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OPTIMIZATION OF RENEWAL INPUT (a, c, b) POLICY WORKING VACATION QUEUE WITH CHANGE OVER TIME AND BERNOULLI SCHEDULE VACATION INTERRUPTION

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Abstract. This paper presents a renewal input single working vacation queue with change over time and Bernoulli schedule vacation interruption under (a, c, b) policy. The service and vacation times are exponentially distributed. The server begins service if there are at least c units in the queue and the service takes place in batches with a minimum of size a and a maximum of size b $(a \leq c \leq b)$. The change over period follows if there are $(a-1)$ customers at service completion instants. The steady state queue length distributions at arbitrary and pre-arrival epochs are obtained. An optimal cost policy is presented along with few numerical experiences. The genetic algorithm and quadratic fit search method are employed to search for optimal values of some important parameters of the system.

Keywords: Single working vacation, vacation interruption, cost, queue, genetic algorithm.

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1. Introduction

Working vacation (WV) models are widely used to analyze many problems in the area of computer communication, manufacturing, production and transportation systems. In these models service is provided during vacation at a rate usually lower than the regular service rate. The concept was introduced by Servi and Finn [1] while analyzing an $M/M/1$ queue with multiple working vacation (MWV), which was extended to $GI/M/1/MWV$ queue by Baba [2]. For single working vacation (SWV) queues, Chae et al. [3] considered the $GI/M/1$ and $GI/Geo/1$ queues. The performance analysis of a $GI/M/1$ queue with SWV has been studied by Li and Tian [4]. Further studies on WV are found in Banik et al. [5], Goswami and Mund [6], Jain and Singh [7], Li et al. [8], Tian and Zhang [9], etc.

Li and Tian [10] first introduced an vacation interruption (VI) in an $M/M/1$ queue. Later, Li et al. [11] generalized their results to a $GI/M/1$ queue with WV and VI. Zhao et al. [12] studied a $GI/M/1$ queue with set-up period, WV and VI. A WV queue with service interruption and multi optional repair was discussed by Jain et al. [13]. Zhang and Hou [14] analyzed a $MAP/G/1$ queue with WV and VI. For the Bernoulli schedule vacation interruption (BS-VI), Zhang and Shi [15] first studied an $M/M/1$ queue with VI under the Bernoulli rule. Recently, Li et al. [16] studied a $GI/Geo/1$ SWV queue with

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start-up period and BS-VI using embedded Markov chain technique. Further, see Gao and Liu [17] for a study on $M/G/1$ queue with SWV and BS-VI.

In many real-life queueing situations jobs are served with a control limit policy. For example, in some manufacturing systems it is possible to process jobs only when the number of units to be processed exceeds a specified level, and when service starts, it is profitable to continue it even when the queue size is less than the specified level but not less than a secondary limit. Tadj et al. [18] have analyzed an optimal control of batch arrival, bulk service queueing system with N policy. The infinite buffer multiple vacations queue with change over times under (a, c, d) policy has been studied by Baburaj and Surendranath [19], where the arrivals and service times are exponentially distributed. A discrete time bulk service (a, c, d) policy queue has been presented by Baburaj [20].

One of the most fundamental objectives in the performance evaluation of queueing models is to search for an optimal value. Many practical design problems are characterized by mixing continuous and discrete variables, discontinuous and non-convex design spaces. In such cases, the standard non linear optimization techniques will be inefficient, computationally expensive, and in most cases, find relative optimum that is closest to the starting point. Genetic algorithm (GA) is well suited for solving such problems and in most cases it finds the global optimum solution with high probability. More details on GA can be found in Haupt & Haupt [21], Lin and Ke ([22], [23]), Rao [24], etc.

Quadratic fit search method (QFSM) is another optimization technique which can be used when the objective function is highly complex and obtaining its derivative is a difficult task. Given a 3-point pattern, one can fit a quadratic function through corresponding functional values that has a unique optimum for the given objective function. For details of QFSM one may refer Rardin [25].

Motivated by the problem of optimization in an (a, c, b) policy queue with vacations, this paper focuses on an infinite buffer renewal input SWV queue with change over time and BS-VI under (a, c, b) policy. The inter-arrival time of customers and service time of batches are respectively, arbitrarily and exponentially distributed. We provide a recursive method using the supplementary variable technique to develop the steady state queue length distributions. Various performance measures and a cost model are developed to determine the optimum service rates in regular busy period and in working vacation, using GA and QFSM. Numerical results are presented to show the effect of model parameters.

The rest of the paper is organized as follows. Model description is given in Section 2. Section 3 presents the computations of the steady state distributions of the number of customers in the queue at arbitrary and pre-arrival epochs. Section 4 discusses various performance measures and the cost analysis. Section 5 contains numerical results and Section 6 concludes the paper.

2. Description of the model

We consider a SWV $GI/M^{(a,c,b)}/1$ queueing system with change over time and BS-VI. It is assumed that the inter-arrival times of customers are independent and identically distributed random variables with cumulative distribution function $A(x)$, probability density function $a(x)$, $x \geq 0$, Laplace-Stiltjes transform (L.-S.T.) $A^*(\theta)$, $Re(\theta) \geq 0$ and mean inter-arrival time $1/\lambda = -A^{*(1)}(0)$, where $h^{(1)}(x_0)$ denotes the first derivative of $h(x)$ at $x = x_0$. The service begins only if there are at least c units in the queue. The customers are served in batches with minimum size a and maximum size b $(a \leq c \leq b)$. The various server states and the activities are given below: At a service completion epoch during regular busy period if the queue size (j) is

- $a \leq j \leq c$: the server continues to serve.

- $0 \leq j \leq a-2$: the server goes for WV.
- $j = a-1$: server will wait for some time in the system called change over time which is exponentially distributed with rate α. It starts service on finding an arrival during this change over time, otherwise it will go for WV.

On returning from working vacation, if the queue size j is

- $0 \leq j \leq c-1$: the server remains dormant in the system until c customers are available in the queue.
- $j \geq c$: $min\{j, b\}$ customers are served according to first-come, first-served rule.

Service times during regular busy period, during vacation and vacation times are exponentially distributed with rate μ, η and ϕ, respectively. During a WV a customer is serviced at a lower rate and at the instants of a service completion, the vacation is interrupted and the server is resumed to a regular busy period with probability $\bar{q} = 1-q$ (if there are at least c customers in the queue), or continues the vacation with probability q. Further, the inter-arrival times, service times, change over times and WV times are mutually independent of each other. The traffic intensity is given by $\rho = \lambda/b\mu$. The state of the system at time t is described by the following random variables:

- $X(t)$ = number of customers present in the queue,
- $U(t)$ = remaining inter-arrival time for the next arrival,
- $Y(t) = \begin{cases} 0, & \text{if the server is in working vacation,} \\ 1, & \text{if the server is busy,} \\ 2, & \text{if the server is in change over time,} \\ 3, & \text{if the server is dormant.} \end{cases}$

At steady state, let us define

$$P_{n,j}(x)dx = \lim_{t\to\infty} Pr\Big\{X(t) = n,\ x < U(t) \leq x + dx, Y(t) = j\Big\},\ j = 0, 1, 2, 3.$$

Let $P^*_{n,0}(\theta),\ P^*_{n,1}(\theta),\ P^*_{n,2}(\theta)$ and $P^*_{n,3}(\theta)$ be the L.-S.T. of $P_{n,0}(x),\ P_{n,1}(x),\ P_{n,2}(x)$ and $P_{n,3}(x)$, respectively so that $P_{n,0} \equiv P^*_{n,0}(0),\ n \geq 0,\ P_{n,1} \equiv P^*_{n,1}(0),\ n \geq 0,\ P_{n,2} \equiv P^*_{n,2}(0),\ n = a-1$ and $P_{n,3} \equiv P^*_{n,3}(0),\ 0 \leq n \leq c-1$ are the steady state probabilities that n customers are in the queue and the server is in working vacation, regular busy period, in change over time and dormancy, respectively, at an arbitrary epoch.

3. Analysis of the model

In this section, we shall discuss the steady state queue length distributions using the supplementary variable technique and the recursive method. We first establish the mathematical equations that govern the system by employing the remaining inter-arrival time as the supplementary variable. Relating the states of the system at two consecutive time epochs t and $t+dt$, and using probabilistic arguments, we set up the following differential-difference equations at steady state:

$$\begin{aligned}
-P^{(1)}_{0,0}(x) &= \mu P_{0,1}(x) + q\eta \sum_{n=c}^{b} P_{n,0}(x) - \phi P_{0,0}(x), \\
-P^{(1)}_{n,0}(x) &= a(x)P_{n-1,0}(0) + \mu P_{n,1}(x) + q\eta P_{n+b,0}(x) - \phi P_{n,0}(x),\ 1 \leq n \leq a-2, \\
-P^{(1)}_{a-1,0}(x) &= a(x)P_{a-2,0}(0) + \alpha P_{a-1,2}(x) + q\eta P_{a+b-1,0}(x) - \phi P_{a-1,0}(x), \\
-P^{(1)}_{n,0}(x) &= a(x)P_{n-1,0}(0) + q\eta P_{n+b,0}(x) - \phi P_{n,0}(x),\ a \leq n \leq c-1, \\
-P^{(1)}_{n,0}(x) &= a(x)P_{n-1,0}(0) + q\eta P_{n+b,0}(x) - (\phi + \eta)P_{n,0}(x),\ n \geq c,
\end{aligned}$$

$$
\begin{aligned}
-P_{0,1}^{(1)}(x) &= a(x)P_{a-1,2}(0) + a(x)P_{c-1,3}(0) + \mu\sum_{n=a}^{b} P_{n,1}(x) + (\phi+\bar{q}\eta)\sum_{n=c}^{b} P_{n,0}(x) \\
&\quad -\mu P_{0,1}(x), \\
-P_{n,1}^{(1)}(x) &= a(x)P_{n-1,1}(0) + \mu P_{n+b,1}(x) + (\phi+\bar{q}\eta)P_{n+b,0}(x) - \mu P_{n,1}(x), n \geq 1, \\
-P_{a-1,2}^{(1)}(x) &= \mu P_{a-1,1}(x) - \alpha P_{a-1,2}(x), \\
-P_{0,3}^{(1)}(x) &= \phi P_{0,0}(x), \\
-P_{n,3}^{(1)}(x) &= a(x)P_{n-1,3}(0) + \phi P_{n,0}(x), 1 \leq n \leq c-1.
\end{aligned}
$$

where $P_{n,j}(0), n \geq 0,\ j = 0,1,2,3$ are the respective probabilities with the remaining inter-arrival time equal to zero, i.e., an arrival is about to occur. Multiplying the above equations by $e^{-\theta x}$, integrating with respect to x from 0 to ∞ yields

$$
\begin{aligned}
(\phi-\theta)P_{0,0}^{*}(\theta) &= \mu P_{0,1}^{*}(\theta) + q\eta\sum_{n=c}^{b} P_{n,0}^{*}(\theta) - P_{0,0}(0), && (1)\\
(\phi-\theta)P_{n,0}^{*}(\theta) &= A^{*}(\theta)P_{n-1,0}(0) + \mu P_{n,1}^{*}(\theta) + q\eta P_{n+b,0}^{*}(\theta) - P_{n,0}(0), \\
&\quad 1 \leq n \leq a-2, && (2)\\
(\phi-\theta)P_{a-1,0}^{*}(\theta) &= A^{*}(\theta)P_{a-2,0}(0) + \alpha P_{a-1,2}^{*}(\theta) + q\eta P_{a+b-1,0}^{*}(\theta) - P_{a-1,0}(0), && (3)\\
(\phi-\theta)P_{n,0}^{*}(\theta) &= A^{*}(\theta)P_{n-1,0}(0) + q\eta P_{n+b,0}^{*}(\theta) - P_{n,0}(0),\ a \leq n \leq c-1, && (4)\\
(\eta+\phi-\theta)P_{n,0}^{*}(\theta) &= A^{*}(\theta)P_{n-1,0}(0) + q\eta P_{n+b,0}^{*}(\theta) - P_{n,0}(0),\ n \geq c, && (5)\\
(\mu-\theta)P_{0,1}^{*}(\theta) &= A^{*}(\theta)P_{a-1,2}(0) + A^{*}(\theta)P_{c-1,3}(0) + \mu\sum_{n=a}^{b} P_{n,1}^{*}(\theta) \\
&\quad +(\phi+\bar{q}\eta)\sum_{n=c}^{b} P_{n,0}^{*}(\theta) - P_{0,1}(0), && (6)\\
(\mu-\theta)P_{n,1}^{*}(\theta) &= A^{*}(\theta)P_{n-1,1}(0) + \mu P_{n+b,1}^{*}(\theta) + (\phi+\bar{q}\eta)P_{n+b,0}^{*}(\theta) - P_{n,1}(0), \\
&\quad n \geq 1, && (7)\\
(\alpha-\theta)P_{a-1,2}^{*}(\theta) &= \mu P_{a-1,1}^{*}(\theta) - P_{a-1,2}(0), && (8)\\
-\theta P_{0,3}^{*}(\theta) &= \phi P_{0,0}^{*}(\theta) - P_{0,3}(0), && (9)\\
-\theta P_{n,3}^{*}(\theta) &= A^{*}(\theta)P_{n-1,3}(0) + \phi P_{n,0}^{*}(\theta) - P_{n,3}(0),\ 1 \leq n \leq c-1. && (10)
\end{aligned}
$$

Adding equations (1) to (10), taking limit as $\theta \to 0$ and using the normalizing condition, $\sum_{n=0}^{\infty}(P_{n,0}+P_{n,1}) + P_{a-1,2} + \sum\limits_{n=0}^{c-1} P_{n,3} = 1$, we obtain

$$
\sum_{n=0}^{\infty}\Big(P_{n,0}(0)+P_{n,1}(0)\Big) + P_{a-1,2}(0) + \sum_{n=0}^{c-1} P_{n,3}(0) = \lambda. \qquad (11)
$$

It may be noted here that the left-hand side of (11) represents the probability that an arrival is about to occur, which is equal to the arrival rate of customers. This is used in the sequel to obtain a relation between an arrival is about to occur and the pre-arrival epoch probabilities.

3.1. **Steady state queue length distribution at pre-arrival epochs.** Let $P_{n,j}^{-}$ denotes the pre-arrival epoch probability, that is, an arrival sees n customers in the queue and the server is in state j at arrival epochs. Applying Bayes' theorem, we have

$$P^-_{n,j} = \lim_{t\to\infty} \frac{P[X(t)=n, Y(t)=j, U(t)=0]}{P[U(t)=0]}.$$

Further, using (11) in the above expression, we obtain

$$P^-_{n,j} = \frac{1}{\lambda} P_{n,j}(0),\ n \geq 0,\ j=0,1;\ n=a-1,\ j=2;\ 0 \leq n \leq c-1,\ j=3. \quad (12)$$

To obtain $P^-_{n,j}$, we need to evaluate $P_{n,j}(0)$, which is discussed below.
We define the displacement operator E as $E^x\omega_n = \omega_{n+x}$, and rewrite equation (5) as

$$(\eta + \phi - \theta - q\eta E^b)P^*_{n,0}(\theta) = (A^*(\theta) - E)P_{n-1,0}(0),\ n \geq c.$$

Setting $\theta = \eta + \phi - q\eta E^b$ in the above equation, we get

$$P_{n,0}(0) = kr^n,\ n \geq c-1. \quad (13)$$

where k is an arbitrary constant and r is a real root inside the unit circle of the equation $A^*(\eta + \phi - q\eta z^b) - z = 0$. We also have,

$$P^*_{n,0}(\theta) = g(n,\theta)k,\ n \geq c, \quad (14)$$

where $g(n,\theta) = \frac{r^{n-1}(A^*(\theta)-r)}{\tau_1-\theta}$ and $\tau_1 = \eta(1-qr^b) + \phi$.
From equation (7), we get

$$P_{n,1}(0) = k_1\xi^n - \frac{(\phi+\bar{q}\eta)kr^{n+b}}{\tau_2},\ n \geq 0, \quad (15)$$

$$P^*_{n,1}(\theta) = g_1(n,\theta)k_1 + g_2(n,\theta)k,\ n \geq 1, \quad (16)$$

where $\tau_2 = \tau_1 - \mu(1-r^b)$, k_1 is an arbitrary constant, ξ is a real root inside the unit circle of the equation $A^*(\mu - \mu z^b) - z = 0$ and

$$g_1(n,\theta) = \frac{\xi^{n-1}(A^*(\theta)-\xi)}{\mu - \theta - \mu\xi^b}\ ,\ g_2(n,\theta) = -\frac{(\phi+\bar{q}\eta)(A^*(\theta)-r)r^{n+b-1}}{\tau_2(\tau_1-\theta)}.$$

Setting $\theta = \alpha$ in equation (8) and using equation (16), one can obtain $P_{a-1,2}(0)$ as

$$P_{a-1,2}(0) = k_1L_1 - kL_2, \quad (17)$$

where

$$L_1 = \frac{\mu(A^*(\alpha)-\xi)\xi^{a-2}}{\mu - \mu\xi^b - \alpha} \text{ and } L_2 = \frac{\mu(A^*(\alpha)-r)(\phi+\bar{q}\eta)r^{a+b-2}}{\tau_2(\tau_1-\alpha)}.$$

Equation (8) together with equations (16) and (17) yields

$$P^*_{a-1,2}(\theta) = g_3(\theta)k_1 + g_4(\theta)k, \quad (18)$$

where

$$g_3(\theta) = \frac{1}{\alpha-\theta}\left(\frac{\mu(A^*(\theta)-\xi)\xi^{a-2}}{\mu-\mu\xi^b-\theta} - L_1\right),$$

$$g_4(\theta) = \frac{1}{\alpha-\theta}\left(L_2 - \frac{\mu(A^*(\theta)-r)(\phi+\bar{q}\eta)r^{a+b-2}}{\tau_2(\tau_1-\theta)}\right).$$

Now inserting $\theta = \phi$ and using equations (13) and (14) in equation (4), we get

$$P_{n,0}(0) = kr^{c-1}h(n),\ a-1 \leq n \leq c-2, \quad (19)$$

where

$$h(n) = A^*(\phi)^{n-c+1}\left(1 - \{q\eta(A^*(\phi)-r)r^{b-c+n+1}\sum_{i=0}^{c-n-2} r^i A^*(\phi)^{c-n-2-i}\}/(\tau_1-\phi)\right).$$

Setting $\theta = \phi$ in (3), we get

$$P_{a-2,0}(0) = kL_3 - k_1L_4, \tag{20}$$

where

$$L_3 = \frac{1}{A^*(\phi)}\left(r^{c-1}h(a-1) - \frac{\alpha}{\alpha-\phi}\big(L_2 - \frac{\mu(A^*(\phi)-r)(\phi+\bar{q}\eta)r^{b+a-2}}{\tau_2(\tau_1-\phi)}\big) - \frac{q\eta(A^*(\phi)-r)r^{a+b-2}}{\tau_1-\phi}\right),$$

$$L_4 = \frac{\alpha}{A^*(\phi)(\alpha-\phi)}\left(\frac{\mu(A^*(\phi)-\xi)\xi^{a-2}}{\mu-\phi-\mu\xi^b} - L_1\right).$$

Setting $\theta = \phi$ in equation (2), we obtain

$$P_{n,0}(0) = kh_1(n) - k_1h_2(n),\ 0 \le n \le a-3, \tag{21}$$

where

$$h_1(n) = A^*(\phi)^{n-a+2}\left(L_3 + \frac{(A^*(\phi)-r)r^{b+n}\Big(\mu(\phi+\bar{q}\eta) - q\eta\tau_2\Big)\sum_{i=0}^{a-3-n} r^i A^*(\phi)^{a-3-n-i}}{\tau_2(\tau_1-\phi)}\right),$$

$$h_2(n) = A^*(\phi)^{n-a+2}\left(L_4 + \frac{\mu(A^*(\phi)-\xi)\xi^n\sum_{i=0}^{a-3-n}\xi^i A^*(\phi)^{a-3-n-i}}{\mu-\phi-\mu\xi^b}\right).$$

Inserting $\theta = \mu$ in equation (6), we get

$$P_{c-1,3}(0) = \frac{k_1}{A^*(\mu)}\big(1 - A^*(\mu)L_1 + \frac{(A^*(\mu)-\xi)(\xi^{a-1}-\xi^b)}{\xi^b(1-\xi)}\big) + \frac{k}{A^*(\mu)}\left(A^*(\mu)L_2 - \frac{(\phi+\bar{q}\eta)r^b}{\tau_2} + \frac{(A^*(\mu)-r)(\phi+\bar{q}\eta)\big(\mu(r^{b+a-1}-r^{2b}) - \tau_2(r^{c-1}-r^b)\big)}{\tau_2(\tau_1-\mu)(1-r)}\right) \tag{22}$$

Putting $\theta = 0$ in equation (6), we get

$$\mu P_{0,1} = k_1L_5 + kL_6, \tag{23}$$

where

$$L_5 = \frac{1}{A^*(\mu)} + (\xi^{a-1}-\xi^b)\left(\frac{A^*(\mu)-\xi}{A^*(\mu)\xi^b(1-\xi)} + \frac{1}{1-\xi^b}\right) - 1,$$

$$L_6 = \frac{\phi+\bar{q}\eta}{\tau_2}\left(\frac{r^b(A^*(\mu)-1)}{A^*(\mu)} + \Big[\mu(r^{b+a-1}-r^{2b}) - \tau_2(r^{c-1}-r^b)\Big] \times \Big(\frac{A^*(\mu)-r}{A^*(\mu)(\tau_1-\mu)(1-r)} - \frac{1}{\tau_1}\Big)\right).$$

Setting $\theta = 0$ in equation (1), we obtain

$$P_{0,0} = \frac{k_1}{\phi}\Big(L_5 + h_2(0)\Big) + \frac{k}{\phi}\Big(L_6 + \frac{q\eta(r^{c-1}-r^b)}{\tau_1} - h_1(0)\Big). \tag{24}$$

Put $\theta = 0$ in equation (2) to get

$$\begin{aligned} P_{n,0} &= kh_3(n) + k_1 h_4(n),\ 1 \le n \le a-3, && (25)\\ P_{a-2,0} &= \frac{k}{\phi}\Big(h_1(a-3) - L_3 - \frac{(1-r)r^{b+a-3}\Big(\mu(\phi+\bar{q}\eta) - q\eta\tau_2\Big)}{\tau_2\tau_1}\Big)\\ &\quad + \frac{k_1}{\phi}\Big(\frac{(1-\xi)\xi^{a-3}}{1-\xi^b} + L_4 - h_2(a-3)\Big), && (26)\end{aligned}$$

where

$$\begin{aligned} h_3(n) &= \frac{1}{\phi}\Big(h_1(n-1) - h_1(n) - \frac{(1-r)r^{b+n-1}\Big(\mu(\phi+\bar{q}\eta) - q\eta\tau_2\Big)}{\tau_2\tau_1}\Big),\\ h_4(n) &= \frac{1}{\phi}\Big(\frac{(1-\xi)\xi^{n-1}}{1-\xi^b} + h_2(n) - h_2(n-1)\Big).\end{aligned}$$

Using equation (3) for $\theta = 0$, we obtain

$$\begin{aligned} P_{a-1,0} &= \frac{k}{\phi}\Big(L_3 + L_2 - \frac{(1-r)r^{a+b-2}\Big(\mu(\phi+\bar{q}\eta) - q\eta\tau_2\Big)}{\tau_2\tau_1} - r^{c-1}h(a-1)\Big)\\ &\quad + \frac{k_1}{\phi}\Big(\frac{(1-\xi)\xi^{a-2}}{1-\xi^b} - L_1 - L_4\Big). && (27)\end{aligned}$$

Inserting $\theta = 0$ in equation (4), we get

$$\begin{aligned} P_{n,0} &= \frac{k}{\phi}\Big(r^{c-1}\Big(h(n-1) - h(n)\Big) + \frac{q\eta(1-r)r^{b+n-1}}{\tau_1}\Big),\ a \le n \le c-2, && (28)\\ P_{c-1,0} &= \frac{kr^{c-1}}{\phi}\Big(h(c-2) - 1 + \frac{q\eta(1-r)r^{b-1}}{\tau_1}\Big). && (29)\end{aligned}$$

From equations (9) and (10), we get

$$P_{n,3}(0) = \phi\sum_{j=0}^{n} P_{j,0},\ 0 \le n \le c-1.$$

Thus using (24) to (28), we have

$$\begin{aligned} P_{0,3}(0) &= k\Big(L_6 + \frac{q\eta(r^{c-1} - r^b)}{\tau_1} - h_1(0)\Big) + k_1\Big(L_5 + h_2(0)\Big), && (30)\\ P_{n,3}(0) &= kh_5(n) + k_1 h_6(n),\ 1 \le n \le a-3, && (31)\\ P_{a-2,3}(0) &= k_1\Big(h_6(a-3) + \frac{(1-\xi)\xi^{a-3}}{1-\xi^b} + L_4 - h_2(a-3)\Big) + k\Big(h_5(a-3) + h_1(a-3)\\ &\quad - L_3 - \frac{(1-r)r^{b+a-3}\Big(\mu(\phi+\bar{q}\eta) - q\eta\tau_2\Big)}{\tau_1\tau_2}\Big), && (32)\\ P_{a-1,3}(0) &= k_1 L_7 + k\Big(h_5(a-3) + h_1(a-3) - \frac{(1-r^2)r^{b+a-3}\Big(\mu(\phi+\bar{q}\eta) - q\eta\tau_2\Big)}{\tau_1\tau_2}\\ &\quad + L_2 - r^{c-1}h(a-1)\Big), && (33)\\ P_{n,3}(0) &= k_1 L_7 + kh_7(n),\ a \le n \le c-2, && (34)\end{aligned}$$

where

$$\begin{aligned}
h_5(n) &= L_6 + \frac{q\eta(r^{c-1}-r^b)}{\tau_1} - h_1(0) + \phi\sum_{j=1}^{n} h_3(j),\\
h_6(n) &= L_5 + h_2(0) + \phi\sum_{j=1}^{n} h_4(j),\\
L_7 &= h_6(a-3) + \frac{(1-\xi^2)\xi^{a-3}}{1-\xi^b} - h_2(a-3) - L_1,\\
h_7(n) &= h_5(a-3) + h_1(a-3) - \frac{(1-r^2)r^{b+a-3}\Big(\mu(\phi+\bar{q}\eta)-q\eta\tau_2\Big)}{\tau_1\tau_2} + L_2\\
&\quad -r^{c-1}h(n) + \frac{q\eta r^{b+a-1}(1-r^{n-a+1})}{\tau_1}.
\end{aligned}$$

Using the normalization condition, we have

$$k_1L_8 + kL_9 = \lambda, \tag{35}$$

where

$$\begin{aligned}
L_8 &= \sum_{n=1}^{a-3}\Big(h_6(n)-h_2(n)\Big) + \frac{1}{A^*(\mu)} + \frac{(A^*(\mu)-\xi)(\xi^{a-1}-\xi^b)}{A^*(\mu)\xi^b(1-\xi)} + L_5 + h_6(a-3)\\
&\quad -h_2(a-3) + \frac{(1-\xi)\xi^{a-3}}{1-\xi^b} + (c-a)L_7 + \frac{1}{1-\xi},\\
L_9 &= \sum_{n=1}^{a-3}\Big(h_1(n)+h_5(n)\Big) + r^{c-1}\sum_{n=a}^{c-2}h(n) + \sum_{n=a}^{c-2}h_7(n) + \frac{r^{c-1}}{1-r} + \frac{q\eta(r^{c-1}-r^b)}{\tau_1} + L_2\\
&\quad +\frac{\phi+\bar{q}\eta}{\tau_2}\left(\frac{(A^*(\mu)-r)\Big(\mu(r^{b+a-1}-r^{2b})-\tau_2(r^{c-1}-r^b)\Big)}{A^*(\mu)(\tau_1-\mu)(1-r)} - r^b\Big(\frac{1}{1-r}+\frac{1}{A^*(\mu)}\Big)\right)\\
&\quad +L_6 + 2\Big(h_1(a-3)+h_5(a-3)\Big) - \frac{(1-r)r^{b+a-3}\Big(\mu(\phi+\bar{q}\eta)-q\eta\tau_2\Big)}{\tau_1\tau_2}.
\end{aligned}$$

Setting $\theta=\phi$ in equations (1) and (6), and after simplification, we get

$$L_{10}k_1 + L_{11}k = 0, \tag{36}$$

where

$$\begin{aligned}
L_{10} &= \frac{\mu}{\mu-\phi}\left(\frac{A^*(\phi)-A^*(\mu)}{A^*(\mu)} + \frac{\xi^{a-1}-\xi^b}{1-\xi}\Big(\frac{A^*(\phi)(A^*(\mu)-\xi)}{A^*(\mu)\xi^b} + \frac{\mu(A^*(\phi)-\xi)}{\mu-\phi-\mu\xi^b}\Big)\right)\\
&\qquad +h_2(0),\\
L_{11} &= \frac{q\eta(A^*(\phi)-r)(r^{c-1}-r^b)}{(\tau_1-\phi)(1-r)} - h_1(0) + \frac{\mu(\phi+\bar{q}\eta)}{\tau_2(\mu-\phi)}\Bigg[\frac{(A^*(\mu)-A^*(\phi))r^b}{A^*(\mu)}\\
&\quad +\frac{\Big(\mu(r^{b+a-1}-r^{2b})-\tau_2(r^{c-1}-r^b)\Big)}{1-r}\Big(\frac{A^*(\phi)(A^*(\mu)-r)}{A^*(\mu)(\tau_1-\mu)} - \frac{A^*(\phi)-r}{\tau_1-\phi}\Big)\Bigg].
\end{aligned}$$

Solving equations (35) and (36), we have

$$k_1 = \frac{-\lambda L_{11}}{L_9L_{10}-L_8L_{11}} \text{ and } k = \frac{\lambda L_{10}}{L_9L_{10}-L_8L_{11}}.$$

We are now in a position to obtain the pre-arrival epoch probabilities $P^-_{n,j}$ from the probabilities $P_{n,j}(0)$.

Theorem 3.1. *The pre-arrival epoch probabilities $P^-_{n,0}$ that an arrival sees n customers in the queue and the server is in vacation, $P^-_{n,1}$ that the server is busy, $P^-_{a-1,2}$ that the server is in change over time and $P^-_{n,3}$ that the server is in dormancy are given by*

$$
\begin{aligned}
P^-_{0,0} &= [kh_1(n) - k_1h_2(n)]/\lambda,\ 0 \le n \le a-3,\\
P^-_{a-2,0} &= [kL_3 - k_1L_4]/\lambda,\\
P^-_{n,0} &= kr^{c-1}h(n)/\lambda,\ a-1 \le n \le c-2,\\
P^-_{n,0} &= kr^n/\lambda,\ n \ge c-1,\\
P^-_{n,1} &= \left[k_1\xi^n - \frac{(\phi+\bar{q}\eta)kr^{n+b}}{\tau_2}\right]/\lambda,\ n \ge 0,\\
P^-_{a-1,2} &= [k_1L_1 - kL_2]/\lambda,\\
P^-_{0,3} &= \left[k\Big(L_6 + \frac{q\eta(r^{c-1}-r^b)}{\tau_1} - h_1(0)\Big) + k_1\Big(L_5 + h_2(0)\Big)\right]/\lambda,\\
P^-_{n,3} &= [kh_5(n) + k_1h_6(n)]/\lambda,\ 1 \le n \le a-3,\\
P^-_{a-2,3} &= \frac{k_1}{\lambda}\Big(h_6(a-3) + \frac{(1-\xi)\xi^{a-3}}{1-\xi^b} + L_4 - h_2(a-3)\Big)\\
&\quad + \frac{k}{\lambda}\Big(h_5(a-3) + h_1(a-3) - L_3 - \frac{(1-r)r^{b+a-3}\Big(\mu(\phi+\bar{q}\eta) - q\eta\tau_2\Big)}{\tau_1\tau_2}\Big),\\
P^-_{a-1,3} &= \Bigg[k_1L_7 + k\Big(h_5(a-3) + h_1(a-3) + L_2 - r^{c-1}h(a-1)\\
&\quad - \frac{(1-r^2)r^{b+a-3}\Big(\mu(\phi+\bar{q}\eta) - q\eta\tau_2\Big)}{\tau_1\tau_2}\Big)\Bigg]/\lambda,\\
P^-_{n,3} &= [k_1L_7 + kh_7(n)]/\lambda,\ a \le n \le c-2,\\
P^-_{c-1,3} &= \frac{k_1}{\lambda A^*(\mu)}\Big(1 - A^*(\mu)L_1 + \frac{(A^*(\mu)-\xi)(\xi^{a-1}-\xi^b)}{\xi^b(1-\xi)}\Big) + \frac{k}{\lambda A^*(\mu)}\Big(A^*(\mu)L_2\\
&\quad - \frac{(\phi+\bar{q}\eta)r^b}{\tau_2} + \frac{(A^*(\mu)-r)(\phi+\bar{q}\eta)\Big(\mu(r^{b+a-1}-r^{2b}) - \tau_2(r^{c-1}-r^b)\Big)}{\tau_2(\tau_1-\mu)(1-r)}\Big).
\end{aligned}
$$

Proof. Using (12) in (13), (15), (17), (19) to (22) and (30) - (34), we obtain the result of the theorem. □

3.2. **Steady state distribution at arbitrary epochs.** The queue length distribution at arbitrary epochs are summarized in the following theorem.

Theorem 3.2. *The arbitrary epoch probabilities are given by*

$$
\begin{aligned}
P_{n,0} &= r^{n-1}(1-r)k/\tau_1,\ n \ge c,\\
P_{n,1} &= \frac{(1-\xi)k_1\xi^{n-1}}{\mu(1-\xi^b)} - \frac{k(\phi+\bar{q}\eta)(1-r)r^{n+b-1}}{\tau_2\tau_1},\ n \ge 1,\\
P_{a-1,2} &= \frac{k_1}{\alpha}\left(\frac{\mu(1-\xi)\xi^{a-2}}{\mu(1-\xi^b)} - L_1\right) + \frac{k}{\alpha}\left(L_2 - \frac{\mu(1-r)(\phi+\bar{q}\eta)r^{a+b-2}}{\tau_2\tau_1}\right),
\end{aligned}
$$

$$
\begin{aligned}
P_{0,1} &= (k_1L_5 + kL_6)/\mu,\\
P_{0,0} &= \frac{k_1}{\phi}\Big(L_5 + h_2(0)\Big) + \frac{k}{\phi}\Big(L_6 + \frac{q\eta(r^{c-1} - r^b)}{\tau_1} - h_1(0)\Big),\\
P_{n,0} &= kh_3(n) + k_1h_4(n),\ 1 \le n \le a-3,\\
P_{a-2,0} &= \frac{k}{\phi}\Bigg(h_1(a-3) - L_3 - \frac{(1-r)r^{b+a-3}\Big(\mu(\phi+\bar{q}\eta) - q\eta\tau_2\Big)}{\tau_2\tau_1}\Bigg)\\
&\quad + \frac{k_1}{\phi}\Big(\frac{(1-\xi)\xi^{a-3}}{1-\xi^b} + L_4 - h_2(a-3)\Big),\\
P_{a-1,0} &= \frac{k}{\phi}\Bigg(L_3 + L_2 - \frac{(1-r)r^{a+b-2}\Big(\mu(\phi+\bar{q}\eta) - q\eta\tau_2\Big)}{\tau_2\tau_1} - r^{c-1}h(a-1)\Bigg)\\
&\quad + \frac{k_1}{\phi}\Big(\frac{(1-\xi)\xi^{a-2}}{1-\xi^b} - L_1 - L_4\Big)\Big),\\
P_{n,0} &= \frac{k}{\phi}\Bigg(r^{c-1}\Big(h(n-1) - h(n)\Big) + \frac{q\eta(1-r)r^{b+n-1}}{\tau_1}\Bigg),\ a \le n \le c-2,\\
P_{c-1,0} &= \frac{kr^{c-1}}{\phi}\Bigg(h(c-2) - 1 + \frac{q\eta(1-r)r^{b-1}}{\tau_1}\Bigg).
\end{aligned}
$$

Proof. Setting $\theta = 0$ in (14), (16), (18), (6) and (1) to (4) we obtain the above result. □

One may note that from Theorem 3.2 we can not get $\{P_{n,3}\}_0^{c-1}$. However, these can be obtained using the following theorem.

Theorem 3.3. *The arbitrary epoch probabilities* $\{P_{n,3}\}_0^{c-1}$ *are given by*

$$
\begin{aligned}
P_{0,3} &= -k_1\Bigg(\frac{1}{\mu}\Big[\frac{1-A^*(\mu)}{A^*(\mu)} + (\xi^{a-1} - \xi^b)\Big(\frac{(A^*(\mu)-\xi)(\lambda-\mu)}{\lambda A^*(\mu)\xi^b(1-\xi)} + \frac{1}{1-\xi^b}\Big)\Big] - \frac{1}{\lambda A^*(\mu)}\\
&\quad + \mu\sum_{j=a}^{b}\frac{\partial g_1}{\partial\theta}(j,0) + \frac{L_5 + h_2(0)}{\phi}\Bigg) - k\Big[\frac{q\eta(r^{c-1} - r^b) + \tau_1(L_6 - h_1(0))}{\tau_1\phi}\\
&\quad + (\phi+\bar{q}\eta)\Bigg(\frac{1}{\lambda\tau_2A^*(\mu)}\Big(r^b - \frac{(A^*(\mu)-r)[\mu(r^{b+a-1} - r^{2b}) - \tau_2(r^{c-1} - r^b)]}{(\tau_1-\mu)(1-r)}\Big)\\
&\quad + \sum_{j=c}^{b}\frac{\partial g}{\partial\theta}(j,0)\Bigg) + \frac{L_6}{\mu} + q\eta\sum_{j=c}^{b}\frac{\partial g}{\partial\theta}(j,0) + \mu\sum_{j=a}^{b}\frac{\partial g_2}{\partial\theta}(j,0)\Big],\\
P_{1,3} &= k_1\Big[\frac{L_5}{\lambda} - h_4(1) - \mu\frac{\partial g_1}{\partial\theta}(1,0)\Big] + k\Big[\frac{L_6}{\lambda} + \frac{q\eta(r^{c-1} - r^b)}{\lambda\tau_1} - h_3(1) - \mu\frac{\partial g_2}{\partial\theta}(1,0)\\
&\quad - q\eta\frac{\partial g}{\partial\theta}(b+1,0)\Big],\\
P_{n,3} &= k_1\Big[\frac{h_6(n-1)}{\lambda} - h_4(n) - \frac{h_2(n-1)}{\lambda} - \mu\frac{\partial g_1}{\partial\theta}(n,0)\Big]\\
&\quad + k\Big[\frac{h_5(n-1)}{\lambda} - h_3(n) + \frac{h_1(n-1)}{\lambda} - \mu\frac{\partial g_2}{\partial\theta}(n,0) - q\eta\frac{\partial g}{\partial\theta}(b+n,0)\Big],\ 2 \le n \le a-3,
\end{aligned}
$$

$$
\begin{aligned}
P_{a-2,3} &= k_1\Big[\frac{h_6(a-3)}{\lambda} - \frac{(1-\xi)\xi^{a-3}}{\phi(1-\xi^b)} - \frac{L_4}{\phi} + \frac{(\lambda-\phi)h_2(a-3)}{\lambda\phi} - \mu\frac{\partial g_1}{\partial\theta}(a-2,0)\Big] \\
&\quad + k\Big[\frac{h_5(a-3)}{\lambda} + \frac{(\phi-\lambda)h_1(a-3)}{\lambda\phi} + \frac{L_3}{\phi} + \frac{(1-r)r^{b+a-3}\Big(\mu(\phi+\bar{q}\eta)-q\eta\tau_2\Big)}{\tau_2\tau_1\phi} \\
&\quad -\mu\frac{\partial g_2}{\partial\theta}(a-2,0) - q\eta\frac{\partial g}{\partial\theta}(b+a-2,0)\Big], \\
P_{a-1,3} &= k_1\Big[\frac{h_6(a-3)-h_2(a-3)}{\lambda} + \frac{L_1+L_4}{\phi} + \frac{(1-\xi)(\phi-\xi\lambda)\xi^{a-3}}{\lambda\phi(1-\xi^b)} - \alpha g_3^{(1)}(0)\Big] \\
&\quad + k\Big[\frac{h_5(a-3)+h_1(a-3)}{\lambda} - \frac{(1-r)r^{b+a-3}\Big(\mu(\phi+\bar{q}\eta)-q\eta\tau_2\Big)(\phi-r\lambda)}{\lambda\phi\tau_1\tau_2} \\
&\quad + \frac{r^{c-1}h(a-1)}{\phi} - \alpha g_4^{(1)}(0) - q\eta\frac{\partial g}{\partial\theta}(b+a-1,0) - \frac{L_2+L_3}{\phi}\Big], \\
P_{a,3} &= \frac{k_1L_7}{\lambda} + k\Big[\frac{h_5(a-3)+h_1(a-3)+L_2}{\lambda} - \frac{r^{c-1}[h(a-1)-h(a)]}{\phi} \\
&\quad - q\eta\Big(\frac{(1-r)r^{b+a-1}}{\tau_1\phi} + \frac{\partial g}{\partial\theta}(b+a,0)\Big) - \frac{(1-r^2)r^{b+a-3}[\mu(\phi+\bar{q}\eta-q\eta\tau_2)]}{\tau_1\tau_2\lambda}\Big], \\
P_{n,3} &= \frac{k_1L_7}{\lambda} + k\Big[\frac{h_7(n-1)}{\lambda} - \frac{r^{c-1}[h(n-1)-h(n)]}{\phi} - \frac{q\eta(1-r)r^{b+n-1}}{\tau_1\phi} + \frac{r^{c-1}h(n-1)}{\lambda} \\
&\quad - q\eta\frac{\partial g}{\partial\theta}(b+n,0)\Big],\ a+1\le n\le c-2, \\
P_{c-1,3} &= \frac{k_1L_7}{\lambda} + k\Big[\frac{h_7(c-2)}{\lambda} - \frac{r^{c-1}[h(c-2)-1]}{\phi} - \frac{q\eta(1-r)r^{b+c-2}}{\tau_1\phi} \\
&\quad + \frac{r^{c-1}h(c-2)}{\lambda} - q\eta\frac{\partial g}{\partial\theta}(b+c-1,0)\Big].
\end{aligned}
$$

Proof. Using the derivatives of equations (1) to (4), (6) and (9) to (10) at $\theta = 0$ and Theorems 3.1 and 3.2, we get the results of the theorem. □

4. Performance measures and cost model

Once the queue length probabilities at pre-arrival and arbitrary epochs are known, we can evaluate the various performance measures. The average queue length when the server is in working vacation (L_{qv}), busy (L_{qb}), change over time (L_{qc}), dormancy (L_{qd}) and the average number of customers in the queue at an arbitrary epoch (L_q) are given by

$$
\begin{aligned}
L_{qv} &= \sum_{n=0}^{\infty} nP_{n,0},\ L_{qb} = \sum_{n=0}^{\infty} nP_{n,1},\ L_{qc} = (a-1)P_{a-1,2}, \\
L_{qd} &= \sum_{n=0}^{c-1} nP_{n,3},\ L_q = L_{qv}+L_{qb}+L_{qc}+L_{qd}.
\end{aligned}
$$

The average waiting time in the queue (W_q) of a customer using Little's rule is given by $W_q = L_q/\lambda$. The probability that the server is on working vacation (P_{wv}), busy (P_b), in change over time (P_c) and in dormancy (P_d) are respectively, given by

$$P_{wv} = \sum_{n=0}^{\infty} P^{-}_{n,0},\ P_b = \sum_{n=0}^{\infty} P^{-}_{n,1},\ P_c = P^{-}_{a-1,2},\ \text{and } P_d = \sum_{n=0}^{c-1} P^{-}_{n,3}.$$

Cost model

We develop the total expected cost function per unit time with an objective to determine the optimum values of μ and η so that the expected cost is minimum. Let us define
C_1= unit time cost of every customer in the queue when the server is on a working vacation,
C_2= unit cost of every customer in the queue when the server is in normal busy period,
C_3= fixed service cost per unit time during the normal busy period,
C_4= fixed service cost per unit time during a WV period,
C_5= fixed cost per unit time during the change over time.
Let F be the total expected cost per unit time. Using the definitions of each cost element and its corresponding system characteristics, we have

$$F = C_1 L_{qv} + C_2 L_{qb} + C_3\mu + C_4\eta + C_5\alpha.$$

We have considered the following optimization problems:

- Minimize $F(\eta)$ subject to the constraint $0.001 \leq \eta \leq 1.6$.
- Minimize $F(\mu)$ subject to the constraint $0.25 \leq \mu \leq 2.0$.

The numerical searching approach is implemented using QFSM and GA on the computer software Mathematica with μ and η as the decision variables. We used these two optimization techniques so as to ensure the reliability of the results.

5. Numerical results

To demonstrate the applicability of the theoretical investigation made in the previous sections, we present some numerical results in the form of tables and graphs.
We have considered the following cost parameters: $C_1 = 5,\ C_2 = 3,\ C_3 = 18,\ C_4 = 36$ and $C_5 = 8$. In addition to these cost parameters we have taken the parameters $a = 5,\ c = 8,\ b = 10,\ \lambda = 3.5,\ \mu = 0.7,\ \phi = 0.1,\ \eta = 0.3,\ \alpha = 2.0$ for all the tables and figures, unless they are considered as variables or their values are mentioned in the respective figures and tables. In all the cases where hyperexponential inter-arrival time distribution is considered, we have taken $\sigma_1 = 0.4,\ \sigma_2 = 0.6,\ \lambda_1 = 4.07$ and $\lambda_2 = 3.2$.

Using QFSM and GA, the optimal values of η (μ) and the minimum expected cost F^* are shown in Table 1 (2) for various values of ϕ (λ) and for hyperexponential (exponential) inter-arrival distribution with $\mu = 1.8$ ($\eta = 0.2$) for Table 1 (2). From Table 1 we observe that both the optimal mean service rate in working vacation and the minimum cost decrease as ϕ increases. From Table 2 as the mean arrival rate increases both the optimal mean service rate in regular busy period and the minimum cost increase. Note that this increase in the optimal service rate with λ is as expected in view of the stability of the system.

Figure 1 illustrates the influence of the traffic intensity ρ on the expected queue length for different values of VI probability, q, with $\eta = 0.7$ and exponential inter-arrival distribution. Here, L_q increases with ρ. From the figure one can also infer that L_q is lower for the model with vacation interruption when $\mu > \eta$ and the effect is reversed when $\mu < \eta$. Further, q has no effect on L_q when $\mu = \eta$. Therefore, the condition $\mu > \eta$ should be maintained in order to get better performance for the VI model.

Figure 2 provides the expected queue length with a change of service rate in working

TABLE 1. The optimal values η^* and F^* for various values of ϕ.

	$q=1$, SWV				$q=0$, VI			
	QFSM		GA		QFSM		GA	
ϕ	η^*	F^*	η^*	F^*	η^*	F^*	η^*	F^*
0.05	0.7984	113.04	0.7984	113.04	0.5655	103.41	0.5655	103.41
0.1	0.7122	108.30	0.7139	108.30	0.5081	99.96	0.5143	99.96
0.2	0.5322	99.29	0.5324	99.29	0.3912	93.18	0.3893	93.18
0.3	0.3367	90.66	0.3364	90.66	0.2726	86.58	0.2725	86.58
0.4	0.1137	82.14	0.1125	82.14	0.1527	80.16	0.1528	80.16

TABLE 2. The optimal values μ^* and F^* for various values of λ.

	$q=1$, SWV				$q=0$, VI			
	QFSM		GA		QFSM		GA	
λ	μ^*	F^*	μ^*	F^*	μ^*	F^*	μ^*	F^*
1.2	0.2775	52.47	0.2726	52.47	0.3022	51.05	0.3093	51.06
1.4	0.3042	57.03	0.3042	57.03	0.3347	54.44	0.3356	54.44
1.6	0.3304	61.81	0.3304	61.81	0.3666	57.73	0.3764	57.75
1.8	0.3563	66.85	0.3563	66.85	0.3980	60.96	0.3979	60.96
2.0	0.3819	72.14	0.3922	72.17	0.4289	64.15	0.4288	64.15

vacation η at different q for Erlang-3 inter-arrival time distribution. As expected, L_q decreases with the increase of η and the larger the probability q, the larger L_q becomes. That is the model with VI ($q = 0$) performs better than the model without VI ($q = 1$). Moreover, when $\eta = 0$ and $\eta = \mu$ then clearly q has no effect on the average queue length.

Figure 3 shows the expected queue length as a function of the maximum batch size b for different values of ϕ and q with hyper-exponential inter-arrival time distribution. The figure demonstrates that the expected queue length decreases as b and ϕ increase. The decrease of L_q with the increase of ϕ is due to the fact that mean vacation time decreases and the server is available with shorter breaks. As in Figure 2, here also the larger the probability q, the larger L_q becomes.

Figure 4 depicts the influence of the minimum threshold c on the expected queue length for different values of λ and q with deterministic inter-arrival distribution and $b = 18$. Here, L_q increases with c, λ and q. Since c is the minimum threshold to start service, increasing c will result in greater accumulation of customers in the queue thereby increasing L_q. From the figure observe that L_q is lower for the model with VI for fixed λ and c.

Figure 5 demonstrates the impact of the minimum batch size (a) on the expected queue length with deterministic inter-arrival distribution for $c = 15$ and $b = 18$. Here we considered four cases of the model: SWV without change over time (SWV, no CHOV), SWV with change over time (SWV, CHOV), SWV with VI and no change over time (VI, no CHOV) and SWV with VI and change over times (VI, CHOV). For all the cases considered L_q increases as a increases. The results in the figure highlight the model with change over time performs better and the model where both VI and change over time are introduced has best performance.

The impact of α on L_q for $q = 1$ and $q = 0$ is presented in Figures 6 and 7, respectively. In both the cases, L_q increases with α for all inter-arrival time distributions considered.

This is due to the fact that as α gets larger the mean duration of change over times becomes smaller so that the server will go for another vacation without waiting for an arrival for some reasonable duration of time. This contributes for L_q to increase. It is also observed that the system performs better for deterministic inter-arrival time distribution. Figure 8 shows the average cost as a function of μ and η for exponential inter-arrival time distribution. The figure demonstrates the existence of an optimal value (μ^*, η^*).

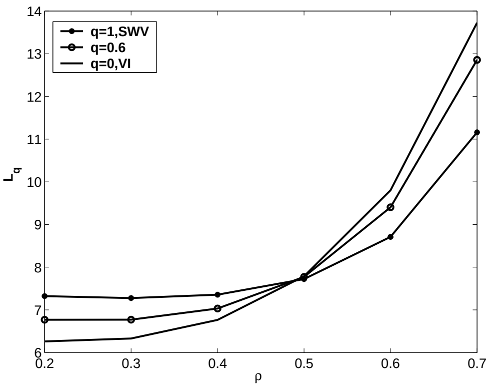

Figure 1. Effect of ρ on L_q.

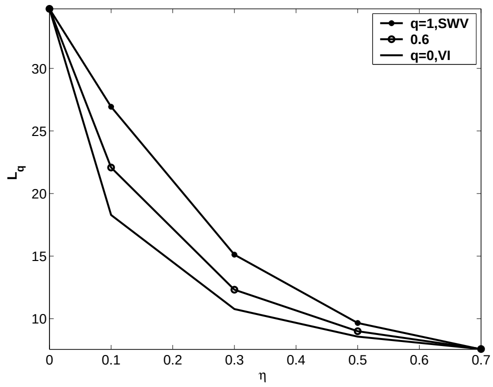

Figure 2. Impact of η on L_q.

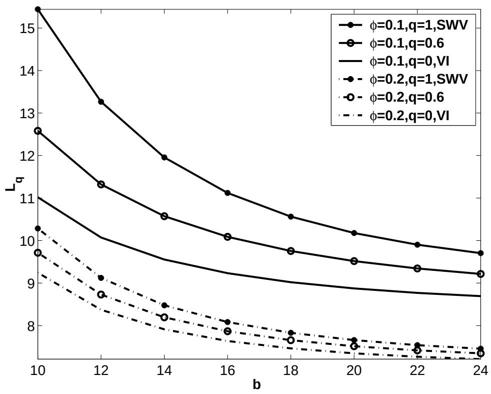

Figure 3. Expected queue length vs b.

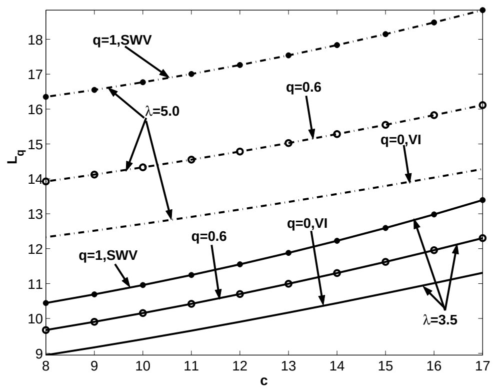

Figure 4. Impact of c on L_q.

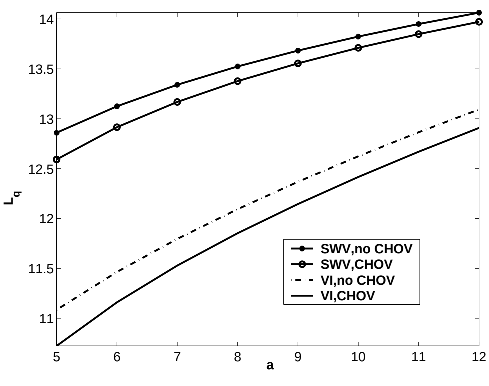

Figure 5. Effect of a on L_q.

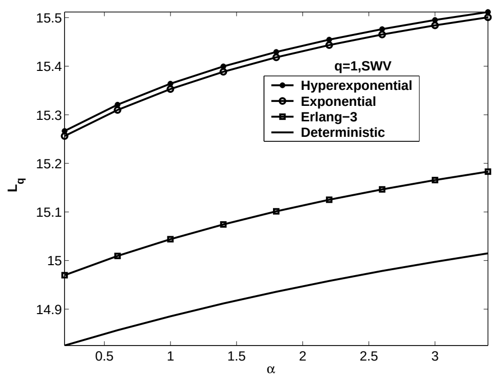

Figure 6. Impact of α on L_q.

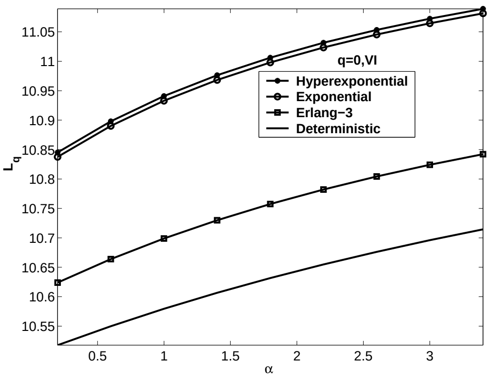

Figure 7. Effect of α on L_q.

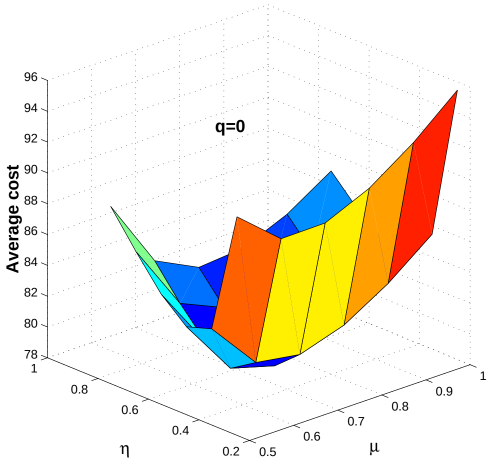

Figure 8. Average cost vs μ and η.

From the above numerical discussion we note the following:
In terms of L_q, the model with vacation interruption performs better. Therefore to offer a better service under the working vacation policy, one can consider vacation interruption policy that utilizes the server and decreases the queue size effectively. It also economizes the average cost.

6. Conclusion

In this paper, we have carried out an analysis of a renewal input infinite buffer batch service queue with SWV with change over time and BS-VI that has potential applications in production, manufacturing, traffic signals and telecommunication systems, etc. We have used a recursive method to obtain the steady state queue length distributions at pre-arrival and arbitrary epochs. Performance measures such as the average queue length and the average waiting time in the queue have been obtained along with a suitable cost function. The quadratic fit search method and genetic algorithm are applied to search for the optimal values of the system parameters. The method of analysis used in this paper can be applied to $GI^X/M^{(a,c,b)}/1$ and $MAP/M^{(a,c,b)}/1$ queues with change over times and BS-VI that are left for future investigation.

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