TWMS J. App. Eng. Math. V.4, No.1, 2014, pp. 7-15.

HARMONIC MAPPINGS RELATED TO STARLIKE FUNCTION OF COMPLEX ORDER α

MELIKE AYDOĞAN[1] §

Abstract. Let S_H be the class of harmonic mappings defined by

$$S_H = \Big\{ f = h(z) + \overline{g(z)} \mid h(z) = z + \sum_{n=2}^{\infty} a_n z^n, g(z) = \sum_{n=1}^{\infty} b_n z^n \Big\}$$

The purpose of this talk is to present some results about harmonic mappings which was introduced by R. M. Robinson [8].

Keywords: Harmonic Mappings, Subordination principle, Distortion theorem, Growth theorem, Coefficient inequality.

AMS Subject Classification: Primary 30C45, Secondary 30C55.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$ be the open unit disc in the complex plane $\mathbb{C}$. A complex-valued harmonic function $f : \mathbb{D} \to \mathbb{C}$ has the representation

$$f = h(z) + \overline{g(z)} \tag{1}$$

where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ and have the following power series expansion,

$$h(z) = \sum_{n=0}^{\infty} a_n z^n,$$

$$g(z) = \sum_{n=0}^{\infty} b_n z^n, z \in \mathbb{D},$$

where a_n,$b_n \in \mathbb{C}$, $n = 0, 1, 2, ...$ Choose (i.e, $b_0 = 0$) so the representation (1) is unique in $\mathbb{D}$ and is called the canonical representation of f.
For the univalent and sense-preserving harmonic functions f in $\mathbb{D}$, it is convenient to make further normalization (without loss of generality), $h(0) = 0$ (i.e. , $a_0 = 0$) and $h'(0) = 1$ (i.e. , $a_1 = 1$). The family of such functions f is denoted by S_H [5] . The family of all functions $f \in S_H$ with the additional property that $g'(0) = 1$ (i.e. , $b_1 = 0$) is denoted by S_H^0 [5] . Observe that the classical family of univalent functions S consists of all functions $f \in S_H^0$ such that $g(z) \equiv 0$. Thus it is clear that $S \subset S_H^0 \subset S_H$ [5] .

Let Ω be the family of functions $\phi(z)$ regular in the open unit disc $\mathbb{D}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for every $z \in \mathbb{D}$.

[1] Department of Mathematics, Işık University, Meşrutiyet Koyu, Şile İstanbul, Turkey
e-mail: melike.aydogan@isikun.edu.tr

§ Submitted for GFTA'13, held in Işık University on October 12, 2013.

For arbitrary fixed numbers A, B, $-1 \leq B < A \leq 1$ denote by $P(A,B)$, the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + ...$ regular in $\mathbb{D}$ and such that $p(z)$ is in $P(A,B)$ if and only if

$$p(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)}, \tag{2}$$

for some function $\phi(z) \in \Omega$ and every $z \in \mathbb{D}$. This class was introduced by Janowski W. [6].

We note that if we give special values to A and B we obtain the following classes.

i. $P(1,-1)$ is the set defined by $Rep(z) > 0$, which is Caratheodory class [1],
ii. $P(1-2\alpha,-1)$, $0 \leq \alpha < 1$ is the set defined by $Rep(z) > \alpha$,
iii. $P(1,0)$ is the set defined by $|p(z) - 1| < 1$
iv. $P(\alpha, 0)$, $0 < \alpha < 1$ is the set defined by $|p(z) - 1| < \alpha$,
v. $P(1, -1 + \frac{1}{M})$, $M > \frac{1}{2}$ is the set defined by $|p(z) - M| < M$, $P(\alpha, -\alpha)$, $0 < \alpha < 1$ is the set defined by $\left|\frac{p(z)-1}{p(z)+1}\right| < \alpha$.

Moreover, let $S^*(1-b)$ be denote the the family of functions
$h(z) = z + a_2 z^2 + a_3 z^3 + ...$ regular in $\mathbb{D}$ and such that $h(z)$ is in $S^*(1-b)$ if and only if

$$1 + \frac{1}{b}(z\frac{h'(z)}{h(z)} - 1) = p(z), b \neq 0, complex \tag{3}$$

for some $p(z) \in P(-1,1)$ and all $z \in \mathbb{D}$. At the same time we note that if we give special values to b we obtain the following subclasses of starlike functions.

i. For $b = 1$, $S^*(0)$ is the class of starlike functions [5],
ii. for $b = 1 - \alpha$, $0 \leq \alpha < 1$, $S^*(\alpha)$ is the class of starlike functions of order α. [5],
iii. for $b = e^{-i\lambda} \cos\lambda$, $|\lambda| < \frac{\pi}{2}$ is the class of $\lambda-$spirallike functions [5],
iv. for $b = (1-\alpha)e^{-i\lambda}\cos\lambda$, $0 \leq \alpha < 1$, $\lambda < \frac{\pi}{2}$ is the class of $\lambda-$spirallike functions of order α, [5], and the class $S^*(1-b)$ was introduced by M. A. Nasr and M. K. Aouf [7].

Let $s_1(z) = z + d_2 z^2 + ...$ and $s_2(z) = z + e_2 z^2 + ...$ be elements of F. If there exists $\phi(z) \in \Omega$ such that $s_1(z) = s_2(\phi(z))$, then we say that $s_1(z)$ is sunbordinate to $s_2(z)$ and we write $s_1(z) \prec s_2(z)$. Specially if $s_2(z)$ is univalent in $\mathbb{D}$, then $s_1(z) \prec s_2(z)$ if and only if $S_1(\mathbb{D}) \subset S_2(\mathbb{D})$ and $S_1(0) = S_2(0)$ implies $S_1(\mathbb{D}_r) \subset S_2(\mathbb{D}_r)$. [3]

In this paper we will investigate the class of harmonic mappings defined by

$$S_H^*(A,B,1-b) = \left\{ f = h(z) + \overline{g(z)} \mid w(z) = \frac{g'(z)}{h'(z)} \prec b_1 \frac{1 + A\phi(z)}{1 + B\phi(z)}, h(z) \in S^*(1-b) \right\} \tag{4}$$

Lemma 1.1. *([5]) Let $\phi(z)$ be regular in the open unit disc $\mathbb{D}$ with $\phi(0) = 0$ and $|\phi(z)| < 1$. If $|\phi(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_1, then we have $z_1.\phi'(z) = k\phi(z_1)$ for some $k \geq 1$.*

2. Main Results

Theorem 2.1. *Let $f = (h(z) + \overline{g(z)})$ be an element of $S_H^*(A,B,1-b)$. Then*

$$\frac{g(z)}{h(z)} \prec b_1 \frac{1 + Az}{1 + Bz}, \tag{5}$$

Proof. Since $f = h(z) + \overline{g(z)}$ be an element of $S_H^*(A, B, 1-b)$, then

$$\frac{g'(z)}{h'(z)} \prec b_1 \frac{1+Az}{1+Bz} \Rightarrow \frac{1}{b_1}\frac{g'(z)}{h'(z)} = \frac{1+A\phi(z)}{1+B\phi(z)} \Rightarrow \tag{6}$$
$$\left|\frac{1}{b_1}\frac{g'(z)}{h'(z)} - \frac{1-ABr^2}{1-B^2r^2}\right| \leq \frac{(A-B)r}{1-B^2r^2} \Rightarrow$$
$$|b_1|\frac{1-Ar}{1-Br} \leq \left|\frac{g'(z)}{h'(z)}\right| \leq |b_1|\frac{1+Ar}{1+Br}$$

Therefore the relations (6) shows that the values of $(\frac{g'(z)}{h'(z)})$ are in the disc

$$D_r(b_1) = \begin{cases} \frac{g'(z)}{h'(z)} \mid \left|\frac{g'(z)}{h'(z)} - \frac{b_1(1-AB)r^2}{1-B^2r^2}\right| \leq \frac{|b_1|(1-AB)r^2}{1-B^2r^2}, & B \neq 0; \\ \\ \frac{g'(z)}{h'(z)} \mid \left|\frac{g'(z)}{h'(z)} - b_1\right| \leq |b_1|\, Ar, & B = 0. \end{cases} \tag{7}$$

Now we define a function $\phi(z)$ by

$$\frac{g(z)}{h(z)} = b_1 \frac{1+A\phi(z)}{1+B\phi(z)} \tag{8}$$

then $\phi(z)$ is analytic in $\mathbb{D}$ and $\phi(0) = 0$. Now we need to show that $|\phi(z)| < 1$. If we take derivative from (8), we obtain

$$\frac{g'(z)}{h'(z)} = b_1[\frac{1+A\phi(z)}{1+B\phi(z)} + \frac{(A-B)z\phi'(z)}{(1+B\phi(z))^2}\frac{h(z)}{zh'(z)}] \tag{9}$$

On the other hand since $h(z) \in S^*(1-b)$, then

$$\left|z\frac{h'(z)}{h(z)} - \frac{1+(2b-1)r^2}{1-r^2}\right| \leq \frac{2\,|b|\,r}{1-r^2} \tag{10}$$

$$z\frac{h'(z)}{h(z)} - \frac{1+(2b-1)r^2}{1-r^2} = \frac{2\,|b|\,r}{1-r^2}e^{i\theta} \Rightarrow \tag{11}$$

$$z\frac{h'(z)}{h(z)} = \frac{1+(2b-1)r^2}{1-r^2} + \frac{2\,|b|\,r}{1-r^2}e^{i\theta} \Rightarrow \tag{12}$$

this shows that the boundary value $\frac{h(z)}{zh'(z)}$ is

$$\frac{1-r^2}{1+2\,|b|\,re^{i\theta}+(2b-1)r^2}$$

If we use Jack's Lemma in this step we can write,

$$w(z_1) = \frac{g'(z_1)}{h'(z_1)} = \begin{cases} b_1(\frac{1+A(\phi_{z_1})}{1+B\phi(z_1)} + \frac{k(A-B)\phi(z_1)}{(1+B\phi(z_1))^2}\frac{1-r^2}{1+2|b|re^{i\theta}+(2b-1)r^2}) \notin w(\mathbb{D}_r(b_1)), & B \neq 0; \\ \\ b_1[(1+A\phi(z_1)) + kA\phi(z_1)\frac{1-r^2}{1+2|b|re^{i\theta}+(2b-1)r^2}] \notin w(\mathbb{D}_r(b_1)), & B = 0. \end{cases} \tag{13}$$

because $|\phi(z_1)| = 1$ and $k \geq 1$. But this is contradiction to the condition

$$\frac{g'(z)}{h'(z)} \prec b_1 \frac{1+Az}{1+Bz}$$

and so have $|\phi(z)| < 1$ for all $z \in \mathbb{D}$.

□

Lemma 2.1. *Let* $f = (h(z) + \overline{g(z)})$ *be an element of* $S^*(1-b)$. *Then*

$$\frac{r(1-r)^{|b|-Reb}}{(1+r)^{|b|+Reb}} \leq |h(z)| \leq \frac{r(1+r)^{|b|-Reb}}{(1-r)^{|b|+Reb}} \tag{14}$$

Proof. Since $f = (h(z) + \overline{g(z)}) \in S^*(1-b)$,

$$Re[1 + \frac{1}{b}(z\frac{h'(z)}{h(z)} - 1)] > 0 \Rightarrow 1 + \frac{1}{b}(z\frac{h'(z)}{h(z)} - 1) = \frac{1+\phi(z)}{1-\phi(z)}$$

$$\frac{1}{b}(z\frac{h'(z)}{h(z)} - 1) = \frac{1+\phi(z) - 1 + \phi(z)}{1-\phi(z)} \Rightarrow$$

$$z\frac{h'(z)}{h(z)} = \frac{2b\phi(z)}{1-\phi(z)} + 1 \Rightarrow$$

$$z\frac{h'(z)}{h(z)} = \frac{1+(2b-1)\phi(z)}{1-\phi(z)} \Rightarrow z\frac{h'(z)}{h(z)} \prec \frac{1+(2b-1)z}{1-z} \tag{15}$$

$$Re[1 + \frac{1}{b}(z\frac{h'(z)}{h(z)} - 1)] > 0 \Rightarrow 1 + \frac{1}{b}(z\frac{h'(z)}{h(z)} - 1) = \frac{1+\phi(z)}{1-\phi(z)} = p(z) \Rightarrow$$

$$1 + \frac{1}{b}(z\frac{h'(z)}{h(z)} - 1) \prec \frac{1+z}{1-z} \Rightarrow$$

$$\left|[1 + \frac{1}{b}(z\frac{h'(z)}{h(z)} - 1)] - \frac{1+r^2}{1-r^2}\right| \leq \frac{2r}{1-r^2} \Rightarrow \tag{16}$$

$$\left|\frac{1}{b}(z\frac{h'(z)}{h(z)} - 1) + (1 - \frac{1+r^2}{1-r^2})\right| \leq \frac{2r}{1-r^2}$$

$$\left|\frac{1}{b}(z\frac{h'(z)}{h(z)} - 1) - \frac{2r^2}{1-r^2}\right| \leq \frac{2r}{1-r^2}$$

$$\left|z\frac{h'(z)}{h(z)} - \frac{1-r^2+2br^2}{1-r^2}\right| \leq \frac{2|b|r}{1-r^2}$$

$$\left|z\frac{h'(z)}{h(z)} - \frac{1+(2b-1)r^2}{1-r^2}\right| \leq \frac{2|b|r}{1-r^2} \tag{17}$$

If we take $b = x + iy$, then we can write the following relations

$$\left|z\frac{h'(z)}{h(z)} - \frac{1+(2(x+iy)-1)r^2}{1-r^2}\right| \leq \frac{2|b|r}{1-r^2}$$

$$\left|z\frac{h'(z)}{h(z)} - (\frac{1+(2x-1)r^2}{1-r^2} + i\frac{yr^2}{1-r^2})\right| \leq \frac{2|b|r}{1-r^2}$$

$$Re(z\frac{h'(z)}{h(z)} - \frac{1+(2x-1)r^2}{1-r^2} - i\frac{yr^2}{1-r^2}) = Re(z\frac{h'(z)}{h(z)} - \frac{1+(2x-1)r^2}{1-r^2}) \tag{18}$$

If we use the following property in the inequality (17) and equality (18) then,

$$-|z| \leq Rez \leq |z|$$

$$-\frac{2|b|r}{1-r^2} \leq Re(z\frac{h'(z)}{h(z)} - \frac{1+(2x-1)r^2}{1-r^2}) \leq \frac{2|b|r}{1-r^2} \tag{19}$$

$$\frac{1-2|b|r+(2x-1)r^2}{1-r^2} \leq Rez\frac{h'(z)}{h(z)} \leq \frac{1+2|b|r+(2x-1)r^2}{1-r^2}. \tag{20}$$

Using the well known equality

$$Rez\frac{h'(z)}{h(z)} = r\frac{\partial}{\partial r}\log|h(z)|$$

we can get

$$\frac{1-2|b|r+(2x-1)r^2}{1-r^2} \leq r\frac{\partial}{\partial r}\log|h(z)| \leq \frac{1+2|b|r+(2x-1)r^2}{1-r^2} \tag{21}$$

After simple calculations and taking integration in inequality (21) from 0 to r, we get the result easily.

□

Lemma 2.2. *Let* $f = (h(z) + \overline{g(z)})$ *be an element of* $S_H^*(A, B, 1-b)$. *Then*

$$F(|b|, Reb, -r)(1-2|b|r-|2b-1|r^2) \leq |h'(z)| \leq F(|b|, Reb, r)(1+2|b|r+|2b-1|r^2) \tag{22}$$

where

$$F(|b|, Reb, r) = \frac{(1+r)^{|b|-Reb-1}}{(1-r)^{|b|+Reb+1}}$$

Proof. If $f = (h(z) + \overline{g(z)})$ be an element of $S_H^*(A, B, 1-b)$ the transformation

$$(\frac{1+(2b-1)z}{1-z})$$

maps $|z| = r$ onto the disc with the centre

$$C(r) = \frac{1+(2b-1)r^2}{1-r^2}$$

and the radius

$$\rho(r) = \frac{2|b|r}{1-r^2}$$

Therefore we can write

$$\left|z\frac{h'(z)}{h(z)} - (\frac{1+(2b-1)z}{1-z})\right| \leq \frac{2|b|r}{1-r^2} \tag{23}$$

If we use the property

$$|z_1| - |z_2| \leq |z_1 - z_2| \leq |z_1| + |z_2|$$

in the inequality (23) then,

$$\left|z\frac{h'(z)}{h(z)}\right| - \left|\frac{1+(2b-1)r^2}{1-r^2}\right| \leq \left|z\frac{h'(z)}{h(z)} - \frac{1+(2b-1)r^2}{1-r^2}\right| \leq \frac{2|b|r}{1-r^2} \tag{24}$$

$$\left|z\frac{h'(z)}{h(z)}\right| \leq \frac{2|b|r}{1-r^2} + \left|\frac{1+(2b-1)r^2}{1-r^2}\right| \tag{25}$$

$$\Rightarrow \left|z\frac{h'(z)}{h(z)}\right| \leq \frac{1+2|b|r+|2b-1|r^2}{1-r^2} \tag{26}$$

On the other hand after simple calculations from inequality (23) then we take,

$$\frac{1-2|b|r-|2b-1|r^2}{1-r^2} \leq \left|z\frac{h'(z)}{h(z)}\right| \tag{27}$$

If we use the inequalities (26) and (27)

$$\frac{1-2|b|r-|2b-1|r^2}{1-r^2} \leq \left|z\frac{h'(z)}{h(z)}\right| \leq \frac{1+2|b|r+|2b-1|r^2}{1-r^2} \tag{28}$$

Here by using Lemma 2.1, we get the result.

□

Theorem 2.2. *Let* $f = (h(z) + \overline{g(z)})$ *be an element of* $S_H^*(A, B, 1-b)$. *Then*

$$\begin{cases} \frac{|b_1|(1-Ar)}{1-Br} F(|b|, Reb, -r)(1-2|b|r - |2b-1|r^2) \leq |g'(z)| \leq \\ \frac{|b_1|(1+Ar)}{1+Br} F(|b|, Reb, r)(1+2|b|r + |2b-1|r^2), & B \neq 0; \\ \\ |b_1|(1-Ar) F(|b|, Reb, -r)(1-2|b|r - |2b-1|r^2) \leq |g'(z)| \leq \\ |b_1|(1+Ar) F(|b|, Reb, r)(1+2|b|r + |2b-1|r^2), & B = 0. \end{cases} \tag{29}$$

$$\begin{cases} \frac{|b_1|(1-Ar)}{1-Br} \frac{r(1-r)^{|b|-Reb}}{(1+r)^{|b|+Reb}} \leq |g(z)| \leq \frac{|b_1|(1+Ar)}{1+Br} \frac{r(1+r)^{|b|-Reb}}{(1-r)^{|b|+Reb}}, & B \neq 0; \\ \\ |b_1|(1-Ar) \frac{r(1-r)^{|b|-Reb}}{(1+r)^{|b|+Reb}} \leq |g(z)| \leq |b_1|(1+Ar) \frac{r(1+r)^{|b|-Reb}}{(1-r)^{|b|+Reb}}, & B = 0. \end{cases} \tag{30}$$

Proof. Using the definition of $S_H^*(A, B, 1-b)$, then we write

$$\begin{cases} \left| \frac{g'(z)}{h'(z)} - \frac{b_1(1-ABr^2)}{1-Br^2} \right| \leq \frac{|b_1|(A-B)r}{1-B^2r^2}, & B \neq 0; \\ \\ \left| \frac{g'(z)}{h'(z)} - b_1 \right| \leq |b_1| Ar, & B = 0. \end{cases} \tag{31}$$

these inequalities can be written in the following form

$$\begin{cases} \frac{|b_1|(1-Ar)}{1-Br} \leq \left| \frac{g'(z)}{h'(z)} \right| \leq \frac{|b_1|(1+Ar)}{1+Br}, & B \neq 0; \\ \\ |b_1|(1-Ar) \leq \left| \frac{g'(z)}{h'(z)} \right| \leq |b_1|(1+Ar), & B = 0. \end{cases} \tag{32}$$

and using Theorem 2.1 we can write

$$\begin{cases} \frac{|b_1|(1-Ar)}{1-Br} \leq \left| \frac{g(z)}{h(z)} \right| \leq \frac{|b_1|(1+Ar)}{1+Br}, & B \neq 0; \\ \\ |b_1|(1-Ar) \leq \left| \frac{g(z)}{h(z)} \right| \leq |b_1|(1+Ar), & B = 0. \end{cases} \tag{33}$$

Finally if we use Lemma 2.1 and Lemma 2.2 respectively in the inequalities (32) and (33), we get the result.

□

Lemma 2.3. *Let* $f = (h(z) + \overline{g(z)})$ *be an element of* $S_H^*(A, B, 1-b)$, *then*

$$\begin{cases} \frac{|b_1|(1-Ar)}{1-Br} \leq |w(z)| \leq \frac{|b_1|(1+Ar)}{1+Br}, & B \neq 0; \\ \\ |b_1|(1-Ar) \leq |w(z)| \leq |b_1|(1+Ar), & B = 0. \end{cases} \tag{34}$$

Proof. Using Theorem 2.1, then we have

$$\begin{cases} \left| \frac{g'(z)}{h'(z)} - \frac{b_1(1-ABr^2)}{1-B^2r^2} \right| \leq \frac{|b_1|(A-B)r}{1-B^2r^2}, & B \neq 0; \\ \\ \left| \frac{g'(z)}{h'(z)} - b_1 \right| \leq |b_1| Ar, & B = 0. \end{cases} \tag{35}$$

□

Corollary 2.1. *Let $f = (h(z) + \overline{g(z)})$ be an element of $S^*(1-b)$, then*

$$\frac{[(1-|b_1|)+(B-|b_1|A)r][(1+|b_1|)+(B+|b_1|A)r]}{(1+Br)^2} \leq (1-|w(z)|^2) \leq$$

$$\frac{[(1-|b_1|)-(B-|b_1|A)r][(1+|b_1|)+(B+|b_1|A)r]}{(1-Br)^2}$$

$$\frac{(1+|b_1|)-(B+|b_1|A)r}{1-Br} \leq (1+|w(z)|) \leq \frac{(1+|b_1|)+(B+|b_1|A)r}{1+Br}$$

$$\frac{(1-|b_1|)+(B-|b_1|A)r}{1+Br} \leq (1-|w(z)|) \leq \frac{(1-|b_1|)-(B-|b_1|A)r}{1-Br}$$

Proof. This corollary is a simple consequence of Lemma 2.3. □

Corollary 2.2. *Let $f = (h(z) + \overline{g(z)})$ be an element of $S^*(A, B, 1-b)$, then*

$$[F(|b|, Reb, -r)]^2(1-2|b|r-|2b-1|r^2)^2\frac{[(1-|b_1|)+(B-|b_1|A)r][(1+|b_1|)+(B+|b_1|A)r]}{(1+Br)^2}$$

$$\leq J_f \leq$$

$$[F(|b|, Reb, r)]^2(1+2|b|r+|2b-1|r^2)^2\frac{[(1-|b_1|)-(B-|b_1|A)r][(1+|b_1|)+(B+|b_1|A)r]}{(1-Br)^2}$$

Proof. Since

$$\begin{aligned} J_f &= \left|h'(z)\right|^2 - \left|g'(z)\right|^2 = \left|h'(z)\right|^2 - \left|h'(z)w(z)\right|^2 \\ &= \left|h'(z)\right|^2(1-|w(z)|^2), \end{aligned}$$

then using Lemma 2.2 and Corollary 2.1 we get it. □

Corollary 2.3. *Let $f = (h(z) + \overline{g(z)})$ be an element of $S^*(A, B, 1-b)$, then*

$$\int_0^r [F(|b|, Reb, -r)](1-2|b|r-|2b-1|r^2)\frac{(1-|b_1|)+(B-|b_1|A)r}{1+Br}dr \leq |f| \leq$$

$$\int_0^r [F(|b|, Reb, r)](1+2|b|r+|2b-1|r^2)\frac{(1-|b_1|)-(B-|b_1|A)r}{1-Br}dr$$

Proof. Since

$$\begin{aligned} (\left|h'(z)\right| - \left|g'(z)\right|)|dz| \leq |df| \leq (\left|h'(z)\right| + \left|g'(z)\right|)|dz| \Rightarrow \\ (\left|h'(z)\right| - \left|h'(z)w(z)\right|)|dz| \leq |df| \leq (\left|h'(z)\right| + \left|h'(z)w(z)\right|)|dz| \Rightarrow \\ \left|h'(z)\right|(1-|w(z)|)|dz| \leq |df| \leq \left|h'(z)\right|(1+|w(z)|)|dz| \end{aligned} \tag{36}$$

Using Lemma 2.2 and Corollary 2.1 in the inequality (36) we get

$$[F(|b|, Reb, -r)](1-2|b|r-|2b-1|r^2)\frac{(1-r)(1-|b_1|)}{(1+|b_1|r)} \leq |df| \leq$$

$$[F(|b|, Reb, r)](1+2|b|r+|2b-1|r^2)\frac{(1+r)(1-|b_1|^2)}{(1+|b_1|r)}$$

After the integration of this inequality we obtain desired result. □

Theorem 2.3. *If $f = (h(z) + \overline{g(z)})$ be an element of $S_H^*(A, B, 1-b)$ then*

$$\sum_{k=1}^{n}|G_k - h_k|^2 \leq (A-B)^2 + \sum_{k=1}^{n-1}|-BG_k + Ah_k|^2 \tag{37}$$

Proof. If $f = (h(z) + \overline{g(z)}) \in S_H^*(A, B, 1-b)$

$$\frac{g'(z)}{h'(z)} \prec b_1 \frac{1+Az}{1+Bz} \Rightarrow \frac{g(z)}{h(z)} \prec b_1 \frac{1+Az}{1+Bz}$$

$$\frac{g(z)}{h(z)} = b_1 \frac{1+Az}{1+Bz}$$

$$\frac{\frac{1}{b_1}(b_1 z + b_2 z^2 + b_3 z^3 + ... + b_n z^n)}{z + a_2 z^2 + a_3 z^3 + ... + a_n z^n} = \frac{1+Az}{1+Bz} \quad (38)$$

$$\frac{G(z)}{h(z)} = \frac{1+Az}{1+Bz} \Rightarrow$$

where

$$G(z) = z + c_2 z^2 + c_3 z^3 + ...$$

and

$$\phi(z) = \sum_{k=1}^{\infty} e_k z^k$$

If we make simple calculations

$$[G(z) - h(z)] = \phi(z)[-BG(z) + Ah(z)]$$

$$\sum_{k=0}^{\infty} c_k z^k - \sum_{k=0}^{\infty} h_k z^k = \sum_{k=1}^{\infty} e_k z^k [\sum_{k=0}^{\infty} -Bc_k z^k + \sum_{k=0}^{\infty} -Bh_k z^k] \quad (39)$$

$$\Rightarrow (c_0 + c_1 z + c_2 z^2 + ... + c_n z^n + ... - h_0 - h_1 z - h_2 z^2 - ... - h_k z^k)$$

$$= (e_1 z + e_2 z^2 + ... + e_k z^k)(-Bc_0 - Bc_1 z - Bc_2 z^2 - ... - Bc_k z^k + ... + Ah_0 + Ah_1 z + Ah_2 z^2 + ... + Ah_k z^k)$$

Then we can write,

$$\Rightarrow [(G_0 - h_0) + \sum_{k=1}^{\infty} (G_k - h_k) z^k] = (\sum_{k=1}^{\infty} e_k z^k)[(-BG_0 + Ah_0) + \sum_{k=1}^{\infty} (-BG_k + Ah_k) z^k]. \quad (40)$$

if we take $G_0 = h_0 = 1$, in the equality (39), then

$$\Rightarrow [\sum_{k=1}^{\infty} (G_k - h_k) z^k] = (\sum_{k=1}^{\infty} e_k z^k)[(A - B) + \sum_{k=1}^{\infty} (-BG_k + Ah_k) z^k]. \quad (41)$$

$$F(z) = \phi(z) G(z), |\phi(z)| < 1.$$

Therefore we have

$$|F(z)|^2 < |G(z)|^2 \Rightarrow \frac{1}{2} \int_0^{2\pi} \left| F(re^{i\theta}) \right|^2 d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \left| G(re^{i\theta}) \right|^2 d\theta$$

$$\Rightarrow \sum_{k=1}^{n} |G_k - h_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |s_k|^2 r^{2k} \leq [(A - B)^2 + \sum_{k=1}^{n-1} |Bh_k + Ah_k|^2 r^{2k}] \quad (42)$$

Since

$$(\sum_{k=n+1}^{\infty} |s_k|^2 r^{2k}) > 0,$$

then the equality (42) can be written in the following form.

$$\sum_{k=0}^{n} (k+1)^2 |b_{k+1}|^2 r^{2k} \leq \sum_{k=0}^{n} (k+1)^2 |a_{k+1}|^2 r^{2k},$$

taking when $r \to 1$ we obtain,

$$\sum_{k=0}^{n}(k+1)^2\left|b_{k+1}\right|^2 \leq \sum_{k=0}^{n}(k+1)^2\left|a_{k+1}\right|^2$$

Here after brief calculations we get the result. We note that the proof of this theorem has been based on the Clunie method [2]. □

References

[1] C. Caratheodory, *Uber den Variabilitatsatsbereich der Koezienten von Potenzrei-hen, die gegebene Werte nicht annehmen*, Math. Ann. 64 (1907), 95-115.

[2] Clunie and T. Sheil-Small, *Harmonic Univalent functions*, Annales Academiae Scientiarum Fennicae, Series A, Vol. 9, pp. 3-35, (1984).

[3] Duren, P. , *Harmonic Mappings in the Plane*, Vol. 156 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge UK, (2004).

[4] Goodman A. W. , *Univalent Functions*, Volume I and Volume II, Mariner publishing Company INC, Tampa Florida, (1983).

[5] Jack, I. S. , *Functions starlike and convex of order* α, J. London Math. Soc. 3(197), no 2.

[6] W. Janowski, *Some extremal problems for certain families of analytic functions I*, Annales Polinici Mathematici 27, (1973), 298-326.

[7] M. A. Nasr and M. K. Aouf, *Starlike functions of complex order*, Jour. of Natural science and Mathematics vol. 25, No.1, (1985), 1-12.

[8] Robinson, R. M . , *Univalent majorants*, Trans. Amer. Math. Soc. 61 (1947), 1-35.