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EXTENSIONS OF KANNAN'S FIXED POINT RESULTS

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Abstract. In this article, we essentially proved some fixed point results for contractive maps type in quasi-pseudometric spaces. The presented theorems are formulated in an asymmetric setting and therefore generalize some existing results in analysis.

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1. Introduction

Banach contraction principle is certainly one of the most cited fixed point results in the literature and a lot of fixed point results are built around it. In 1968, Kannan [1] established a similar result which was followed by different variant of contractive type maps in metric spaces. We extend the result of Kannan, and henceforth many other results in the theory and give new theorems. We use a quasi-pseudometric structure and show that some proofs follow closely the classical proofs in the metric case, but generalize them.

2. Prelimaries

In this section, we recall some elementary definitions from the asymmetric topology which are necessary for a good understanding of the work below.

Definition 2.1. *Let X be a non empty set. A function $d : X \times X \to [0, \infty)$ is called a* ***quasi-pseudometric*** *on X if:*

i) $d(x, x) = 0 \quad \forall\, x \in X$,
ii) $d(x, z) \leq d(x, y) + d(y, z) \quad \forall\, x, y, z \in X$.

*Moreover, if $d(x, y) = 0 = d(y, x) \Longrightarrow x = y$, then d is said to be a T_0-**quasi-pseudometric** or a **di-metric**. The latter condition is referred to as the T_0-condition.*

Remark 2.1.

- *Let d be a quasi-pseudometric on X, then the map d^{-1} defined by $d^{-1}(x, y) = d(y, x)$ whenever $x, y \in X$ is also a quasi-pseudometric on X, called the **conjugate** of d. In the literature, d^{-1} is also denoted d^t or $\bar{d}$.*
- *It is easy to verify that the function d^s defined by $d^s := d \vee d^{-1}$, i.e. $d^s(x, y) = \max\{d(x, y), d(y, x)\}$ defines a **metric** on X whenever d is a T_0-quasi-pseudometric.*

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Example 2.1. *On $\mathbb{R}\times\mathbb{R}$, we define the real valued map d given by*

$$d(a,b) = a\dot{-}b = \max\{a-b,0\}.$$

Then $(\mathbb{R},d)$ is a di-metric space.

Let (X,d) be a quasi-pseudometric space. Then for each $x\in X$ and $\epsilon>0$, the set

$$B_d(x,\epsilon)=\{y\in X : d(x,y)<\epsilon\}$$

denotes the open ϵ-ball at x with respect to d. It should be noted that the collection

$$\{B_d(x,\epsilon) : x\in X, \epsilon>0\}$$

yields a base for the topology $\tau(d)$ induced by d on X. In a similar manner, for each $x\in X$ and $\epsilon\geq 0$, we define

$$C_d(x,\epsilon)=\{y\in X : d(x,y)\leq\epsilon\},$$

known as the closed ϵ-ball at x with respect to d.
Also the collection

$$\{B_{d^{-1}}(x,\epsilon) : x\in X, \epsilon>0\}$$

yields a base for the topology $\tau(d^{-1})$ induced by d^{-1} on X. The set $C_d(x,\epsilon)$ is $\tau(d^{-1})$-closed, but not $\tau(d)$-closed in general.
The balls with respect to d are often called *forward balls* and the topology $\tau(d)$ is called *forward topology*, while the balls with respect to d^{-1} are often called *backward balls* and the topology $\tau(d^{-1})$ is called *backward topology*.

Definition 2.2. *Let (X,d) be a quasi-pseudometric space. The convergence of a sequence (x_n) to x with respect to $\tau(d)$, called d-**convergence** or **left-convergence** and denoted by $x_n \overset{d}{\longrightarrow} x$, is defined in the following way*

$$x_n \overset{d}{\longrightarrow} x \Longleftrightarrow d(x,x_n)\longrightarrow 0. \tag{1}$$

*Similarly, the convergence of a sequence (x_n) to x with respect to $\tau(d^{-1})$, called d^{-1}-**convergence** or **right-convergence** and denoted by $x_n \overset{d^{-1}}{\longrightarrow} x$,*
is defined in the following way

$$x_n \overset{d^{-1}}{\longrightarrow} x \Longleftrightarrow d(x_n,x)\longrightarrow 0. \tag{2}$$

*Finally, in a quasi-pseudometric space (X,d), we shall say that a sequence (x_n) d^s-**converges** to x if it is both left and right convergent to x, and we denote it as $x_n \overset{d^s}{\longrightarrow} x$ or $x_n\longrightarrow x$ when there is no confusion. Hence*

$$x_n \overset{d^s}{\longrightarrow} x \iff x_n \overset{d}{\longrightarrow} x \text{ and } x_n \overset{d^{-1}}{\longrightarrow} x.$$

Definition 2.3. *A sequence (x_n) in a quasi-pseudometric (X,d) is called*

(a) ***left K-Cauchy*** *with respect to d if for every $\epsilon>0$, there exists $n_0\in\mathbb{N}$ such that*

$$\forall\ n,k : n_0\leq k\leq n \quad d(x_k,x_n)<\epsilon;$$

(b) ***right K-Cauchy*** *with respect to d if for every $\epsilon>0$, there exists $n_0\in\mathbb{N}$ such that*

$$\forall\ n,k : n_0\leq k\leq n \quad d(x_n,x_k)<\epsilon;$$

(c) *d^s-**Cauchy** if for every $\epsilon>0$, there exists $n_0\in\mathbb{N}$ such that*

$$\forall n,k\geq n_0 \quad d(x_n,x_k)<\epsilon.$$

Remark 2.2.

- *A sequence is left K-Cauchy with respect to d if and only if it is right K-Cauchy with respect to d^{-1}.*
- *A sequence is d^s-Cauchy if and only if it is both left and right K-Cauchy.*

Definition 2.4. *A quasi-pseudometric space (X, d) is called **left-complete** provided that any left K-Cauchy sequence is d-convergent.*

Definition 2.5. *A quasi-pseudometric space (X, d) is called **right-complete** provided that any right K-Cauchy sequence is d-convergent.*

Definition 2.6. *A T_0-quasi-pseudometric space (X, d) is called **bicomplete** provided that the metric d^s on X is complete.*

Example 2.2. *Let $X = [0; \infty)$. Define for each $x, y \in X$, $n(x, y) = x$ if $x > y$, and $n(x, y) = 0$ if $x \leq y$. It is not difficult to check that (X, n) is a T_0-quasi-pseudometric space. Notice also that for $x, y \in [0; \infty)$, we have $n^s(x, y) = \max\{x, y\}$ if $x \neq y$ and $n^s(x, y) = 0$ if $x = y$. The metric n^s is complete on $[0, \infty)$.*

Definition 2.7. *Let (X, d) be a quasi-pseudometric type space. A function $f : X \to X$ is called d-**sequentially continuous** or **left-sequentially continuous** if for any d-convergent sequence (x_n) with $x_n \xrightarrow{d} x$, the sequence (fx_n) d-converges to fx, i.e. $(fx_n) \xrightarrow{d} fx$.*

Definition 2.8. *Let (X, d) be a quasi-pseudometric space. A mapping $T : X \to X$ is said to be* **left-sequentially convergent** *if we have, for every sequence $(a_n)_n$, if $(Ta_n)_n$ is d-convergent , then $(a_n)_n$ is also d-convergent.*

We conclude this section by recalling Kannan's results.

Theorem 2.1. *(Compare [1]) Let (X, m) be a complete metric space and $S : X \to X$ be a self mapping on X such that*

$$m(Sx, Sy) \leq \lambda[m(x, Sx) + m(Sy, y)] \qquad (x, y \in X) \tag{3}$$

where $\lambda \in [0, 1/2)$. Then S has a unique fixed point.

3. Main Results

First, we extend Kannan's theorem in the following way.

Theorem 3.1. *(Compare [1]) Let (X, d) be a bicomplete T_0-quasi-pseudometric space and $T : (X, d) \to (X, d)$ be a self mapping on X such that*

$$d(Tx, Ty) \leq \lambda[d(x, Tx) + d(Ty, y)] \qquad (x, y \in X) \tag{4}$$

where $\lambda \in [0, 1/2)$. Then T has a unique fixed point.

Proof. It is enough to show that $(T, d^s) \to (T, d^s)$ satisfies the condition (3). Indeed, using the inequality (4), we have

$$\begin{aligned} d^{-1}(Tx, Ty) = d(Ty, Tx) &\leq \lambda[d(y, Ty) + d(Tx, x)] \qquad (x, y \in X) \\ &\leq \lambda[d^{-1}(x, Tx) + d^{-1}(Ty, y)] \qquad (x, y \in X), \end{aligned}$$

i.e. the map $T : (X, d^{-1}) \to (X, d^{-1})$ also satisfies the inequality (4). Hence, we respectively have

$$\begin{aligned} d(Tx, Ty) &\le \lambda[d(x, Tx) + d(Ty, y)] \qquad (x, y \in X) \\ &\le \lambda[d^s(x, Tx) + d^s(Ty, y)] \qquad (x, y \in X), \end{aligned}$$

and

$$\begin{aligned} d^{-1}(Tx, Ty) = d(Ty, Tx) &\le \lambda[d^{-1}(x, Tx) + d^{-1}(Ty, y)] \qquad (x, y \in X), \\ &\le \lambda[d^s(x, Tx) + d^s(Ty, y)] \qquad (x, y \in X), \end{aligned}$$

which entail that

$$d^s(Tx, Ty) \le \lambda[d^s(x, Tx) + d^s(Ty, y)] \qquad (x, y \in X),$$

i.e. the map $(T, d^s) \to (T, d^s)$ satisfies the condition (3). By assumption, (X, d) is bicomplete, hence (X, d^s) is complete. Therefore, by Theorem (2.1), T has a unique fixed point. This completes the proof.

Theorem 3.2. *Let (X, d) be a Hausdorff left-complete T_0-quasi-pseudometric space and $T, S : X \to X$ be self mappings such that S is a d-sequentially continuous function and T left-sequentially convergent. If there exists one $\lambda \in [0, 1/2)$ and*

$$d(TSx, TSy) \le \lambda[d(Tx, TSx) + d(Ty, TSy)] \qquad (x, y \in X) \tag{5}$$

then $Fix(S) \neq \emptyset$ where $Fix(S)$ is the set of fixed points of S.

Proof. Let x_0 be an arbitrary point in X. We define the iterative sequence $(x_n)_n$ by $x_{n+1} = Sx_n$, $n = 1, 2, \cdots$. Using (5), we have

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &= d(TSx_{n-1}, TSx_n) \\ &\le \lambda[d(Tx_{n-1}, TSx_{n-1}) + d(Tx_n, TSx_n)], \end{aligned}$$

which entails that

$$d(Tx_n, Tx_{n+1}) \le \frac{\lambda}{1-\lambda} d(Tx_{n-1}, Tx_n).$$

By induction, we can then write

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\le \frac{\lambda}{1-\lambda} d(Tx_{n-1}, Tx_n) \le \left(\frac{\lambda}{1-\lambda}\right)^2 d(Tx_{n-2}, Tx_{n-1}) \\ &\le \cdots \le \left(\frac{\lambda}{1-\lambda}\right)^n d(Tx_0, Tx_1). \end{aligned}$$

Hence, for every $m, n \in \mathbb{N}$ such that $m > n$ we have,

$$\begin{aligned} d(Tx_n, Tx_m) &\leq d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2}) + \cdots d(Tx_{m-1}, Tx_m) \\ &\leq \left[\left(\frac{\lambda}{1-\lambda}\right)^n + \left(\frac{\lambda}{1-\lambda}\right)^{n+1} + \cdots \left(\frac{\lambda}{1-\lambda}\right)^{m-1} \right] d(Tx_0, Tx_1) \\ &\leq \left(\frac{\lambda}{1-\lambda}\right)^n \frac{1}{1-\frac{\lambda}{1-\lambda}} d(Tx_0, Tx_1). \end{aligned}$$

Letting $n \to \infty$, we obtain that (Tx_n) is a left K-Cauchy and since (X, d) is left-complete, there exists $q \in X$ such that $Tx_n \xrightarrow{d} q$. Since T is left-sequentially convergent, there exists $u \in X$ such that $x_n \xrightarrow{d} u$. S is d-sequentially continuous, so $x_{n+1} \xrightarrow{d} Su$. Since X Hausdorff $u = Su$.

Corollary 3.1. *If moreover, T is injective in the above theorem 3.1, then $card(Fix(S)) = 1$, the cardinality of $Fix(S)$ is 1.*

Proof. For uniqueness, assume by contradiction that there exists another fixed point v. Then

$$d(TSu, TSv) = d(Tu, Tv) \leq \lambda[d(Tu, TSu) + d(Tv, TSv)] = 0,$$

and

$$d(TSv, TSu) = d(Tv, Tu) \leq \lambda[d(Tv, TSv) + d(Tu, TSu)] = 0,$$

which entails that $d(Tu, Tv) = 0 = d(Tv, Tu)$. Using the T_0-condition and injectivity of T, we conclude that $u = v$.

In the following, we have another extension of Kannan's theorem. Let Φ be the class of continuous functions $F : [0, \infty) \to [0, \infty)$ such that $F^{-1}(0) = \{0\}$.

Theorem 3.3. *Let S be a d-sequentially continuous self-mapping on a Hausdorff left-complete T_0-quasi-pseudometric space (X, d) satisfying*

$$F(d(Sx, Sy)) \leq \lambda[F(d(x, Sx)) + F(d(y, Sy))] \qquad (x, y \in X) \tag{6}$$

for some $\lambda \in [0, 1)$ and for some $F \in \Phi$, then S has a unique fixed point. Moreover, for any arbitrary $x_0 \in X$ the orbit $\{S^n x_0, n \geq 1\}$ D-converges to the fixed point.

Proof.
Since $F^{-1}(0) = \{0\}$, $F(\eta) > 0$ for any $\eta > 0$. Let x_0 be an arbitrary point in X. We define the iterative sequence $(x_n)_n$ by $x_{n+1} = Sx_n$, $n = 1, 2, \cdots$. Using (6), we have

$$\begin{aligned} F(d(x_n, x_{n+1})) &= F(d(Sx_{n-1}, Sx_n)) \\ &\leq \lambda[F(d(x_{n-1}, Sx_{n-1}) + F(d(x_n, Sx_n))], \end{aligned}$$

which entails that

$$F(d(x_n, x_{n+1})) \leq \frac{\lambda}{1-\lambda} F(d(x_{n-1}, x_n)).$$

By induction, we can then write

$$F(d(x_n, x_{n+1})) \leq \frac{\lambda}{1-\lambda} F(d(x_{n-1}, x_n)) \leq \left(\frac{\lambda}{1-\lambda}\right)^2 F(d(x_{n-2}, x_{n-1}))$$
$$\leq \cdots \leq \left(\frac{\lambda}{1-\lambda}\right)^n F(d(x_0, x_1)).$$

Hence, for every $m, n \in \mathbb{N}$ such that $m > n$ we have,

$$\begin{aligned} F(d(x_n, x_m)) &= F(d(Sx_{n-1}, Sx_{m-1})) \\ &\leq \lambda[F(d(x_{n-1}, x_n)) + F(d(x_{m-1}, x_m))] \\ &\leq \lambda\left[\left(\frac{\lambda}{1-\lambda}\right)^{n-1} + \left(\frac{\lambda}{1-\lambda}\right)^{m-1}\right] F(d(x_0, x_1)). \end{aligned}$$

Letting $n \to \infty$, we obtain that $F(d(x_n, x_m)) \to 0$. Since $F \in \Phi$, we conclude that (x_n) is a left K-Cauchy and since (X, d) is left-complete, there exists $u \in X$ such that $x_n \overset{d}{\longrightarrow} u$. S is d-sequentially continuous, so $x_{n+1} \overset{d}{\longrightarrow} Su$. Since X Hausdorff $u = Su$.
For uniqueness, assume by contradiction that there exists another fixed point v. Then

$$F(d(Su, Sv)) = F(d(u, v)) \leq \lambda[F(d(u, Su)) + F(d(v, Sv))] = 0,$$

and

$$F(d(Sv, Su)) = F(d(v, u)) \leq \lambda[F(d(v, Sv)) + F(d(u, Su))] = 0,$$

which entails that $d(u, v) = 0 = d(v, u)$. Using the T_0-condition, we conclude that $u = v$.

References

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Yaé Ulrich Gaba for the photography and short autobiography, see TWMS J. App. Eng. Math., V.4, N.2.