu) \in [\hat{\tau}, t_{10} + \delta_1] \times [0, \varepsilon_1] \times V. \end{cases} \tag{2.5}$$

Lemma 2.2. *Let the following conditions hold:*

2.1. *the function $\varphi_0(t), t \in J_1$ is absolutely continuous and the function $\dot{\varphi}_0(t)$ is bounded;*

2.2. *there exist compact sets $K_0 \subset O$ and $U_0 \subset V$ containing neighborhoods of sets $\varphi_0(J_1) \cup x_0([t_{00}, t_{10}])$ and $clu_0(J)$, respectively, such that the function $f(t, x, y, u)$ is bounded on the set $J \times K_0^2 \times U_0$;*

2.3. *there exist the limits*

$$\lim_{t \to t_{00}-} \dot{\varphi}_0(t) = \dot{\varphi}_0^-, \lim_{w \to w_0} f(w, u_0(t)) = f^-,$$

where $w = (t, x, y) \in (a, t_{00}] \times O^2, w_0 = (t_{00}, \varphi_0(t_{00}), \varphi_0(t_{00} - \tau_0))$. *Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1]$ and $\delta_2 \in (0, \delta_1]$ such that*

$$\max_{t \in [\hat{\tau}, t_{10} + \delta_2]} | \Delta x(t) | \leq O(\varepsilon\delta\mu)^1 \tag{2.6}$$

for arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_2] \times V^-$, where $V^- = \{\delta\mu \in V : \delta t_0 \leq 0, \delta\tau \leq 0\}$. Moreover,

$$\Delta x(t_{00}) = \varepsilon\Big[\delta\varphi(t_{00}) + \{\dot{\varphi}_0^- - f^-\}\delta t_0\Big] + o(\varepsilon\delta\mu). \tag{2.7}$$

Lemma 2.3. *Let the conditions 2.1 and 2.2 of Lemma 2.2 hold, and there exist the limits*

$$\lim_{t \to t_{00}+} \dot{\varphi}_0(t) = \dot{\varphi}_0^+, \lim_{w \to w_0} f(w, u_0(t)) = f^+, w \in [t_{00}, b) \times O^2.$$

Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1]$ and $\delta_2 \in (0, \delta_1]$ such that the inequality (2.6) is valid for arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_2] \times V^+$, where $V^+ = \{\delta\mu \in V : \delta t_0 \geq 0, \delta\tau \geq 0\}$. Moreover,

$$\Delta x(t_{00} + \varepsilon\delta t_0) = \varepsilon\Big[\delta\varphi(t_{00}) + \{\dot{\varphi}_0^+ - f^+\}\delta t_0\Big] + o(\varepsilon\delta\mu). \tag{2.8}$$

Lemmas 2.2 and 2.3 can be proved in analogy to Lemma 2.4 (see [11]).

[1]Here and throughout the following, the symbols $O(t; \varepsilon\delta\mu), o(t; \varepsilon\delta\mu)$ stand for quantities (scalar or vector) that have the corresponding order of smallness with respect to ε uniformly with respect to t and $\delta\mu$.

3. Formulation of main results

Theorem 3.1. *Let the conditions of Lemma 2.2 hold. Then there exist numbers* $\varepsilon_2 \in (0, \varepsilon_1]$ *and* $\delta_2 \in (0, \delta_1]$ *such that*

$$\Delta x(t; \varepsilon\delta\mu) = \varepsilon\delta x(t; \delta\mu) + o(t; \varepsilon\delta\mu) \tag{3.1}$$

for arbitrary $(t, \varepsilon, \delta\mu) \in [t_{00}, t_{10} + \delta_2] \times [0, \varepsilon_2] \times V^-$ *and*

$$\delta x(t; \delta\mu) = Y(t_{00}; t)\{\dot{\varphi}_0^- - f^-\}\delta t_0 + \beta(t; \delta\mu), \tag{3.2}$$

where

$$\beta(t; \delta\mu) = Y(t_{00}; t)\delta\varphi(t_{00}) + \int_{t_{00}-\tau_0}^{t_{00}} Y(\xi + \tau_0; t) f_y[\xi + \tau_0]\delta\varphi(\xi) d\xi$$

$$-\Big\{ \int_{t_{00}}^{t} Y(\xi; t) f_y[\xi] \dot{x}_0(\xi - \tau_0) d\xi \Big\}\delta\tau + \int_{t_{00}}^{t} Y(\xi; t) f_u[\xi]\delta u(\xi) d\xi. \tag{3.3}$$

Here $Y(\xi; t)$ *is the* $n \times n$*-matrix function satisfying the linear functional-differential equation with advanced argument*

$$Y_\xi(\xi; t) = -Y(\xi; t) f_x[\xi] - Y(\xi + \tau_0; t) f_y[\xi + \tau_0], \xi \in [t_{00}, t], \tag{3.4}$$

and the condition

$$Y(\xi; t) = \begin{cases} I \ for \ \xi = t, \\ \Theta \ for \ \xi > t, \end{cases} \tag{3.5}$$

$$f_x = \frac{\partial}{\partial x} f, \ f_x[\xi] = f_x(\xi, x_0(\xi), x_0(\xi - \tau_0), u_0(\xi));$$

I is the identity matrix and Θ is the zero matrix.

Some comments. The expression (3.2) is called the variation formula.

c1. Theorem 3.1 corresponds to the case when the variations at the points t_{00} and τ_0 are performed simultaneously on the left.

c2. The summand

$$-\Big\{ \int_{t_{00}}^{t} Y(\xi; t) f_y[\xi] \dot{x}_0(\xi - \tau_0) d\xi \Big\}\delta\tau$$

in formula (3.3) is the effect of perturbation of the delay τ_0.

c3. The expression

$$Y(t_{00}; t)\{\dot{\varphi}_0^- - f^-\}\delta t_0$$

is the effect of continuous initial condition (2.2) and perturbation of the initial moment t_{00}.

c4. The expression

$$Y(t_{00}; t)\delta\varphi(t_{00}) + \int_{t_{00}-\tau_0}^{t_{00}} Y(\xi + \tau_0; t) f_y[\xi + \tau_0]\delta\varphi(\xi) d\xi + \int_{t_{00}}^{t} Y(\xi; t) f_u[\xi]\delta u(\xi) d\xi$$

in formula (3.3) is the effect of perturbations of initial $\varphi_0(t)$ and control $u_0(t)$ functions.

c5. The variation formula allow one to obtain an approximate solution of the perturbed functional-differential equation

$$\dot{x}(t) = f(t, x(t), x(t - \tau_0 - \varepsilon\delta\tau), u_0(t) + \varepsilon\delta u(t))$$

with the perturbed initial condition $x(t) = \varphi_0(t) + \varepsilon\delta\varphi(t), t \in [\hat{\tau}, t_{00} + \varepsilon\delta t_0]$. In fact, for a sufficiently small $\varepsilon \in (0, \varepsilon_2]$ from (3.1) it follows $x(t) \approx x_0(t) + \varepsilon\delta x(t; \delta\mu)$ (see (2.5)).

c6. In the variation formula proved in [12], for the equation (1.1) instead of the expression

$$\int_{t_{00}}^{t} Y(\xi; t) f_u[\xi]\delta u(\xi) d\xi$$

(see (3.3)), we have

$$\int_{t_{00}}^{t} Y(\xi; t)\delta f(\xi, x_0(\xi), x_0(\xi - \tau_0) d\xi,$$

where δf is a perturbation of the right-hand side of the equation (1.1). The local version of the formula (3.2) follows from the formula obtained in [12] if the function f additionally satisfies the conditions : the function $f_u(t, x, y, u_0(t))$ is continuously differentiable with respect to x and y; there exists the limit

$$\lim_{w \to w_{01}} f(w, u_0(t)), w = (t, x, y) \in (a, t_{00} + \tau_0] \times O^2$$

where $w_{01} = (t_{00} + \tau_0, x_0(t_{00} + \tau_0), \varphi_0(t_{00}))$, with $t_{00} + \tau_0 < t_{10}$. In the present work variation formulas are proved without of these conditions.

Theorem 3.2. *Let the conditions of Lemma 2.3 hold. Then for each* $\hat{t}_0 \in (t_{00}, t_{10})$ *there exist numbers* $\varepsilon_2 \in (0, \varepsilon_1]$ *and* $\delta_2 \in (0, \delta_1]$ *such that for arbitrary* $(t, \varepsilon, \delta\mu) \in [\hat{t}_0, t_{10} + \delta_2] \times [0, \varepsilon_2] \times V^+$, *formula (3.1) holds and*

$$\delta x(t; \delta\mu) = Y(t_{00}; t)\{\dot{\varphi}_0^+ - f^+\}\delta t_0 + \beta(t; \delta\mu). \tag{3.6}$$

Theorem 3.2 corresponds to the case when the variations at the points t_{00} and τ_0 are performed simultaneously on the right. Theorems 3.1 and 3.2 are proved by a scheme given in [3]. The following assertion is a corollary to Theorems 3.1 and 3.2.

Theorem 3.3. *Let the assumptions of Theorems 3.1 and 3.2 be fulfilled. Moreover,* $\dot{\varphi}_0^- - f^- = \dot{\varphi}_0^+ - f^+ =: \hat{f}$. *Then for each* $\hat{t}_0 \in (t_{00}, t_{10})$ *there exist numbers* $\varepsilon_2 \in (0, \varepsilon_1]$ *and* $\delta_2 \in (0, \delta_1]$ *such that for arbitrary* $(t, \varepsilon, \delta\mu) \in [\hat{t}_0, t_{10} + \delta_2] \times [0, \varepsilon_2] \times V$, *formula (3.1) holds, where*

$$\delta x(t; \delta\mu) = Y(t_{00}; t)\hat{f}\delta t_0 + \beta(t; \delta\mu).$$

Theorem 3.3 corresponds to the case when at the points t_{00} and τ_0 two-sided variations are simultaneously performed. All assumptions of Theorem 2.3 are satisfied if the function $f(t, x, y, u)$ is continuous, the function $\varphi_0(t)$ is continuously differentiable and the function $u_0(t)$ is continuous at the point t_{00}. Clearly, in this case $\hat{f} = \dot{\varphi}_0(t_{00}) - f(t_{00}, \varphi_0(t_{00}), \varphi_0(t_{00} - \tau_0), u_0(t_{00}))$.

4. Proof of theorem 3.1

Here and in what follows we shall assume that $t_0 = t_{00} + \varepsilon\delta t_0, \tau = \tau_0 + \varepsilon\delta\tau, \varphi(t) = \varphi_0(t) + \varepsilon\delta\varphi(t), u(t) = u_0(t) + \varepsilon\delta u(t)$. Let $\varepsilon_2 \in (0, \varepsilon_1]$ be so small (see Lemma 2.2) that for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_2] \times V^-$ the following inequalities hold $t_{00} - \tau \le t_0, t_0 + \tau \ge t_{00}$. The function $\Delta x(t)$ (see (2.5)) satisfies the equation

$$\dot{\Delta} x(t) = f(t, x_0(t) + \Delta x(t), x_0(t - \tau) + \Delta x(t - \tau), u(t)) - f[t]$$

$$= f_x[t]\Delta x(t) + f_y[t]\Delta x(t-\tau_0) + \varepsilon f_u[t]\delta u(t) + r(t;\varepsilon\delta\mu) \tag{4.1}$$

on the interval $[t_{00}, t_{10} + \delta_2]$, where

$$r(t;\varepsilon\delta\mu) = f(t, x_0(t) + \Delta x(t), x_0(t-\tau) + \Delta x(t-\tau), u(t)) - f[t]$$

$$-f_x[t]\Delta x(t) - f_y[t]\Delta x(t-\tau_0) - \varepsilon f_u[t]\delta u(t), \tag{4.2}$$

$$f[t] = f(t, x_0(t), x_0(t-\tau_0), u_0(t)).$$

By using the Cauchy formula ([3],p.21), one can represent the solution of equation (4.1) in the form

$$\Delta x(t) = Y(t_{00};t)\Delta x(t_{00}) + \varepsilon\int_{t_{00}}^{t} Y(\xi;t)f_u[\xi]\delta u(\xi)d\xi$$

$$+\sum_{i=0}^{1} R_i(t;t_{00},\varepsilon\delta\mu), t \in [t_{00}, t_{10}+\delta_2], \tag{4.3}$$

where

$$\begin{cases} R_0(t;t_{00},\varepsilon\delta\mu) = \int_{t_{00}-\tau_0}^{t_{00}} Y(\xi+\tau_0;t)f_y[\xi+\tau_0]\Delta x(\xi)d\xi, \\ R_1(t;t_{00},\varepsilon\delta\mu) = \int_{t_{00}}^{t} Y(\xi;t)r(\xi;\varepsilon\delta\mu)d\xi \end{cases} \tag{4.4}$$

and $Y(\xi;t)$ is the matrix function satisfying equation (3.4) and condition (3.5). The function $Y(\xi;t)$ is continuous on the set $\Pi = \{(\xi,t):\ a \le \xi \le t, t \in J\}$ ([3], Lemma 2.1.7). Therefore,

$$Y(t_{00};t)\Delta x(t_{00}) = \varepsilon Y(t_{00};t)\Big[\delta\varphi(t_{00}) + \{\dot\varphi_0^- - f^-\}\delta t_0\Big] + o(t;\varepsilon\delta\mu) \tag{4.5}$$

(see (2.7)). One can readily see that

$$R_0(t;t_{00},\varepsilon\delta\mu) = \varepsilon\int_{t_{00}-\tau_0}^{t_0} Y(\xi+\tau_0;t)f_y[\xi+\tau_0]\delta\varphi(\xi)d\xi$$

$$+\int_{t_0}^{t_{00}} Y(\xi+\tau_0;t)f_y[\xi+\tau_0]\Delta x(\xi)d\xi = \varepsilon\int_{t_{00}-\tau_0}^{t_{00}} Y(\xi+\tau_0;t)f_y[\xi+\tau_0]\delta\varphi(\xi)d\xi$$

$$+o(t;\varepsilon\delta\mu) \tag{4.6}$$

(see (2.5) and (2.6)). We introduce the notations:

$$f[t;s,\varepsilon\delta\mu] = f(t, x_0(t) + s\Delta x(t), x_0(t-\tau_0) + s\{x_0(t-\tau) - x_0(t-\tau_0)$$

$$+\Delta x(t-\tau)\}, u_0(t) + s\varepsilon\delta u(t)), \sigma(t;s,\varepsilon\delta\mu) = f_x[t;s,\varepsilon\delta\mu] - f_x[t],$$

$$\rho(t;s,\varepsilon\delta\mu) = f_y[t;s,\varepsilon\delta\mu] - f_y[t], \theta(t;s,\varepsilon\delta\mu) = f_u[t;s,\varepsilon\delta\mu] - f_u[t].$$

It is easy to see that

$$f(t, x_0(t) + \Delta x(t), x_0(t-\tau) + \Delta x(t-\tau), u_0(t) + \varepsilon\delta u(t)) - f[t]$$

$$= \int_0^1 \frac{d}{ds} f[t;s,\varepsilon\delta\mu]ds = \int_0^1 \Big\{ f_x[t;s,\varepsilon\delta\mu]\Delta x(t) + f_y[t;s,\varepsilon\delta\mu]\{x_0(t-\tau)$$

$$-x_0(t-\tau_0) + \Delta x(t-\tau)\} + \varepsilon f_u[t;s,\varepsilon\delta\mu]\delta u(t) \Big\} ds$$

$$= \Big[\int_0^1 \sigma(t;s,\varepsilon\delta\mu)ds \Big] \Delta x(t) + \Big[\int_0^1 \rho(t;s,\varepsilon\delta\mu)ds \Big] \{x_0(t-\tau)$$

$$-x_0(t-\tau_0) + \Delta x(t-\tau)\} + \varepsilon \Big[\int_0^1 \theta(t;s,\varepsilon\delta\mu)ds \Big] \delta u(t)$$

$$+f_x[t]\Delta x(t) + f_y[t]\{x_0(t-\tau) - x_0(t-\tau_0) + \Delta x(t-\tau)\} + \varepsilon f_u[t]\delta u(t).$$

By taking account of last relation for $t \in [t_{00}, t_{10} + \delta_2]$ we have

$$R_1(t;t_{00},\varepsilon\delta\mu) = \sum_{i=2}^{6} R_i(t;t_{00},\varepsilon\delta\mu),$$

where

$$R_2(t;t_{00},\varepsilon\delta\mu) = \int_{t_{00}}^{t} Y(\xi;t)\sigma_1(\xi;\varepsilon\delta\mu)\Delta x(\xi)d\xi,$$

$$\sigma_1(\xi;\varepsilon\delta\mu) = \int_0^1 \sigma(\xi;s,\varepsilon\delta\mu)ds,\ R_3(t;t_{00},\varepsilon\delta\mu)$$

$$= \int_{t_{00}}^{t} Y(\xi;t)\rho_1(\xi;\varepsilon\delta\mu)\{x_0(\xi-\tau) - x_0(\xi-\tau_0) + \Delta x(\xi-\tau)\}d\xi,$$

$$\rho_1(\xi;\varepsilon\delta\mu) = \int_0^1 \rho(\xi;s,\varepsilon\delta\mu)ds, R_4(t;t_{00},\varepsilon\delta\mu)$$

$$= \varepsilon \int_{t_{00}}^{t} Y(\xi;t)\theta_1(\xi;\varepsilon\delta\mu)\delta u(\xi)d\xi,\ \theta_1(\xi;\varepsilon\delta\mu)$$

$$= \int_0^1 \theta(\xi;s,\varepsilon\delta\mu)ds,\ R_5(t;t_{00},\varepsilon\delta\mu) = \int_{t_{00}}^{t} Y(\xi;t)f_y[\xi]\{x_0(\xi-\tau)$$

$$-x_0(\xi-\tau_0)\}d\xi,\ R_6(t;t_{00},\varepsilon\delta\mu) = \int_{t_{00}}^{t} Y(\xi;t)f_y[\xi]\{\Delta x(\xi-\tau)$$

$$-\Delta x(\xi-\tau_0)\}d\xi$$

(see (4.2)). The function $x_0(t), t \in [\hat{\tau}, t_{10}+\delta_2]$ is absolutely continuous, then for each fixed Lebesgue point $\xi \in (t_{00}, t_{10}+\delta_2)$ of function $\dot{x}_0(\xi-\tau_0)$ we get

$$x_0(\xi-\tau) - x_0(\xi-\tau_0) = \int_{\xi}^{\xi-\varepsilon\delta\tau} \dot{x}_0(s-\tau_0)ds$$

$$= -\varepsilon\dot{x}_0(\xi-\tau_0)\delta\tau + \gamma(\xi;\varepsilon\delta\mu), \tag{4.7}$$

with

$$\lim_{\varepsilon\to 0} \frac{\gamma(\xi;\varepsilon\delta\mu)}{\varepsilon} = 0 \ \text{ uniformly for } \delta\mu \in V^-. \tag{4.8}$$

Thus, (4.8) is valid for almost all points of the interval $(t_{00}, t_{10}+\delta_2)$. From (4.7) taking into account boundedness of the function

$$\dot{x}_0(t) = \begin{cases} \dot{\varphi}_0(t), t \in [\hat{\tau}, t_{00}], \\ f[t], t \in (t_{00}, t_{10}+\delta_2] \end{cases}$$

it follows

$$| x_0(\xi-\tau) - x_0(\xi-\tau_0) | \le O(\varepsilon\delta\mu) \text{ and } \Big| \frac{\gamma(\xi;\varepsilon\delta\mu)}{\varepsilon} \Big| \le const. \tag{4.9}$$

It is clear that

$$| \Delta x(\xi-\tau) - \Delta x(\xi-\tau_0) | = \begin{cases} o(\xi;\varepsilon\delta\mu) \text{ for } \xi \in [t_{00}, t_0+\tau], \\ O(\xi;\varepsilon\delta\mu) \text{ for } \xi \in [t_0+\tau, t_{00}+\tau_0] \end{cases} \tag{4.10}$$

(see (2.6)). We note that there exists $L(\cdot) \in L(J, [0,\infty))$ such that

$$| f(t,x_1,y_1,u_1) - f(t,x_2,y_2,u_2) | \le L(t)(| x_1-x_2 | + | y_1-y_2 | + | u_1-u_2 |),$$
$$t \in J, (x_i,y_i,u_i) \in K_0^2 \times U_0, i=1,2 \text{ ([3], Lemma 2.1.2).}$$

Let $\xi \in [t_{00}+\tau_0, t_{10}+\delta_2]$ then $\xi-\tau \ge t_{00}, \xi-\tau_0 \ge t_{00}$, therefore

$$| \Delta x(\xi-\tau) - \Delta x(\xi-\tau_0) | \le \int_{\xi-\tau_0}^{\xi-\tau} | \dot{\Delta} x(s) | ds$$
$$\le \int_{\xi-\tau_0}^{\xi-\tau} L(s)\Big\{ | \Delta x(s) | + | x_0(s-\tau) - x_0(s-\tau_0) |$$
$$+ | \Delta x(s-\tau) | + \varepsilon | \delta u(s) | \Big\} ds = o(\xi;\varepsilon\delta\mu) \tag{4.11}$$

(see (4.1),(2.4),(2.6) and (4.9)). It is not difficult to see that for the expressions $R_i(t;t_{00},\varepsilon\delta\mu), i=\overline{2,6}$ we get

$$| R_2(t;t_{00},\varepsilon\delta\mu) | \le \| Y \| O(\varepsilon\delta\mu)\sigma_2(\varepsilon\delta\mu), \; | R_3(t;t_{00},\varepsilon\delta\mu) |$$
$$\le \| Y \| O(\varepsilon\delta\mu)\rho_2(\varepsilon\delta\mu), \; | R_4(t;t_{00},\varepsilon\delta\mu) | \le \varepsilon\alpha \| Y \| \theta_2(\varepsilon\delta\mu), R_5(t;t_{00},\varepsilon\delta\mu)$$
$$= -\varepsilon\Big[\int_{t_{00}}^{t} Y(\xi;t) f_y[\xi] \dot{x}_0(\xi-\tau_0) d\xi \Big] \delta\tau + \gamma_1(t;\varepsilon\delta\mu), | R_6(t;t_{00},\varepsilon\delta\mu) |$$
$$\le \| Y \| \int_{t_{00}}^{t_{10}+\delta_2} | f_y[\xi] || \Delta x(\xi-\tau) - \Delta x(\xi-\tau_0) | d\xi = o(\varepsilon\delta\mu) \tag{4.12}$$

(see (2.6),(2.3),(4.7) and (4.9)-(4.11)). Here

$$\| Y \| = \sup\Big\{ | Y(\xi;t) | : (\xi,t) \in \Pi \Big\}, \gamma_1(t;\varepsilon\delta\mu) = \int_{t_{00}}^{t} Y(\xi;t) f_y[\xi] \gamma(\xi;\varepsilon\delta\mu) d\xi$$

$$\sigma_2(\varepsilon\delta\mu) = \int_{t_{00}}^{t_{10}+\delta_2} | \sigma_1(\xi;\varepsilon\delta\mu) | d\xi, \; \rho_2(\varepsilon\delta\mu) = \int_{t_{00}}^{t_{10}+\delta_2} | \rho_1(\xi;\varepsilon\delta\mu) |,$$

$$\theta_2(\varepsilon\delta\mu) = \int_{t_{00}}^{t_{10}+\delta_2} \Big\{ \sum_{i=1}^{k} \mid \delta u_i(\xi) \mid \Big\} \mid \theta_1(\xi;\varepsilon\delta\mu) \mid d\xi,$$

Obviously,

$$\Big|\frac{\gamma_1(t;\varepsilon\delta\mu)}{\varepsilon}\Big| \leq \parallel Y \parallel \int_{t_{00}}^{t_{10}+\delta_2} \mid f_y[\xi] \mid \Big|\frac{\gamma(\xi;\varepsilon\delta\mu)}{\varepsilon}\Big| d\xi.$$

By the Lebesguer theorem on passing to the limit under the integral sign, we have

$$\lim_{\varepsilon\to 0} \sigma_2(\varepsilon\delta\mu) = 0,\ \lim_{\varepsilon\to 0} \rho_2(\varepsilon\delta\mu) = 0,\ \lim_{\varepsilon\to 0} \theta_2(\varepsilon\delta\mu) = 0,\ \lim_{\varepsilon\to 0} \Big|\frac{\gamma_1(t;\varepsilon\delta\mu)}{\varepsilon}\Big| = 0$$

uniformly for $(t,\delta\mu) \in [t_{00}, t_{10}+\delta_2] \times V^-$ (see(4.9)). Thus,

$$R_i(t;t_{00},\varepsilon\delta\mu) = o(t;\varepsilon\delta\mu), i = 2,3,4; \tag{4.13}$$

and

$$R_5(t;t_{00},\varepsilon\delta\mu) = -\varepsilon\Big[\int_{t_{00}}^{t} Y(\xi;t) f_y[\xi]\dot{x}_0(\xi-\tau_0)d\xi\Big]\delta\tau + o(t;\varepsilon\delta\mu). \tag{4.14}$$

On the basis of (4.12)-(4.14) we obtain

$$R_1(t;t_{00},\varepsilon\delta\mu) = -\varepsilon\Big[\int_{t_{00}}^{t} Y(\xi;t) f_y[\xi]\dot{x}_0(\xi-\tau_0)d\xi\Big]\delta\tau + o(t;\varepsilon\delta\mu). \tag{4.15}$$

From (4.3) by virtue of (4.5),(4.6) and (4.15) we obtain (3.1), where $\delta x(t;\delta\mu)$ has form (3.2).

5. Proof of theorem 3.2

Let $\hat{t}_0 \in (t_{00}, t_{10})$ be a fixed point, and let $\varepsilon_2 \in (0, \varepsilon_1]$ be so small (see Lemma 2.3) that $t_0 < \hat{t}_0$ and $t_0 - \tau_0 < t_{00}$, for arbitrary $(\varepsilon,\delta\mu) \in (0,\varepsilon_1] \times V^+$. The function $\Delta x(t)$ satisfies equation (4.1) on the interval $[t_0, t_{10}+\delta_2]$. By using the Cauchy formula, we can represent it in the form

$$\Delta x(t) = Y(t_0;t)\Delta x(t_0) + \varepsilon \int_{t_0}^{t} Y(\xi;t) f_u[\xi]\delta u(\xi)d\xi + \sum_{i=0}^{1} R_i(t;t_0,\varepsilon\delta\mu), \tag{5.1}$$

(see (4.4)). The matrix function $Y(\xi;t)$ is continuous on $[t_{00},\hat{t}_0] \times [\hat{t}_0, t_{10}+\delta_2] \subset \Pi$, therefore

$$Y(t_0;t)\Delta x(t_0) = \varepsilon Y(t_{00};t)[\delta\varphi(t_{00}) + (\dot{\varphi}_0^+ - f^+)]\delta t_0 + o(\varepsilon\delta\mu) \tag{5.2}$$

(see (2.8)). Now let us transform $R_0(t;t_0,\varepsilon\delta\mu)$. It is not difficult to see that

$$R_0(t;t_0,\varepsilon\delta\mu) = \varepsilon \int_{t_0-\tau_0}^{t_{00}} Y(\xi+\tau_0;t) f_y[\xi+\tau_0]\delta\varphi(\xi)d\xi$$

$$+ \int_{t_{00}}^{t_0} Y(\xi+\tau_0;t) f_y[\xi+\tau_0]\Delta x(\xi)d\xi$$

$$= \varepsilon \int_{t_{00}-\tau_0}^{t_{00}} Y(\xi+\tau_0;t) f_y[\xi+\tau_0]\delta\varphi(\xi)d\xi + o(t;\varepsilon\delta\mu). \tag{5.3}$$

In a similar way, for $t \in [\hat{t}_0, t_{10}+\delta_2]$ one can prove

$$R_1(t;t_0,\varepsilon\delta\mu) = -\varepsilon\Big[\int_{t_{00}}^{t} Y(\xi;t) f_y[\xi]\dot{x}_0(\xi-\tau_0)d\xi\Big]\delta\tau + o(t;\varepsilon\delta\mu). \tag{5.4}$$

Finally, we note that

$$\varepsilon \int_{t_0}^{t} Y(\xi;t) f_u[\xi] \delta u(\xi) d\xi = \varepsilon \int_{t_{00}}^{t} Y(\xi;t) f_u[\xi] \delta u(\xi) d\xi + o(t;\varepsilon\delta\mu). \tag{5.5}$$

for $t \in [\hat{t}_0, t_{10} + \delta_2]$. By taking account of (5.2)-(5.5), from (5.1), we obtain (3.1), where $\delta x(t;\varepsilon\delta\mu)$ has form (3.6).

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References

[1] Gamkrelidze,R. V., (1978), Principles of optimal control theory, Plenum Press-New York and London.

[2] Kharatishvili, G. L., Machaidze, Z. A., Markozashvili, N. I. and Tadumadze, T. A., (1973), Abstract Variational Theory and Its Application to Optimal Problems with Delays, Metsniereba, Tbilisi, (in Russian).

[3] Kharatishvili, G. L. and Tadumadze, T. A., (2007), Variation formulas of solutions and optimal control problems for differential equations with retarded argument, J. Math. Sci.(N.Y.), 104(1), 1-175.

[4] Kharatishvili, G. and Tadumadze, T., (2008), Variation formulas for solution of a nonlinear differential equation with time delay and mixed initial condition, J. Math. Sci. (N.Y.), 148 (3), 302-330.

[5] Kharatishvili, G., Tadumadze, T. and Gorgodze, N., (2000), Continuous dependence and differentiability of solution with respect to initial data and right-hand side for differential equations with deviating argument, Mem. Differential Equations Math. Phys., 19, 3-105.

[6] Neustadt, L. W., (1976), Optimization: A theory of necessary conditions, Princeton Univ.Press, Princeton, New York.

[7] Ogustoreli, N. M., (1966), Time-Delay Control Systems, Academic Press, New York-London.

[8] Ramishvili, I. and Tadumadze, T., (2004), Formulas of variation for a solution of neutral differential equations with continuous initial condition, Georgian Math. J., 11:1, 155-175.

[9] Tadumadze, T. A., (1983), Some topics of qualitative theory of optimal control, Tbilisi State University Press, Tbilisi, (in Russian).

[10] Tadumadze, T., (2009), Formulas of variation for solutions for some classes of functional-differential equations and their applications, Nonlinear Analysis, 71(12), e706-e710.

[11] Tadumadze, T., and Alkhazishvili, L., (2004), The formulas of variation of solution for non-linear controlled delay differential equation with continuous initial condition, Mem. Diff. Eq. Math. Phys., 31, 83-97.

[12] Tadumadze, T., (2010), Variation formulas of solution for nonlinear delay differential equation with taking into account delay perturbation and discontinuous initial condition, Georgian International Journal of Sciences and Technology, v. 3, no. 1, 53-71.

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