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ON THE SOLUTIONS OF FUZZY FRACTIONAL DIFFERENTIAL EQUATION

DJURDJICA TAKAČI[1], ARPAD TAKAČI[1], ALEKSANDAR TAKAČI[2] §

Abstract. In this paper the exact and the approximate solutions of fuzzy fractional differential equation, in the sense of Caputo Hukuhara differentiability, with a fuzzy condition are constructed by using the fuzzy Laplace transform. The obtained solutions are expressed in the form of the fuzzy Mittag-Leffler function. The presented procedure is visualized and the graphs of the obtained approximate solutions are drawn by using the *GeoGebra* package.

Keywords: Fractional, Fuzzy calculus, Laplace transform.

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1. Introduction

In recent literature, there are many authors analyzing fuzzy differential equations and fuzzy fractional differential equations. In the paper [9] the solutions of fuzzy fractional differential equations, in the sense of Riemann-Liouville Hukuhara-differentiability, are constructed by using fuzzy the Laplace transform. The properties, main notations and notions of fuzzy the Laplace transform are given in the paper [9]. In our research we use them for determining the solution of a similar problem, but with fractional derivative considered in the sense of the Caputo Hukuhara-differentiability.

In Section 2, the most important notation and notions necessary for the understanding the procedure of constructing solutions are listed.

In Section 3, the exact and the approximate solutions of fractional fuzzy differential equations, with fractional derivative of order $0 < \beta \leq 1$, are constructed, by using fuzzy Laplace transform. The fuzzy Mittag-Leffler function is introduced in order to represent the solution of the considered problem, analogously as the classical Mittag-Leffler function represents solution of the corresponding fractional differential equation. At the end of the paper an example is given and visualized.

The techniques used in this paper were presented in our previous papers as [10], for problems without any fractional derivative, and in the paper [11] with the fractional one.

[1] Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia,
e-mail: djtak@dmi.uns.ac.rs

[2] Faculty of Technology, University of Novi Sad Cara Lazara 1, 21000 Novi Sad, Serbia,
e-mail: stakaci@tf.uns.ac.rs

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2. Notions and notations

2.1. **Fuzzy calculus.** Fuzzy set theory was introduced by L. Zadeh in [13]. In this paper the fuzzy numbers are considered in parametric form, and their arithmetic operations are analyzed in the paper [8].

- Fuzzy function $f : [0,a] \to E$, where E is the set of fuzzy numbers, has the following parametric representation:
$$f(t,\lambda) = (f_1(t,\lambda), f_2(t,\lambda)), \quad t \in [0,a], \;\; 0 \le \lambda \le 1. \tag{1}$$
- Fuzzy integral of fuzzy function f can be defined as:
$$\int f(t,\lambda)dt = \left(\int f_1(t,\lambda)dt, \int f_2(t,\lambda)dt\right), \quad t \in [0,a], \;\; 0 \le \lambda \le 1. \tag{2}$$
- Fuzzy derivative of fuzzy function f can be defined as:
$$(f(t,\lambda))' = \left(f_1'(t,\lambda), f_2'(t,\lambda)\right), \quad t \in [0,a]. \;\; 0 \le \lambda \le 1, \tag{3}$$
- Fuzzy fractional integral of order $0 \le \beta \le 1$ of fuzzy function f, in the Riemann-Liouville sense (see[3], [5], [6], [12]) can be defined by:
$$J^\beta f(t,\lambda) = \left(J^\beta f_1(t,\lambda), \quad J^\beta f_2(t,\lambda)\right). \tag{4}$$
- Fuzzy fractional derivative in Caputo Hukuhara differentiability sense, [3], of order $0 < \beta < 1$, of fuzzy function f, can be defined by:
$$D^\beta f(t,\lambda) = \left(D^\beta f_1(t,\lambda), \;\; D^\beta f_2(t,\lambda)\right). \tag{5}$$

2.2. **Fuzzy Laplace transform.** The classical Laplace transform $F(s)$ of a function f is defined by:
$$F(s) = \mathfrak{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$
provided that this integral converges.

Fuzzy Laplace transform of a fuzzy function given by (1) was defined and analyzed in the paper [9] in the parametric form as:
$$\mathfrak{L}[f(t,\lambda)] = (\mathfrak{L}[f_1](t,\lambda), \mathfrak{L}[f_2](t,\lambda)), \tag{6}$$
where
$$\mathfrak{L}[f_i] =, \quad i = 1,2 \tag{7}$$
are classical Laplace transforms of functions f_i, $i = 1,2$.

The multiplication of fuzzy Laplace transforms $F(s) = (F_1(s), F_2(s))$ and $G(s) = (G_1(s), G_2(s))$ of the fuzzy functions $f(t) = (f_1(t), f_2(t))$ and $g(t) = (g_1(t), g_2(t))$ can be expressed as:
$$F(s)\cdot G(s) = (F_1(s)\cdot G_1(s), F_2(s)\cdot G_2(s)) = (f_1(t) * g_1(t), f_2(t) * g_2(t)),$$
where the sign $*$ stands for the convolution of the appropriate functions.

In that sense the fuzzy Laplace transform of fuzzy fractional integral given by (4) can be written as:
$$\mathfrak{L}\{J^\beta f(t,\lambda)\} = \left(\frac{1}{s^\beta} f_1(t,\lambda), \quad \frac{1}{s^\beta} f_2(t,\lambda)\right).$$
The fuzzy Laplace transform of fuzzy fractional derivative given by (5) can be written as:
$$\mathfrak{L}\{D^\beta f(t,\lambda)\} = \left(s^\beta f_1(t,\lambda) - s^{1-\beta} f_1(0,\lambda), \quad s^\beta f_2(t,\lambda) - s^{1-\beta} f_2(0,\lambda)\right).$$

3. The solution of fuzzy fractional differential equation

Let us consider the fuzzy fractional differential equation with fuzzy coefficients and fuzzy condition:

$$D^{\beta}x(t,\lambda) = A(\lambda)x(t,\lambda) + f(t,\lambda), \quad 0 < \beta < 1, \quad 0 \leq \lambda \leq 1, \tag{8}$$

with the fuzzy initial condition

$$x(0,\lambda) = x_0(\lambda), \quad 0 \leq \lambda \leq 1 \tag{9}$$

where x and f are fuzzy functions with crisp variable t, and x_0 and A are fuzzy numbers. As noticed before the fuzzy fractional derivative is considered in Caputo H-differentiability sense, as in relation (5).

Applying the fuzzy Laplace transform to the problem (8), (9), the following fuzzy equation is obtained:

$$s^{\beta}x^{\lambda} - A^{\lambda}x^{\lambda} = s^{\beta-1}x_0^{\lambda} + F^{\lambda}, \tag{10}$$

where F^{λ} is the fuzzy Laplace transform of the fuzzy function f, $0 \leq \lambda \leq 1$.

Let us remark that the equation in (10) is a fuzzy algebraic equation and its solution can be determined by applying usual algebraic operations taking care about the arithmetic of fuzzy numbers (see [8]). In that sense the solution of equation (10) can be written in the form:

$$x^{\lambda} = \frac{\ell^{1-\beta}x_0^{\lambda} + F^{\lambda}}{s^{\beta} - A^{\lambda}} = \frac{\ell x_0^{\lambda} + \ell^{\beta}F^{\lambda}}{1 - A^{\lambda}\ell^{\beta}} = (\ell x_0^{\lambda} + \ell^{\beta}F^{\lambda})\sum_{i=0}^{\infty}(A^{\lambda})^{i}\ell^{\beta i}, \tag{11}$$

where $\ell = 1/s$.

For our purposes we need to analyze the expressions in relation (11).

It is well known that the classical Laplace transform of classical Mittag-Leffler function is:

$$\mathfrak{L}\left\{\sum_{i=0}^{\infty}\frac{t^{\beta i}}{\Gamma(\beta i+1)}\right\} = \sum_{i=0}^{\infty}s^{-\beta i-1}.$$

Next we introduce the fuzzy Mittag-Leffler function as:

$$\sum_{i=0}^{\infty}a_i(\lambda)\frac{t^{\beta i}}{\Gamma(\beta i+1)} = \left(\sum_{i=0}^{\infty}a_{1i}(\lambda)\frac{t^{\beta i}}{\Gamma(\beta i+1)}, \sum_{i=0}^{\infty}a_{2i}(\lambda)\frac{t^{\beta i}}{\Gamma(\beta i+1)}\right), \quad 0 \leq \lambda \leq 1,$$

for fuzzy numbers $a_i(\lambda) = (a_{1i}(\lambda), a_{2i}(\lambda))$, $i = 1, 2, \ldots$, and $0 \leq \lambda \leq 1$.

The fuzzy Laplace transform of the fuzzy Mittag-Leffler function can be written as:

$$\mathfrak{L}\left\{\sum_{i=0}^{\infty}a_i(\lambda)\frac{t^{\beta i}}{\Gamma(\beta i+1)}\right\} = \sum_{i=0}^{\infty}(a^{\lambda})_i s^{-\beta i-1},$$

where a_i, $i = 1, 2, \ldots$, and $0 \leq \lambda \leq 1$.

The multiplications and powers of fuzzy numbers, appearing in relation (11), are fuzzy numbers. This implies that the expressions in relation (11) represent two fuzzy Mittag-Leffler functions. It is important to note that the infinite series representing fuzzy Mittag-Leffler functions, converge for every t, in the sense of the results in the paper [4]. Therefore, the fuzzy function $x(t,\lambda)$, corresponding to x^{λ}, in relation (11), represents the exact solution of the problem (8), (9).

The approximate solution of the considered problem is obtained by taking finite sum in Mittag-Leffler functions.

3.1. **An example.** Let us consider the equation (8) with $A = 1$ (crisp number) and $f(t, \lambda) = (u_1(\lambda)t, u_2(\lambda)t)$, for the sake of simplicity. Then the fuzzy Laplace transform of the fuzzy equation (8) and the condition (9) can be written as:

$$s^\beta x^\lambda - Ax^\lambda = s^{\beta-1}x_0^\lambda + \frac{1}{s^2}u^\lambda,$$

and its solution has the form:

$$x^\lambda = (\ell x_0^\lambda + \ell^{2+\beta}u^\lambda)\sum_{i=0}^{\infty}\ell^{\beta i},$$

where $\ell = 1/s$.

From the previous relation the fuzzy solution of the fuzzy problem (8), (9) can be expressed as:

$$\begin{aligned} x(t,\lambda) &= x_0(\lambda)\sum_{i=0}^{\infty}\frac{t^{\beta i}}{\Gamma(\beta i+1)} + u(\lambda)\sum_{i=0}^{\infty}\frac{t^{(i+1)\beta+2}}{\Gamma((i+1)\beta+2)} \\ &=: x_0(\lambda)M(t,\beta) + u(\lambda)L(t,\beta), \end{aligned} \tag{12}$$

for $0 \le \lambda \le 1$, and M and L are the corresponding clasical Mittag-Leffler functions with crisp variable t.

In particular, if fuzzy numbers $u(\lambda)$ and $x(0, \lambda)$ are triangular fuzzy numbers given by:

$$\begin{aligned} x(0,\lambda) &= (1,2,3) = (2-(1-\lambda), 2+(1-\lambda)), \quad 0 \le \lambda \le 1 \\ u(\lambda) &= (0,1,2) = (1-(1-\lambda), 1+(1-\lambda)), \quad 0 \le \lambda \le 1, \end{aligned}$$

then the solution of corresponding problem expressed by relation (11) has the form:

$$x^\lambda = \frac{(1,2,3)\ell + (0,1,2)\ell^{2+\beta}}{1-\ell} = \left((1,2,3) + (0,1,2)\ell^{1+\beta}u^\lambda\right)\sum_{i=0}^{\infty}\ell^{\beta i+1}.$$

Then the fuzzy solution of the problem (8), (9) is obtained from previous relation, and it can be written as:

$$\begin{aligned} x(t,\lambda) &= (1,2,3)M(t,\beta) + (0,1,2)L(t,\beta) \\ &= ((1+\lambda, 3-\lambda)M(t,\beta) + (\lambda, 2-\lambda)L(t,\beta)) \\ &= ((1+\lambda)M(t,\beta) + \lambda L(t,\beta), (3-\lambda)M(t,\beta) + (2-\lambda)L(t,\beta)), \end{aligned}$$

where the notations M and L are introduced in relation (12).

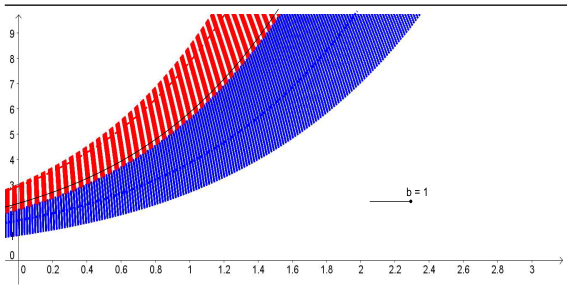

Figure 1.

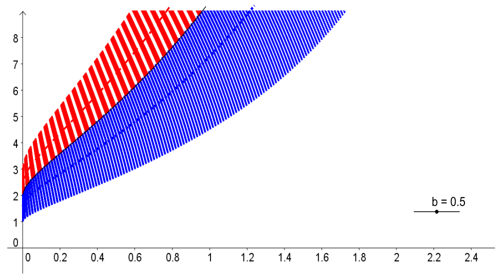

Figure 2.

On Figures 1 and 2 we visualized the approximate solutions $x_n(t, \lambda)$, for $n = 7$, (by taking seven addends in Mitag-Leffler function), and $\beta = 1$, and $\beta = 0.5$, respectively. Let us remark that if $\beta = 1$, then the equation (8) is fuzzy differential equation and for $\beta = 0.5$, the equation (8) is a fuzzy fractional differential equation, for all $0 \le \lambda \le 1$. The dots curves (lower curves, blue) are the graphs of the function $\lambda M_7(t,\beta) + (1+\lambda)L_7(t,\beta)$ and the dash (uper curves, red) curves are the graphs of the function $(2-\lambda)M_7(t,\beta) + (3-\lambda)L_7(t,\beta)$.

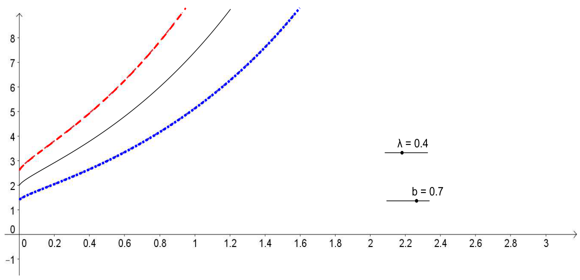

Figure 3.

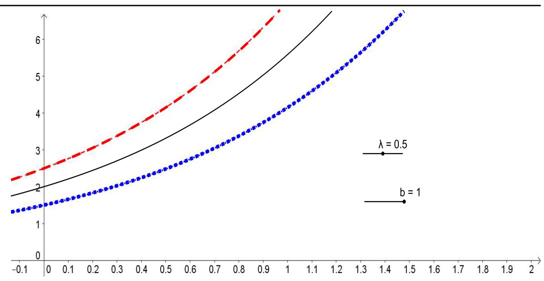

Figure 4.

On Figures 3 and 4, we visualized the approximate solutions for $n = 7$, for $\beta = 0.7$, $\lambda = 0.4$, and $\beta = 1$, $\lambda = 0.5$, respectively. The dots curves (lower curves, blue) is the graphs of the function $\lambda M_7(t,\beta) + (1+\lambda)L_7(t,\beta)$ and the dash (upper curves, red) curves is the graphs of the function $(2-\lambda)M_7(t,\beta) + (3-\lambda)L_7(t,\beta)$. The curve in the middle is the graph of the solution for $\lambda = 1$. Let us remark that if $\beta = 1$, and $\lambda = 1$ then the equation (8) is ordinary differential equation $x'(t) = x(t) + t$ with the condition $x(0) = 2$. Its solution is $x(t) = 3e^t - t - 1$, and the graph of its approximation is drawn on Figure 4, the middle line.

In previous example it is shown that the obtained solutions of the considered fuzzy fractional equation differential in special cases represent:

- the solution of corresponding fuzzy differential equation, if the order of fractional derivative $\beta = 1$;
- the solution of corresponding fractional differential equation, for $0 < \beta \le 1$, and $\lambda = 1$;
- the solution of corresponding differential equation, if the order of fractional derivative $\beta = 1$, and $\lambda = 1$.

3.2. **Future work.** In the following we shall consider the fuzzy fractional differential equation with fuzzy coefficients:

$$x^{(\beta_N)}(t,\lambda) - \sum_{k=0}^{N-1} A_k(\lambda) x^{(\beta_k)}(t,\lambda) = f(t,\lambda) \tag{13}$$

for $0 < \beta_1 < \beta_2 < \ldots < \beta_N$, $\quad 0 \le \lambda \le 1$, $\quad t > 0$, and with the appropriate with the initial conditions. In (13) x and f are fuzzy functions and A_k, $\quad k = 0, \ldots N-1$ are triangular fuzzy numbers. Fractional derivatives are considered in Caputo H-differentiability sense. By using fuzzy Laplace transform the exact and the approximate solutions will be constructed.

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