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SINGULAR PERTURBATIONS FOR ABSTRACT ELLIPTIC OPERATORS AND APPLICATIONS

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Abstract. Dirichlet problem for parameter depended elliptic differential-operator equation with variable coefficients in smooth domains is studied. The uniform maximal regularity, Fredholmness and the positivity of this problem in vector-valued L_p-spaces are obtained. It is proven that the corresponding differential operator is positive and is a generator of an analytic semigroup. In application, the maximal regularity properties of Cauchy problem for abstract parabolic equation and anisotropic elliptic equations with small parameters are established.

Keywords: Boundary value problems, differential-operator equations, Banach-valued function spaces, operator-valued multipliers, interpolation of Banach spaces, semigroup of operators.

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1. Introduction, notations and background

In last years, the maximal regularity properties of boundary value problems (BVPs) for differential-operator equations (DOEs) have found many applications in PDE, psedo DE and in the different physical process (see for references $[1-4]$, [8], [10], $[12-23]$, $[26-27]$). The main objective of the present paper is to discuss the maximal L_p regularity properties of BVP for elliptic variable coefficient DOE with small parameter

$$\begin{aligned}(L_\varepsilon+\lambda)\,u &= \sum_{i,j=1}^{n}\varepsilon a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i\partial x_j}-(A(x)+\lambda)\,u(x)+\sum_{k=1}^{n}\varepsilon^{\frac{1}{2}}A_k(x)\frac{\partial u(x)}{\partial x_k} \qquad (1)\\ &= f(x), \quad x\in G,\ u\mid_\Gamma=0,\end{aligned}$$

where Γ is a boundary of region $G\subset R^n$, ε is a positive small parameter and λ is a spectral parameter, a_{ij} are complex-valued functions, A and A_k are possible unbounded linear operators in a Banach space E.

Maximal regularity of partial DOE in L_p spaces was studied in [1], [4], [7], $[18-23]$. The results in [4] and $[18-23]$ are restricted to rectangular domains and equations do not contain mixed derivatives in leading part. Moreover problems investigated in [1] and [8] involves only bounded operator coefficients.

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In contrast to all above we study elliptic problems with unbounded operator coefficients in general domains with smooth boundaries.

We say that the problem (1) is uniform maximal L_p-regular (or separable in L_p) if:

(1) for all $f \in L_p(G;E)$ there exists a unique solution $u_\varepsilon = u \in W_p^2(G;E(A),E)$ satisfying (1) a.e. on G;

(2) there exists a positive constant C independent of f and ε such that

$$\sum_{i,j=1}^{n} \varepsilon \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L_p(G;E)} + \|Au\|_{L_p(G;E)} \leq C \|f\|_{L_p(G;E)} .$$

Let O be an operator generated by the problem (1) i.e.

$$\begin{aligned} D(O_\varepsilon) &= W_p^2(G;E(A),E,L_\Gamma) \\ &= \left\{u : u \in W_p^2(G;E) \cap L_p(G;E(A)),\ u\mid_\Gamma = 0\right\}, \quad O_\varepsilon u = L_\varepsilon u. \end{aligned}$$

We show that the problem (1) is maximal L_p regular, which implies that there is a bounded inverse of O from L_p to $W_p^2(G;E(A),E)$. Moreover, we prove that the operator O is positive and generator of analytic semigroup in L_p.

Since (1) involves unbounded operator coefficients, it becomes difficult to obtain global estimates for solutions of (1). Therefore to show O has a left inverse and that its range coincides with L_p, we use covering and flattening arguments, representation formulas, operator-valued Fourier multiplier results, abstract embedding theorems (Theorems A_1,A_2) and the separability properties of local differential operators with constant coefficients (both on plane and half plane). Then by using these results along with qualitative properties of some embedding operators we prove discreetness of spectrum and completeness of roots elements of O. In applications well posedness of abstract parabolic Cauchy problem and optimal regularity of anisotropic elliptic equations in $L_{\mathbf{p}}$, $\mathbf{p} = (p_1, p)$ (i.e. Lebesgue spaces with mixed norm) and maximal regularity for infinite systems of elliptic equations are derived.

The paper is organized as follows: Section 2 collects definitions and background materials, embedding theorems of Sobolev-Lions spaces, maximal regularity properties for elliptic DOE in line and halfline and estimates of approximation numbers. In Section3, the L_p-separability and Fredholmness results for (1) are presented. Finally Section 4-5 are devoted abstract parabolic Cauchy problem and to some applications, respectively.

A Banach space E is called UMD-space if the Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon \to 0} \int\limits_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\, dy$$

is bounded in $L_p(R,E)$, $p \in (1,\infty)$ (see. e.g. [6]). UMD spaces include e.g. L_p, l_p spaces and Lorentz spaces L_{pq}, p, $q \in (1,\infty)$.

Let $\mathbf{C}$ be the set of complex numbers and

$$S_\varphi = \{\ \lambda \in \mathbf{C},\ |\arg \lambda| \leq \varphi\} \cup \{0\}, 0 \leq \varphi < \pi.$$

A linear operator A is said to be a positive in a Banach space E with bound $M > 0$ if $D(A)$ is dense on E and

$$\left\| (A + \lambda I)^{-1} \right\|_{L(E)} \leq M (1 + |\lambda|)^{-1}$$

with $\lambda \in S_\varphi, \varphi \in (0,\pi]$, where I is an identity operator in E and $L(E)$ is the space of all bounded linear operators in E. Sometimes instead of $A + \lambda I$ will be written $A + \lambda$ and denoted by A_λ. It is known that ([24, §1.15.1]) there exist fractional powers A^θ of positive operator A. Let $E\left(A^\theta\right)$ denote the space $D\left(A^\theta\right)$ with graphical norm

$$\|u\|_{E(A^\theta)} = \left(\|u\|^p + \left\|A^\theta u\right\|^p\right)^{\frac{1}{p}}, 1 \leq p < \infty, \ -\infty < \theta < \infty.$$

Let E_1 and E_2 be two Banach spaces. $(E_1, E_2)_{\theta,p}$, $0 < \theta < 1, 1 \leq p \leq \infty$ will denote the interpolation spaces defined by K method [24, §1.3.1].

A set $W \subset B(E_1, E_2)$ is called R-bounded (see [7], [24]) if there is a constant $C > 0$ such that for all $T_1, T_2, ..., T_m \in W$ and $u_1, u_2, ..., u_m \in E_1, m \in N$

$$\int_0^1 \left\|\sum_{j=1}^m r_j(y) T_j u_j\right\|_{E_2} dy \leq C \int_0^1 \left\|\sum_{j=1}^m r_j(y) u_j\right\|_{E_1} dy,$$

where $\{r_j\}$ is a sequence of independent symmetric $\{-1, 1\}$-valued random variables on $[0, 1]$.

Let $S(R^n; E)$ denote the Schwartz class, i.e. the space of all E-valued rapidly decreasing smooth functions on R^n. Let F denote the Fourier transformation. A function $\Psi \in C(R^n; B(E_1, E_2))$ is called a Fourier multiplier from $L_p(R^n; E_1)$ to $L_p(R^n; E_2)$ if the map $u \to \Lambda u = F^{-1}\Psi(\xi) Fu$, $u \in S(R^n; E_1)$ is well defined and extends to a bounded linear operator

$$\Lambda : \ L_p(R^n; E_1) \to \ L_p(R^n; E_2).$$

The set of all multipliers from $L_p(R^n; E_1)$ to $L_p(R^n; E_2)$ will be denoted by $M_p^p(E_1, E_2)$. For $E_1 = E_2 = E$ it denotes by $M_p^p(E)$.

Let

$$\begin{aligned} \xi &= (\xi_1, \xi_2, ..., \xi_n), \ \xi \in R^n, \ \beta = (\beta_1, \beta_2, ..., \beta_n), \\ \xi^\beta &= \xi_1^{\beta_1} \xi_2^{\beta_2} ... \xi_n^{\beta_n}, U_n = \{\beta = (\beta_1, \beta_2, ..., \beta_n), \beta_k \in \{0, 1\}\}. \end{aligned}$$

Definition 1.1. *A Banach space E is said to be a space satisfying a multiplier condition if for any $\Psi \in C^{(n)}(R^n; B(E))$ the R-boundedness of the set $\left\{\xi^\beta D_\xi^\beta \Psi(\xi) : \xi \in R^n \backslash 0, \beta \in U_n\right\}$ implies that Ψ is a Fourier multiplier in $L_p(R^n; E)$, i.e. $\Psi \in M_p^p(E)$ for any $p \in (1, \infty)$.*

Definition 1.2. *The φ-positive operator A is said to be an R-positive in a Banach space E if there exists $\varphi \in [0\ \pi)$ such that the set*

$$L_A = \left\{A(A + \xi)^{-1} : \xi \in S_\varphi\right\}$$

is R-bounded.

A linear operator $A(x)$ is said to be positive in E uniformly in x if $D(A(x))$ is independent of x, $D(A(x))$ is dense in E and

$$\left\|(A(x) + \lambda I)^{-1}\right\| \leq \frac{M}{1 + |\lambda|}$$

for all $\lambda \in S(\varphi)$, $\varphi \in [0\ \pi)$.

Let $\sigma_\infty(E_1, E_2)$ denote the space of all compact operators from E_1 to E_2. For $E_1 = E_2 = E$ it is denoted by $\sigma_\infty(E)$.

Let E_0 and E be two Banach spaces and E_0 continuously and densely embedded into E. Let m be a natural number.

Let $W_p^m(G;E_0,E)$ denote a function space of all functions $u \in L_p(G;E_0)$ possess the generalized derivatives $D_k^m u = \frac{\partial^m u}{\partial x_k^m}$ such that $D_k^m u \in L_p(G;E)$ with the norm

$$\|u\|_{W_p^m(G;E_0,E)} = \|u\|_{L_p(G;E_0)} + \sum_{k=1}^n \|D_k^m u\|_{L_p(G;E)} < \infty.$$

We will called it Sobolev-Lions type space. For $E_0 = E$ the space $W_p^m(G;E_0,E)$ will denoted by $W_p^m(G;E)$. It is clear to see that

$$W_p^m(G;E_0,E) = W_p^m(G;E) \cap L_p(G;E_0).$$

Let we define the following parametrized norm

$$\|u\|_{W_{p,\varepsilon}^m(G;E_0,E)} = \|u\|_{L_p(G;E_0)} + \sum_{k=1}^n \varepsilon \|D_k^m u\|_{L_p(G;E)} < \infty.$$

Let G be a domain in R^n with the sufficiently smooth boundary Γ. The space $W_p^s(\Gamma;E_0,E)$ is defined as the trace space of $W_p^m(G;E_0,E)$ as in a scalar case (see [15, § 1.7.3] or [24, § 3.6.1]) i.e. for $E_0 = E = \mathbf{C}$ replacing the space $L_p\left(R^{n-1}\right)$ by $L_p\left(R^{n-1};E\right)$.

From [22] it follows the following theorem.

Theorem 1.1. *Suppose the following conditions are satisfied:*

(1) E is a Banach space satisfying the multiplier condition with respect to $p \in (1,\infty)$ and A is an R-positive operator in E;

(2) $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ are n tuples of nonnegative integer numbers such that $|\alpha| = \sum\limits_{k=1}^n \alpha_k$, $\varkappa = \frac{|\alpha|}{m} \leq 1$ and $0 < h \leq h_0 < \infty$, $0 < \mu \leq 1 - \varkappa$;

(3) $\Omega \in R^n$ is a region such that there exists a bounded linear extension operator from $W_p^m(G;E(A),E)$ to $W_p^m(R^n;E(A),E)$.

Then an embedding

$$D^\alpha W_p^m(G;E(A),E) \subset L_p\left(G;E\left(A^{1-\varkappa-\mu}\right)\right)$$

is continuous and there exists a positive constant C_μ such that

$$\varepsilon^{\frac{|\alpha|}{m}} \|D^\alpha u\|_{L_p(G;E(A^{1-\varkappa-\mu}))} \leq C_\mu \left[h^\mu \|u\|_{W_p^m(G;E(A),E)} + h^{-(1-\mu)} \|u\|_{L_p(G;E)}\right]$$

for all $u \in W_p^m(G;E(A),E)$.

Theorem 1.2. *Suppose all conditions of the Theorem 1.1 are satisfied and suppose G is a bounded region in R^n, $A^{-1} \in \sigma_\infty(E)$. Then for $0 < \mu \leq 1-\varkappa$ an embedding*

$$D^\alpha W_p^m(G;E(A),E) \subset L_p\left(G;E\left(A^{1-\varkappa-\mu}\right)\right)$$

is compact.

Consider at first, the following DOE in all space R^n

$$(L+\lambda)u = \sum_{k=1}^n \varepsilon a_k \frac{\partial^m u(x)}{\partial x_k^m} + (A+\lambda)u = f(x),\ x \in R^n, \tag{2}$$

where ε is a positive small parameter and a_k are complex numbers.

We set $L(\xi) = \sum\limits_{k=1}^n a_k \xi_k^m$, for $\xi = (\xi_1, \xi_2, ..., \xi_n) \in R^n$. From [15, § 1.7.3] or [24, § 3.6.1] it follows the following theorem.

Theorem 1.3. *The maps* $u \to u^{(j)}\left(x^1, 0\right)$ *are bounded linear and surjective from* $W_p^m\left(R_+^n; E(A), E\right)$ *to* F_j.

Then, by [4], we get

Theorem 1.4. *Let* E *be a Banach space satisfying the multiplier condition with respect to* $p \in (1, \infty)$ *and* A *is a R-positive operator in* E *for a* $\varphi \in [0\ \pi)$. *Moreover*

$$|L(\xi)| \geq M \sum_{k=1}^{n} |\xi_k|^m,\ L(\xi) \in S(\varphi).$$

Then the problem (2) *has a unique solution* $u \in W_p^m(R^n; E(A), E)$ *for* $f \in L_p(R^n; E)$, $|\arg \lambda| \leq \varphi$ *and the uniform coercive estimate holds*

$$\sum_{k=1}^{n} \sum_{i=0}^{m} |\lambda|^{1-\frac{i}{m}} \varepsilon^{\frac{i}{m}} \left\|D_k^i u\right\|_{L_p(R^n;E)} + \|Au\|_{L_p(R^n;E)} \leq M \|f\|_{L_p(R^n;E)}.$$

Considering BVPs for DOE

$$(L+\lambda)u = \sum_{k=1}^{n} \varepsilon a_k D_k^m u(x) + A_\lambda u = f(x), x \in R_+^n, \tag{3}$$

$$L_j u = \sum_{i=0}^{m_j} \varepsilon^{\sigma_i} \alpha_{ji} u^{(i)}\left(x^1, 0\right) = f_j,\ j = 1, 2, ...m$$

where $\sigma_i = \frac{1}{m}\left(i + \frac{1}{p}\right)$, $m_j \in \{0, 1, ..., m-1\}$, a, α_{ji} are complex numbers and $A_\lambda = A + \lambda$, A is a possible unbounded operator in E. Let ω_j, $j = 1, 2, ..., m$ be roots of an equation

$$a_n \omega^m + 1 = 0.$$

Let

$$F_j = \left(W_p^m\left(R^{n-1}; E(A), E\right), L_p\left(R^{n-1}; E\right)\right)_{\theta_j, p}, \theta_j = \frac{pm_j + 1}{mp}, j = 1, 2, ..., m,$$

$$L_0(\xi) = \sum_{j=1}^{n-1} a_j \xi_j^m.$$

By virtue of [5] and trace Theorem 1.3 we obtain

Theorem 1.5. *Let* E *be a Banach space satisfying the multiplier condition with respect to* $p \in (1, \infty)$ *and* A *is a R-positive operator in* E *for a* $\varphi \in \left(0, \frac{\pi}{2}\right)$. *Let* $|\arg\ \omega_j - \pi| \leq \frac{\pi}{2} - \varphi$, $j = 1, 2, ..., d$, $|\arg\ \omega_j| \leq \frac{\pi}{2} - \varphi$, $j = d+1, d+2, ..., m$, $0 < d < m, \alpha_{km_k} \neq 0$ *and*

$$|L_0(\xi)| \geq M \sum_{k=1}^{n-1} |\xi_k|^m, L_0(\xi) \in S(\varphi).$$

Then the operator $u \to \{[L+\lambda]u, L_1 u, L_2 u, ...L_m\}$ *is an isomorphism (algebraic and topological) from* $W_p^m\left(R_+^n; E(A), E\right)$ *onto* $L_p\left(R_+^n; E\right) \times \prod_{j=1}^{m} F_j$. *Moreover, for* $\lambda \in S(\varphi)$ *and sufficiently large* $|\lambda|$ *the uniform coercive estimate holds*

$$\sum_{k=1}^{n} \sum_{i=0}^{m} \varepsilon^{\frac{i}{m}} |\lambda|^{1-\frac{i}{m}} \left\|D_k^i u\right\|_{L_p\left(R_+^n;E\right)} + \|Au\|_{L_p\left(R_+^n;E\right)} \leq$$

$$M\left[\|f\|_{L_p\left(R_+^n;E\right)}+\sum_{j=1}^{m}\left(\|f_j\|_{F_j}+|\lambda|^{1-\theta_k}\|f_k\|_E\right)\right].$$

Proof. Since $L_p\left(R_+^n;E\right)=L_p\left(R_+;L_p\left(R^{n-1};E\right)\right)$, the problem (3) can be express as

$$Lu=\varepsilon a_n D_{x_n}^m u\left(x_n\right)+(B+\lambda)u=f_k\left(y\right),\ L_k u=f_k,\ k=1,2,$$

where B is a differential operator in $L_p\left(R^{n-1};E\right)$ generated by problem (2). By virtue of [4, Theorem 3.1] the operator B is R positive in $L_p\left(R^{n-1};E\right)$. By [1, Theorem 4.5.2], $L_p\left(R^{n-1};E\right)\in UMD$ provided $E\in UMD$, $p\in(1,\infty)$. Then by virtue [25], $L_p\left(R^{n-1};E\right)$ is the space satisfying the multiplier condition. Then by virtue of [2, Theorem 2] we get the assertion. □

2. Partial DOE with variable coefficients

Consider the inhomogenous problem (1), i.e.

$$\begin{aligned}(L_\varepsilon+\lambda)u &= \sum_{i,j=1}^{n}\varepsilon a_{ij}\left(x\right)\frac{\partial^2 u\left(x\right)}{\partial x_i\partial x_j}-\left(A\left(x\right)+\lambda\right)u\left(x\right)+\sum_{k=1}^{n}\varepsilon^{\frac{1}{2}}A_k\left(x\right)\frac{\partial u\left(x\right)}{\partial x_k} \qquad (4)\\ &= f\left(x\right),\quad x\in G,\ L_{\Gamma\varepsilon}u=\varepsilon^{\frac{1}{2p}}u\mid_\Gamma=g,\end{aligned}$$

where the second equality is in the trace sense.

First we obtain coercive estimate for strong solutions of the problem (4).

Condition 1. Suppose the following conditions are satisfied:

(1) $a_{ij}=a_{ji}$ and there is $\mu>0$ such that

$$\mu^{-1}|\xi|^2\le L_0\left(x,\xi\right)\le\mu|\xi|^2\ \text{ for } x\in G,\ L_0\left(x,\xi\right)=\sum_{i,j=1}^{n}a_{ij}\left(x\right)\xi_i\xi_j;$$

(2) $\Gamma\in C^{(2)}$ $[10,\ \S\ 6.2]$.

Let

$$F=\left(W_p^2\left(\Gamma;E\left(A\right),E\right),L_p\left(\Gamma;E\right)\right)_{\frac{1}{2p},p}.$$

Theorem 2.1. *Let the Condition 1 be satisfied and:*

(1) E is a Banach space satisfying the multiplier condition with respect to $p\in(1,\infty)$;

(2) $A\left(x\right)$ is a R-positive in E uniformly with respect to $x\in\bar{G}$, $A\left(x\right)A^{-1}\left(x^0\right)\in C\left(\bar{G};B\left(E\right)\right)$;

(3) for any $\delta>0$ there is $C\left(\delta\right)>0$ such that for a.e. $x\in G$ and for $u\in\left(E\left(A\right),E\right)_{\frac{1}{2},\infty}$

$$\|A_k\left(x\right)u\|\le\delta\|u\|_{\left(E\left(A\right),E\right)_{\frac{1}{2},\infty}}+C\left(\delta\right)\|u\|.$$

Then for $u\in W_p^2\left(G;E\left(A\right),E\right)$ and for $\lambda\in S\left(\varphi\right)$ and sufficiently large $|\lambda|$ we have

$$\sum_{k=1}^{n}\sum_{i=0}^{2}\varepsilon^{\frac{i}{2}}|\lambda|^{1-\frac{i}{2}}\left\|D_k^i u\right\|_{L_p\left(G;E\right)}\le M\left[\|\left(L_\varepsilon+\lambda\right)u\|_{L_p\left(G;E\right)}+\|L_{\Gamma\varepsilon}u\|_F\right]. \qquad (5)$$

Proof. Let $G_1,G_2,...,G_N$ be regions in R^n and $\varphi_1,\varphi_2,...,\varphi_N$ be a corresponding partition of unity that functions φ_j are smooth on R, $\sigma_j=\mathrm{supp}\varphi_j\subset G_j$ and $\sum\limits_{j=1}^{N}\varphi_j\left(x\right)=1$.Then for $u \in W_p^2(G;E(A),E)$ we have $u(x) = \sum\limits_{j=1}^{N} u_j(x)$, where $u_j(x) = u(x)\varphi_j(x)$. Let $u \in W_p^2(G;E(A),E)$ then, from the equation (4) we obtain

$$\begin{aligned}(L_\varepsilon + \lambda) u_j &= \sum_{k,i=1}^{n} \varepsilon a_{ki} D_{ki}^2 u_j(x) - (A(x) + \lambda) u_j(x) = f_j(x), && (6)\\ L_{\Gamma\varepsilon} u_j &= g_j,\ j = 1,2,...,N. && (7)\end{aligned}$$

where

$$f_j = f\varphi_j + \sum_{k,i=1}^{n} \varepsilon a_{ki}\left[\varphi_j D_{ki}^2 u + D_k u D_k \varphi_j + \varphi_j D_i u + u D_{ki}^2 \varphi_j\right] - \qquad (8)$$

$$\sum_{k=1}^{n} \varepsilon^{\frac{1}{2}} \varphi_j A_k(x) \frac{\partial u(x)}{\partial x_k}, \qquad j = 1,2,...,N.$$

Let $G_j \cap G = G_j$. Since the boundary Γ is sufficient smooth, then (see e.g. [15,§ 1.7.3] there are differentiable diffeomorphism Ψ_j on neighborhood of G_j transform G_j to $\tilde{G}_j$ with plain boundary and such that $L_\Gamma u_j$ are transformed to $\tilde{L}_\Gamma \tilde{u}_j = \tilde{u}_j(y)\mid_{y_n=0}$, where $\tilde{\upsilon}(y) = \upsilon(\Psi_j(y))$ for $\upsilon \in W_p^2(G_j;E(A),E)$. For these transformations the space $W_p^2(G_j;E(A),E)$ are isomorphically mapped to spaces $W_p^2\left(\tilde{G}_j;E(A),E\right)$ and under these maps the equation (6) is transforms to

$$(L + \lambda) \tilde{u}_j = \sum_{k,,i=1}^{n} \varepsilon \tilde{a}_{kij} D_{ki}^2 \tilde{u}_j(y) - \tilde{A}_{j\lambda}(y)\tilde{u}_j(y) = \tilde{f}_j(y).$$

Moreover by virtue of condition (1), there is a linear mapping which transforms the expression $\sum\limits_{k,,i=1}^{n} \tilde{a}_{ki} D_{kij}^2 \tilde{u}_j(y) + \tilde{A}_{j\lambda}(y)\tilde{u}_j(y)$ to

$$\sum_{k}^{n} \varepsilon \tilde{a}_{kj} D_k^2 \tilde{u}_j(y) - \tilde{A}_{j\lambda}(y)\tilde{u}_j(y),\ \tilde{a}_{kj} > 0\ .$$

After redenoting y by x, $\tilde{G}$ by $G_j, \tilde{a}_{kj}(y)$ by $a_{kj}(x)$, $\tilde{A}_{j\lambda}(y)$ by $A_\lambda(x)$ and $\tilde{u}_j(y)$ by $u_j(x)$ et.c, and freezing coefficients in transformed equation (6) we obtain that

$$\sum_{k}^{n} \varepsilon a_{kj}(x_{j0}) D_k^2 u_j(x) + (A(x_{j0}) + \lambda) u_j(x) = F_j(x), \qquad (9)$$

$$L_{\Gamma\varepsilon} u_j = V_j\left(x^1\right), \qquad (10)$$

where

$$F_j = f_j + [A(x_{j0}) - A(x)] u_j + \sum_{k}^{n} \varepsilon [a_k(x) - a_{ki}(x_{j0})] D_k^2 u_j(x), \qquad (11)$$

By Theorem 1.4 we obtain that the problem $(9) - (10)$ has a unique solution $u_j \in W_p^2(G_j;E(A),E)$ and for $\lambda \in S(\varphi)$ and for sufficiently large $|\lambda|$ the following coercive estimate holds

$$\sum_{k=1}^{n}\sum_{i=0}^{2}\varepsilon^{\frac{i}{2}}\left|\lambda\right|^{1-\frac{i}{2}}\left\|D_k^i u_j\right\|_{G_j,p}+\left\|Au_j\right\|_{G_j,p}\leq C\left\|F_j\right\|_{G_j,p}+\left\|\Phi_j\right\|_F \tag{12}$$

In a similar way, by virtue of Theorem 1.3 we obtain estimates of type (12) for regions $G_j\subset G$. Whence, using properties of the smoothness of coefficients of equation (11) and by virtue of Theorem 1.1 and Theorem 1.3 choosing diameters of σ_j sufficiently small, we get that

$$\left\|F_j\right\|_{G_j,p}\leq\delta\left\|u_j\right\|_{W_p^2(G_j;E(A),E)}+C\left(\delta\right)\left\|u_j\right\|_{G_j,p}, \tag{13}$$

$$\left\|V_j\right\|_F\leq C\left\|g_j\right\|_F+\delta\left\|u_j\right\|_{W_p^2(G_j;E(A),E)}+C\left(\delta\right)\left\|u_j\right\|_{G_j,p}, \tag{14}$$

where δ is a sufficiently small and $C\left(\delta\right)$ is a continuous function. From (12)- (14) we get

$$\begin{aligned}\sum_{k=1}^{n}\sum_{i=0}^{1}\varepsilon^{\frac{i}{2}}\left|\lambda\right|^{1-\frac{i}{2}}\left\|D_k^i u_j\right\|_{G_j,p} &+ \left\|Au_j\right\|_{G_j,p}\\ &\leq C\left\|f\right\|_{G_j,p}+\left\|g_j\right\|_F+\varepsilon\left\|u_j\right\|_{W_p^2}+C\left(\varepsilon\right)\left\|u_j\right\|_{G_j,p}.\end{aligned}$$

Choosing $\varepsilon<1$ from the above inequality we have

$$\sum_{k=1}^{n}\sum_{i=0}^{2}\varepsilon^{\frac{i}{2}}\left|\lambda\right|^{1-\frac{i}{2}}\left\|D_k^i u_j\right\|_{G_j,p}+\left\|Au_j\right\|_{G_j,p}\leq C\left[\left\|f\right\|_{G_j,p}+\left\|u_j\right\|_{G_j,p}+\left\|g_j\right\|_F\right]. \tag{15}$$

By a similar manner we also obtain estimates (15) for regions G_j entirely belonging to G. Then by virtue of the estimate (15) for $u\in W_p^2\left(G;E\left(A\right),E\right)$ we have

$$\sum_{k=1}^{n}\sum_{i=0}^{2}\varepsilon^{\frac{i}{2}}\left|\lambda\right|^{1-\frac{i}{2}}\left\|D_k^i u\right\|_p+\left\|Au\right\|_p\leq C\left[\left\|\left(L_\varepsilon+\lambda\right)u\right\|_p+\left\|u\right\|_p+\left\|g_j\right\|_F\right]. \tag{16}$$

Let $u\in W_p^2\left(G;E\left(A\right),E\right)$ be the solution of the problem (4). Then for $\lambda\in S\left(\varphi\right)$ we get

$$\left\|u\right\|_p=\left\|\left(L_\varepsilon+\lambda\right)u-L_\varepsilon u\right\|_p\leq\frac{1}{\lambda}\left[\left\|\left(L_\varepsilon+\lambda\right)u\right\|_p+\left\|u\right\|_{W_p^2}\right]. \tag{17}$$

Then by Theorem 1.1, by virtue of (16), (17) for sufficiently large $\left|\lambda\right|$ and for $u\in W_p^2\left(G;E\left(A\right),E\right)$ we get the estimate (5). □

Consider now the BVP (1).

Theorem 2.2. *Let all conditions of Theorem 2.1 are satisfied. Then for all $f\in L_p\left(G;E\right)$, for $\lambda\in S\left(\varphi\right)$ and sufficiently large $\left|\lambda\right|$ there is a unique solution of the problem* (1) *and the following uniform coercive estimate holds*

$$\sum_{k=1}^{n}\sum_{i=0}^{2}\varepsilon^{\frac{i}{2}}\left|\lambda\right|^{1-\frac{i}{2}}\left\|D_k^i u_j\right\|_{L_p(G;E)}+C\left(\varepsilon\right)\left\|u_j\right\|_{G_j,p},\leq M\left\|f\right\|_{L_p(G;E)}. \tag{18}$$

Proof. From Theorem we obtain the estimate (18). The estimate (18) implies that the problem (1) has only a unique solution and the operator $O_\varepsilon+\lambda$ has an invertible operator in its rank space. We need to show that this rank space coincide with the space $L_p\left(G;E\right)$. We consider the smooth functions $g_j=g_j\left(x\right)$ with respect to the partition of the unique $\varphi_j=\varphi_j\left(y\right)$ on the region G that equal one on $\operatorname{supp}\varphi_j$, where $\operatorname{supp}g_j\subset G_j$ and $\left|g_j\left(x\right)\right|<1$.

Let us construct for all j the function u_j, that are defined on the regions $\Omega_j = G \cap G_j$ and satisfying the problem (1). Consider first when G_j adjoin to the boundary points. The problem (1) can be express in the form

$$\sum_{k,i=1}^{n} \varepsilon a_{ki}(x_j) D_{ki}^2 u_j(x) + A_\lambda(x_j) u_j(x) = \tag{19}$$

$$g_j \left\{ f + [A(x_j) - A(x)] u_j \sum_{k,i=1}^{n} \varepsilon [a_{ki}(x_j) - a_{ki}(x)] D_{ki}^2 u_j - \sum_{k=1}^{n} \varepsilon^{\frac{1}{2}} A_k(x) \frac{\partial u_j(x)}{\partial x_k} \right\},$$

$$L_\Gamma u_j = 0, \quad j = 1, 2, .., N.$$

Consider operators $O_j(\varepsilon) + \lambda$ in $L_p(G_j; E)$ generated by BVPs (19) when G_j partially belong to G. By virtue of Theorem 1.4 for all $f \in L_p(G_j; E)$, for $\lambda \in S(\varphi)$ and sufficiently large $|\lambda|$ we have

$$\sum_{k=1}^{n} \sum_{i=0}^{2} \varepsilon^{\frac{i}{2}} |\lambda|^{1-\frac{i}{2}} \left\| D_k^i [O_j(\varepsilon) + \lambda]^{-1} f \right\|_p + \left\| A [O_j(\varepsilon) + \lambda]^{-1} f \right\|_p \le C \|f\|_p. \tag{20}$$

Extending u_j zero on the outside of supp φ_j and passing substitutions $u_j = [O_j(\varepsilon) + \lambda]^{-1} \upsilon_j$ we obtain from (19) operator equations;

$$\upsilon_j = K_{j\lambda\varepsilon} \upsilon_j + g_j f, \; j = 1, 2, ..., N, \tag{21}$$

where

$$K_{j\lambda\varepsilon} = g_j \Big\{ f + [A(x_{0j}) - A(x)] [O_j(\varepsilon) + \lambda]^{-1} +$$

$$\sum_{i,j=1}^{n} \varepsilon [a_{ij}(x) - a_{ij}(x_{0j})] \frac{\partial^2}{\partial x_i \partial x_j} [O_j(\varepsilon) + \lambda]^{-1} - \sum_{k=1}^{n} \varepsilon^{\frac{1}{2}} A_k(x) \frac{\partial}{\partial x_k} [O_j(\varepsilon) + \lambda]^{-1} \Big\}.$$

By virtue of Theorem 1.1 and the estimate (20), in view of the smoothness of the coefficients of the expression $K_{j\lambda\varepsilon}$, and in view of condition (4) for $\lambda \in S(\varphi)$ and sufficiently large $|\lambda|$ we have $\|K_{j\lambda\varepsilon}\| < \delta$, where δ is sufficiently small. Consequently, equations (21) have unique solutions $\upsilon_j = [I - K_{j\lambda\varepsilon}]^{-1} g_j f$ and

$$\|\upsilon_j\|_p = \left\| [I - K_{j\lambda\varepsilon}]^{-1} g_j f \right\|_p \le \|f\|_p.$$

Whence, $[I - K_{j\lambda\varepsilon}]^{-1} g_j$ are bounded linear operators from $L_p(G; E)$ to $L_p(G_j; E)$. Thus, we obtain that the functions

$$u_j = U_{j\lambda\varepsilon} f = [O_j(\varepsilon) + \lambda]^{-1} [I - K_{j\lambda\varepsilon}]^{-1} g_j f$$

are solutions of the problem (19). In a similar way, by using Theorem 1.4 we can construct the solutions u_j for problems (19) with respect to regions entirely belonging to G. Consider a linear operator $(U_\varepsilon + \lambda)$ in $L_p(G; E)$ such that

$$(U_\varepsilon + \lambda) f = \sum_{j=1}^{N} \varphi_j(y) U_{j\lambda\varepsilon} f.$$

It is clear from the constructions U_j and the estimate (20) that operators $U_{j\lambda}$ are bounded linear from $L_p(G; E)$ to $W_p^2(G; E(A), E)$ and for $\lambda \in S(\varphi)$ and sufficiently large $|\lambda|$ we have

$$\sum_{k=1}^{n}\sum_{i=0}^{2}\varepsilon^{\frac{i}{2}}\left|\lambda\right|^{1-\frac{i}{2}}\left\|D_k^i U_{j\lambda}^{-1}f\right\|_p+\left\|AU_{j\lambda}^{-1}f\right\|_p\leq C\left\|f\right\|_p. \tag{22}$$

Therefore $(U+\lambda)$ is a bounded linear operator from L_p to L_p. Then act of $O_\varepsilon+\lambda$ to $u=\sum\limits_{j=1}^{N}\varphi_j U_{j\lambda\varepsilon}f$ gives $O_\lambda u=f+\sum\limits_{j=1}^{N}\Phi_{j\lambda\varepsilon}f$, where $\Phi_{j\lambda\varepsilon}$ are linear operators defined by

$$\Phi_{j\lambda\varepsilon}=\left\{\sum_{j=1}^{N}\sum_{i,k=1}^{n}\frac{\partial^2\varphi_j}{\partial x_i\partial x_k}U_{j\lambda\varepsilon}+\frac{\partial\varphi_j}{\partial x_k}\frac{\partial}{\partial x_i}U_{j\lambda\varepsilon}+\frac{\partial\varphi_j}{\partial x_i}\frac{\partial}{\partial x_k}U_{j\lambda\varepsilon}\sum_{i}^{n}\varepsilon^{\frac{1}{2}}A_i\frac{\partial\varphi_j}{\partial x_i}U_{j\lambda\varepsilon}\right\}.$$

By virtue of embedding Theorem 1.1 and the estimate (22) from the expression $\Phi_{j\lambda\varepsilon}$ we obtain that operators $\Phi_{j\lambda\varepsilon}$ are bounded linear from $L_p(G;E)$ to $L_p(G_j;E)$ and $\|\Phi_{j\lambda\varepsilon}\|<1$. Therefore, there exists a bounded linear invertible operator

$$\left(I+\sum_{j=1}^{N}\Phi_{j\lambda\varepsilon}\right)^{-1}.$$

Whence, we obtain that for all $f\in L_p(G;E)$ BVP (1) have a unique solution

$$u(x)=O_\lambda^{-1}f=\sum_{j=1}^{N}\varphi_j O_{j\lambda}^{-1}\left[I-K_{j\lambda}\right]^{-1}g_j\left(I+\sum_{j=1}^{N}\Phi_{j\lambda}\right)^{-1}f$$

i.e. we obtain the assertion. □

Result 1. Theorem 2.1 implies that the operator O has a resolvent $(O+\lambda)^{-1}$ for $\lambda\in S(\varphi)$ and the following estimate holds

$$\sum_{k=1}^{n}\sum_{i=0}^{2}\varepsilon^{\frac{i}{2}}\left|\lambda\right|^{1-\frac{i}{2}}\left\|D_k^i\left(O_\varepsilon+\lambda\right)^{-1}\right\|_{B(L_p(G;E))}+\left\|A\left(O_\varepsilon+\lambda\right)^{-1}\right\|_{B(L_p(G;E))}\leq C. \tag{23}$$

Remark 1. The estimate (18) and the embedding Theorem 1.1 implies that under conditions of Theorem 2.2 the following estimate

$$\sum_{i,j=1}^{n}\varepsilon\left\|\frac{\partial^2 u}{\partial x_i\partial x_j}\right\|_{L_p(G;E)}+\|Au\|_{L_p(G;E)}\leq C\|f\|_{L_p(G;E)}$$

holds for the solution of the problem (1). I.e the problem (1) is separable in $L_p(G;E)$.

Remark 2. Result 1 implies that the operator O is positive in $L_p(G;E)$. Moreover in view of (22) and by virtue of [24, § 1.14.5] the operator O generates an analytic semigroup when $\varphi\in\left(\frac{\pi}{2},\pi\right)$.

Theorem 2.3. *Let all conditions of Theorem 2.1 hold and $A^{-1}\in\sigma_\infty(E)$. Then the operator O is Fredholm from $W_p^2(G;E(A),E)$ into $L_p(G;E)$.*

Proof. Theorem 2.2 implies that the operator $O_\varepsilon+\lambda$ for sufficiently large $|\lambda|$ have a bounded inverse $(O_\varepsilon+\lambda)^{-1}$ from $L_p(G;E)$ to $W_p^2(G;E(A),E)$, that is the operator $O_\varepsilon+\lambda$ is Fredholm from $W_p^2(G;E(A),E)$ into $L_p(G;E)$. By virtue of Theorem 1.2 the embedding $W_p^2(G;E(A),E)\subset L_p(G;E)$ is compact. Then by perturbation theory of linear operators we obtain that the operator O is Fredholm from $W_p^2(G;E(A),E)$ into $L_p(G;E)$. □

Result 2. If we put $a_{ij}(x) = 1,\ A(x) = q(x),\ A_k(x) = 0,\ k = 1, 2, .., n$ in (1) then we obtain from Theorem 2.3 that the following operator with small parameter

$$S_\varepsilon u = -\sum_{i,j=1}^{n} \varepsilon \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + q(x)\, u(x),\ u\mid_\Gamma = 0$$

is positive and is a generator of analytic semigroup in $L_p(G)$.

3. Abstract Cauchy problem for degenerate parabolic equation

Consider now the following abstract parobolic problem with small parameter

$$\frac{\partial u}{\partial t} + \varepsilon \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} - A(x)\, u \quad + \tag{24}$$

$$\varepsilon^{\frac{1}{2}} \sum_{k=1}^{n} A_k(x) \frac{\partial u}{\partial x_k} - au \quad = \quad f(t,x), \qquad x \in G,\ t \in R_+$$

$$u(t,x)\mid_\Gamma = 0,\ u(0,x) = 0,$$

where Γ is a boundary of region $G \subset R^n$, ε is a positive small parameter, a_{ij} are complex-valued functions, $a > 0$, A and A_k are possible unbounded linear operators in a Banach space E.

In this section we obtain the well-posedness of problem (23).

If $G_+ = R_+ \times G$, $\mathbf{p} = (p, p_1)$, $L_{\mathbf{p}}(G_+; E)$ will be denote the space of all $\mathbf{p}$-summable scalar-valued functions with mixed norm (see e.g. [7, §1] for $E = \mathbf{C}$), i.e. the space of all measurable functions f defined on G, for which

$$\|f\|_{L_{\mathbf{p}}(G_+)} = \left(\int_{R_+} \left(\int_{G} \|f(x,y)\|_E^p\, dx \right)^{\frac{p_1}{p}} dy \right)^{\frac{1}{p_1}} < \infty.$$

Analogously, $W_{\mathbf{p}}^m(G_+; E)$ denotes the E-valued Sobolev space with corresponding mixed norm for (see e.g. [7, §10] for $E = \mathbf{C}$).

First, let us show that the operator O_ε is R-positive in $L_p(G; E)$.

Theorem 3.1. *Let all conditions of Theorem 2.1 are hold. Then, the operator O_ε is an R-positive in $L_p(G; E)$.*

Proof. Really, by virtue of Theorem 2.2 we obtain that for $f \in L_p(G; E)$ the BVP (1) have a unique solution exspressing in the form

$$u(x) = (O_\varepsilon + \lambda)^{-1} f = \sum_{j=1}^{N} \varphi_j \left(O_j(\varepsilon) + \lambda\right)^{-1} \left[I - K_{j\lambda\varepsilon}\right]^{-1} g_j \left(I + \sum_{j=1}^{N} \Phi_{j\lambda\varepsilon} \right)^{-1} f,$$

where $O_j(\varepsilon)$ are local operators generated by BVPs with small parameter of type (9) − (10) for $V_j = 0$ and $K_{j\lambda\varepsilon}, \Phi_{j\lambda\varepsilon}$ are uniformly bounded operators in $L_p(G; E)$ defined in Theorem 2.2. By virtue of [4] the operators $O_j(\varepsilon)$ are R-positive. Then by using the above representation and by virtue Theorem 2.2 we obtain the assertions. □

Theorem 3.2. *Let all conditions of Theorem 2.1 hold. Then for all $f \in L_{\mathbf{p}}(G_+;E)$ and sufficiently large $a>0$ the problem (23) has a unique solution belonging to $W_{\mathbf{p}}^{1,2}(G_+;E(A),E)$ and the estimate holds*

$$\left\|\frac{\partial u}{\partial t}\right\|_{L_{\mathbf{p}}(G_+;E)}+\sum_{i,j=1}^{n}\varepsilon\left\|\frac{\partial^2 u}{\partial x_i\partial x_j}\right\|_{L_{\mathbf{p}}(G_+;E)}+\|Au\|_{L_{\mathbf{p}}(G_+;E)}\leq C\|f\|_{L_{\mathbf{p}}(G_+;E)}.$$

Proof. The problem (23) can be express as the following Cauchy problem

$$\frac{du}{dt}+(O_\varepsilon+a)\,u(t)=f(t),\ u(0)=0. \tag{25}$$

The Theorem 2.3 implies that the operator O_ε is R-positive and generator of analytic semigroup in $E_0=L_p(G;E)$. Then by virtue of [25, Theorem 4.2] we obtain that for all $f\in L_{p_1}(R_+;E_0)$ the problem (24) has a unique solution belonging to $W_{p_1}^1(R_+;D(O_\varepsilon),E_0)$ and the uniform estimate holds

$$\left\|\frac{du}{dt}\right\|_{L_{p_1}(R_+;E_0)}+\|O_\varepsilon u\|_{L_{p_1}(R_+;E_0)}\leq C\|f\|_{L_{p_1}(R_+;E_0)}.$$

Since $L_{p_1}(R_+;E_0)=L_{\mathbf{p}}(G_+;E)$, by Theorem 2.2 we have $\|(O_\varepsilon+a)\,u\|_{L_{p_1}(R_+;F)}=D(O_\varepsilon)$. These relations and the above estimate implies the assertion. □

4. Boundary value problems for anisotropic elliptic equations with parameters

The Fredholm property of BVPs for elliptic equations with parameters in smooth domains were studied in e.g. [5], [8] also for non smooth domains these questions were investigated e.g. in [12].

Let $\Omega\subset R^n$ be an open connected set with compact C^{2m}-boundary $\partial\Omega$. Let us consider the boundary value problems on cylindrical domain $\tilde{\Omega}=G\times\Omega$ for the following anisotropic elliptic equation

$$\begin{aligned} L_\varepsilon u &= \sum_{i,j=1}^{n}\varepsilon a_{ij}(x)\frac{\partial^2 u(x,y)}{\partial x_i\partial x_j}+\sum_{k=1}^{n}d_k\frac{\partial u(x,y)}{\partial x_k}+\sum_{|\alpha|\leq 2m}a_\alpha(y)\,D_y^\alpha u(x,y) \\ &= f(x,y),\quad x\in G\ \ y\in\Omega, \end{aligned} \tag{26}$$

$$u(x,y)\,|_\Gamma=0 \tag{27}$$

$$B_j u=\sum_{|\beta|\leq m_j}b_{j\beta}(y)\,D_y^\beta u(x,y)=0,\ x\in G,\ y\in\partial\Omega,\ j=1,2,...,m. \tag{28}$$

where Γ is the boundary of the region $G\in R^n$ and a_{ij} are complex-valued function on G. $D_j=-i\frac{\partial}{\partial y_j}$, $m_k\in\{0,1\}$, $y=(y_1,...,y_n)$.

If $\tilde{\Omega}=G\times\Omega$, $\mathbf{p}=(p_1,p)$, $L_{\mathbf{p}}\left(\tilde{\Omega}\right)$ will be denote the space of all $\mathbf{p}$-summable scalar-valued functions with mixed norm i.e. the space of all measurable functions f defined on $\tilde{\Omega}$, for which

$$\|f\|_{L_{\mathbf{p}}(\tilde{\Omega})}=\left(\int_G\left(\int_\Omega|f(x,y)|^{p_1}dy\right)^{\frac{p}{p_1}}dx\right)^{\frac{1}{p}}<\infty.$$

Analogously, $W_{\mathbf{p}}^{2,2m}\left(\tilde{\Omega}\right)$ denotes the anisotropic Sobolev space with corresponding mixed norm.

Theorem 4.1. *Let the following conditions be satisfied;*

(1) The Condition 1 holds;

(2) $a_\alpha \in C\left(\bar{\Omega}\right)$ *for each* $|\alpha| = 2m$ *and* $a_\alpha \in [L_\infty + L_{r_k}](\Omega)$ *for each* $|\alpha| = k < 2m$ *with* $r_k \geq q$ *and* $2m - k > \frac{l}{r_k}$;

(3) $b_{j\beta} \in C^{2m-m_j}(\partial\Omega)$ *for each* j, β *and* $m_j < 2m$, $\sum\limits_{j=1}^{m} b_{j\beta}\left(y^{'}\right)\sigma_j \neq 0$, *for* $|\beta| = m_j$, $y^{'} \in \partial G$, *where* $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n) \in R^n$ *is a normal to* $\partial\Omega$;

(4) for $y \in \bar{\Omega}$, $\xi \in R^n$, $\lambda \in S(\varphi)$, $\varphi \in (0, \pi)$, $|\xi| + |\lambda| \neq 0$ *let* $\lambda + \sum\limits_{|\alpha|=2m} a_\alpha(y)\xi^\alpha \neq 0$;

(5) for each $y_0 \in \partial\Omega$ *local BVP in local coordinates corresponding to* y_0

$$\lambda + \sum_{|\alpha|=2m} a_\alpha(y_0) D^\alpha \vartheta(y) = 0,$$

$$B_{j0}\vartheta = \sum_{|\beta|=m_j} b_{j\beta}(y_0) D^\beta u(y) = h_j,\ j = 1, 2, ..., m$$

has a unique solution $\vartheta \in C_0(R_+)$ *for all* $h = (h_1, h_2, ..., h_m) \in R^m$, *and for* $\xi^{'} \in R^{n-1}$ *with* $|\xi^{'}| + |\lambda| \neq 0$.

Then:

a) Then for all $f \in L_{\mathbf{p}}\left(\tilde{\Omega}\right)$, $|\arg\lambda| \leq \varphi$ *and sufficiently large* $|\lambda|$ *problem* $(25) - (27)$ *has a unique solution* u *that belongs to* $W_{\mathbf{p}}^{2,2m}\left(\tilde{\Omega}\right)$ *and the coercive uniform estimate hold*

$$\sum_{k=1}^{n}\sum_{i=0}^{2} \varepsilon^{\frac{i}{2}} |\lambda|^{1-\frac{i}{2}} \left\| \frac{\partial^i u}{\partial x_k} \right\|_{L_{\mathbf{p}}(\tilde{\Omega})} + \sum_{|\beta|=2m} \left\| D_y^\beta u \right\|_{L_{\mathbf{p}}(\tilde{\Omega})} \leq C \|f\|_{L_{\mathbf{p}}(\tilde{\Omega})};$$

b) the problem $(25) - (27)$ *is Fredholm in* $L_{\mathbf{p}}\left(\tilde{\Omega}\right)$.

Proof. Let $E = L_{p_1}(\Omega)$. Then by virtue of [25, Theorem 3.6] the (1) part of Theorem 2.2 is satisfied. Consider the operator A defined by

$$D(A) = W_{p_1}^{2m}(\Omega; B_j u = 0),\ Au = \sum_{|\alpha|\leq 2m} a_\alpha(y) D^\alpha u(y).$$

For $x \in \Omega$ also consider operators

$$A_k(x) u = d_k(x, y) u(y),\ k = 1, 2, ..., n.$$

The problem $(25) - (27)$ can be rewritten in the form (1), where $u(x) = u(x, .)$, $f(x) = f(x, .)$ are functions with values in $E = L_{p_1}(\Omega)$. By virtue of [5] the problem

$$\lambda u(y) + \sum_{|\alpha|\leq 2m} a_\alpha(y) D_y^\alpha u(y) = f(y),$$

$$B_j u = \sum_{|\beta|\leq m_j} b_{j\beta}(y) D_y^\beta u(y) = 0,\ j = 1, 2, ..., m$$

has a unique solution for $f \in L_{p_1}(\Omega)$ and arg $\lambda \in S(\varphi_0)$, $|\lambda| \to \infty$. Moreover, in view of [8, Theorem 8.2] the differential operator A is R-positive in L_{p_1}. It is known that the embedding $W_{p_1}^{2m}(\Omega) \subset L_{p_1}(\Omega)$ is compact (see e.g. [24, Theorem 3.2.5]). Then by using interpolation properties of Sobolev spaces (see e.g. [24, § 4]) it is clear to see that condition (3) of the Theorem 2.2 is fulfilled too. Then from Theorems 2.2 and 2.3 the assertions are obtained. □

5. BVPs for infinite systems of elliptic equations with small parameter

Consider the following infinity systems of boundary value problem

$$\sum_{i,j=1}^{n} \varepsilon a_{ij}(x) \frac{\partial^2 u_m(x)}{\partial x_i \partial x_j} + (d_m(x) + \lambda) u_m(x) \quad + \tag{29}$$

$$\sum_{k=1}^{n}\sum_{j=1}^{\infty} \varepsilon^{\frac{1}{2}} d_{kjm}(x) \frac{\partial u_m(x)}{\partial x_k} \quad = \quad f_m(x), \ x \in G, \ m = 1, 2, ..., \infty,$$

$$u_m(x) \mid_{\Gamma} = 0, \tag{30}$$

where Γ is the boundary of the region $G \in R^n$ and a_{ij} are complex-valued function on G.

Let

$$d(x) = \{d_m(x)\}, \ d_m > 0, \ u = \{u_m\}, \ Du = \{d_m u_m\}, \ m = 1, 2, ...\infty,$$

$$l_q(D) = \left\{ u : u \in l_q, \|u\|_{l_q(d)} = \|Du\|_{l_q} = \left(\sum_{m=1}^{\infty} |d_m u_m|^q \right)^{\frac{1}{q}} < \infty \right\},$$

$$x \in G, \ 1 < q < \infty.$$

Let Q_ε denote a differential operator in $L_p(G; l_q)$ generated by problem $(28) - (29)$. Let

$$B = B(L_p(G; l_q)).$$

Theorem 5.1. *Let the following condition hold:*

(1) The Condition1 holds;
(2) $d_j \in C(\bar{G}), d_{kmj} \in L_\infty(G)$ and

$$\max_k \sup_m \sum_{j=1}^{\infty} d_{kmj}(x) d_j^{-\left(\frac{1}{2}-\mu\right)} < M \text{ for all } x \in G \text{ and } 0 < \mu < \frac{1}{2}.$$

a.e. for $x \in G$ and $1 < p < \infty$.

Then:

(a) for all $f(x) = \{f_m(x)\}_1^\infty \in L_p(G; l_q)$, for $\lambda \in S(\varphi)$, $\varphi \in (0, \pi)$ and for sufficiently large $|\lambda|$ the problem $(28) - (29)$ has a unique solution $u = \{u_m(x)\}_1^\infty$ belonging to $W_p^2(G, l_q(D), l_q)$ and the coercive estimate holds

$$\sum_{k=1}^{n} \varepsilon \left[\int_G \left(\sum_{m=1}^{\infty} \left| D_k^2 u_m(x) \right|^q \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}} + \left[\int_G \left(\sum_{m=1}^{\infty} |d_m u_m(x)|^q \right) dx \right]^{\frac{1}{p}} \tag{31}$$

$$\leq C \left[\int_G \left(\sum_{m=1}^{\infty} |f_m(x)|^q \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}}.$$

(b) For sufficiently large $|\lambda| > 0$ *there exists a resolvent* $(Q_\varepsilon + \lambda)^{-1}$ *of operator* Q_ε *and*

$$\sum_{k=1}^{n}\sum_{j=0}^{2}\varepsilon^{\frac{i}{2}}|\lambda|^{1-\frac{j}{2}}\left\|D_k^j(Q_\varepsilon+\lambda)^{-1}\right\|_B+\left\|A(Q_\varepsilon+\lambda)^{-1}\right\|_B\leq M; \tag{32}$$

Proof. Really, let $E = l_q$, A and $A_k(x)$ be infinite matrices, such that

$$A = [d_m(x)\,\delta_{jm}]\,,\ A_k(x) = [d_{kjm}(x)]\,,\ m,\ j = 1, 2, ...\infty.$$

It is clear to see that this operator A is R-positive in l_q. Therefore, by virtue of Theorem 3.1 we obtain that the problem $(28) - (29)$ for all $f \in L_p(G; l_q)$, for $\lambda \in S(\varphi)$ and sufficiently large $|\lambda|$ has a unique solution u that belongs to space $W_p^2(G; l_q(D), l_q)$ and

$$\sum_{k=1}^{n}\varepsilon\left\|D_k^2u\right\|_{L_p(G;l_q)}+\|Du\|_{L_p(G;l_q)}\leq C\|f\|_{L_p(G;l_q)}\,.$$

From the above we obtain (30). The estimate (31) is obtained from Result 1. □

Remark 4. There are many positive operators in the different concrete Banach spaces. Therefore, putting concrete Banach spaces and concrete positive operators (i.e. pseudo-differential operators or finite or infinite matrices for instance) instead of E and A, respectively, by virtue of Theorem 2.2 - 3.1 we can obtain different class of maximal regular BVPs for partial differential or pseudo-differential equations or its finite and infinite systems with parameters.

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