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GENERAL RELATED JENSEN TYPE INEQUALITIES FOR FUZZY INTEGRALS

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Abstract. In this paper, related inequalities to Jensen type inequality for the seminormed fuzzy integrals are studied. Several examples are given to illustrate the validity of theorems. Some results on Jensen type inequalities are obtained.

Keywords: Fuzzy measure, Jensen's inequality, Seminormed fuzzy integral, Semiconormed fuzzy integral.

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1. Introduction

The properties and applications of the Sugeno integral have been studied by many authors, including Ralescu and Adams [11], Román-Flores et al. [12, 13] and Wang and Klir [17].

In connection with the topics that will be discussed in this article, there are several other theoretical and applied papers related to fuzzy measure theory on metric spaces, such as Li et al. [8, 9] and Narukawa et al. [10]. Many authors generalized the Sugeno integral by using some other operators to replace the special operator(s) $\wedge$ and/or $\vee$ (see, e.g., [1, 2, 3, 4, 6, 8, 18]). Suárez and Gil Álvarez presented two families of fuzzy integrals, the so-called seminormed fuzzy integrals and semiconormed fuzzy integrals [15].

This paper is organized as follows: in Section 2, some preliminaries and summarization of some previous known results are briefly recalled. Section 3 deals with a general related Jensen type inequality for the fuzzy integrals.

2. Preliminaries

In this section we recall some basic definitions and previous results that will be used in the next section.

Let X be a non-empty set, Σ be a σ-algebra of subsets of X. Throughout this paper, all considered subsets are supposed to belong to Σ.

Definition 2.1 (Sugeno [16]). *A set function $\mu : \Sigma \to [0, 1]$ is called a fuzzy measure if the following properties are satisfied:*

(FM1) $\mu(\emptyset) = 0$ *and* $\mu(X) = 1$;
(FM2) $A \subset B$ *implies* $\mu(A) \leq \mu(B)$;
(FM3) $A_n \to A$ *implies* $\mu(A_n) \to \mu(A)$.

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When μ is a fuzzy measure, the triple (X, Σ, μ) is called a fuzzy measure space.

Let (X, Σ, μ) be a fuzzy measure space, and $\mathcal{F}_+(X) = \{f | f : X \to [0,1]$ is measurable with respect to $\Sigma\}$. In what follows, all considered functions belong to $\mathcal{F}_+(X)$. For any $\alpha \in [0,1]$, we will denote the set $\{x \in X | f(x) \geq \alpha\}$ by F_α and $\{x \in X | f(x) > \alpha\}$ by $F_{\bar{\alpha}}$. Clearly, both F_α and $F_{\bar{\alpha}}$ are non-increasing with respect to α, i.e., $\alpha \leq \beta$ implies $F_\alpha \supseteqq F_\beta$ and $F_{\bar{\alpha}} \supseteqq F_{\bar{\beta}}$.

Definition 2.2 (Sugeno [16])**.** *Let (X, Σ, μ) be a fuzzy measure space, and $A \in \Sigma$, the Sugeno integral of f over A, with respect to the fuzzy measure μ, is defined by*

$$\fint_A f d\mu = \bigvee_{\alpha \in [0,1]} (\alpha \wedge \mu(A \cap F_\alpha)).$$

When A=X, then

$$\fint_X f d\mu = \fint f d\mu = \bigvee_{\alpha \in [0,1]} (\alpha \wedge \mu(F_\alpha)).$$

Note in the above definition, $\wedge$ is just the prototypical t-norm minimum and $\vee$ the prototypical t-conorm maximum.

Definition 2.3 ([7])**.** *A t-norm is a function T : [0,1]$\times$[0,1] $\to$ [0,1] satisfying the following conditions:*

(A) $T(x, 1) = T(1, x) = x \quad \forall x \in [0,1]$.
(B) $\forall x_1, x_2, y_1, y_2$ *in* $[0,1]$, *if* $x_1 \leq x_2, y_1 \leq y_2$ *then* $T(x_1, y_1) \leq T(x_2, y_2)$.
(C) $T(x, y) = T(y, x)$.
(D) $T(T(x, y), z) = T(x, T(y, z))$.

Definition 2.4 ([7])**.** *A function $S : [0,1] \times [0,1] \to [0,1]$ is called a t-conorm, if there is a t-norm T such that $S(x, y) = 1 - T(1 - x, 1 - y)$. Evidently, a t-conorm S satisfies: (A') $S(x, 0) = S(0, x) = x$, $\forall x \in [0,1]$ as well as conditions (B), (C) and (D).*

A binary operator T (S) on [0,1] is called a t-seminorm (t-semiconorm), if it satisfies the above conditions (A) and (B) ((A') and (B))[15]. Notice that in the literature, a t-seminorm is also called a semicopula. By using the concepts of t-seminorm and t-semiconorm, Suárez and Gil (1986) proposed two families of fuzzy integrals:

Definition 2.5. *Let T be a t-seminorm, then the seminormed fuzzy integral of f over A with respect to T and the fuzzy measure μ is defined by*

$$\int_{T,\ A} f d\mu = \bigvee_{\alpha \in [0,1]} T(\alpha, \mu(A \cap F_\alpha)). \tag{1}$$

Definition 2.6. *Let S be a t-semiconorm, then the semiconormed fuzzy integral of f over A with respect to S and the fuzzy measure μ is defined by*

$$\int_{S,\ A} f d\mu = \bigwedge_{\alpha \in [0,1]} S(\alpha, \mu(A \cap F_{\bar{\alpha}})).$$

Remark 2.1. *Maybe one can define the so-called conormed-seminormed fuzzy integral $\int_{TS,\ A} f d\mu$ by using an arbitrary t-conorm S to replace the special t- conorm "$\bigvee$" in* (1)*. However, as Suárez and Gil Álvarez* [15] *pointed out that the conormed-seminormed fuzzy integral satisfies the essential property*

$$\int_{TS,\ X} a d\mu = a, \quad \forall a \in [0,1]$$

if and only if $S = \bigvee$. The case of semiconormed fuzzy integral is similar. It is easy to see that the Sugeno integral is a special seminormed fuzzy integral. Moreover, Kandel and Byatt in (see [5]*) showed another expression of the Sugeno integral as follows:*

$$\fint_A f d\mu = \bigwedge_{\alpha\in[0,1]} (\alpha \vee \mu(A \cap F_{\bar{\alpha}})).$$

So the semiconormed fuzzy integrals also generalize the concept of the Sugeno integral. Notice that the seminormed fuzzy integral is just the family of the weakest universal fuzzy integrals. Note that if $\int_{T,A} f d\mu = a$, then $T(\alpha, \mu(A \cap F_\alpha)) \leq a$ for all $\alpha \in [0,1]$ and, for any $\varepsilon > o$ there exists α_ε such that $T(\alpha_\varepsilon, \mu(A \cap F_{\alpha_\varepsilon})) \geq a - \varepsilon$. Also, if $\int_{S,A} f d\mu = a$, then $S(\alpha, \mu(A \cap F_{\bar{\alpha}})) \geq a$ for all $\alpha \in [0,1]$ and, for any $\varepsilon > o$ there exists α_ε such that $S(\alpha_\varepsilon, \mu(A \cap F_{\bar{\alpha}_\varepsilon})) \leq a + \varepsilon$. The well-known Jensen inequality is a part of the classical mathematical analysis, (see [14]*).*

Theorem 2.1. *Let $h : [0,1] \to (a,b)$ be real and integrable function and Φ a convex function on (a,b). Then*

$$\Phi\left(\int_0^1 h(x)dx\right) \leq \int_0^1 \Phi(h(x))dx. \tag{2}$$

In [3], Daraby et al. have proved general version of the Jensen type inequality as follows:

Theorem 2.2. *Let (X, Σ, μ) be a fuzzy measure space and $f : X \to [0,1]$ be a measurable function. Let T be a seminorm such that $\int_{T,A} f d\mu = a$ for any $A \in \Sigma$. Let $\Phi : [0,1] \to [0,1]$ be a continuous and strictly increasing function such that $\Phi(x) \leq x$, for every $x \in [0,a]$, then:*

$$\Phi\left(\int_{T,A} f d\mu\right) \leq \int_{T,A} \Phi(f)d\mu. \tag{3}$$

3. Main results

3.1. **Some results on Jensen type seminormed fuzzy integrals.** In this section, we will show some results on general Jensen type inequality. At the first, we show that conditions "Φ strictly increasing" and "$\Phi(x) \leq x$, for every $x \in [0, \int_{T,A} f d\mu]$" on the function Φ in Theorem 2.2 cannot be avoided, as we will illustrate in the following examples.

Example 3.1. *We observe that "Φ strictly increasing" condition of function Φ in Theorem 2.2 cannot be avoided. Let μ be the Lebesgue measure on $\mathbb{R}$ and T be defined as*

$$T_D(x,y) = \begin{cases} 0 & \textit{if } \max\{x,y\} < 1, \\ \min\{x,y\} & \textit{if } \max\{x,y\} = 1. \end{cases}$$

and consider $f(x) = x\chi_{[0,1]}(x)$. If we define:

$$\Phi(x) = \begin{cases} x & if \quad 0 \leq x < \frac{1}{2} \\ (1-x)\chi_{[0,1]}(x) & if \quad x \geq \frac{1}{2}. \end{cases}$$

Thus:

If $\max\{x,y\} < 1$, then $T_D(x,y) = 0$. So we have $\int_{T,A} f dm = 0$.
Consequently,

$$\Phi\left(\int_{T,A} f dm\right) = 0.$$

Hence

$$\Phi\left(\int_{T,A} f dm\right) = 0 \leq \int_{T,A} \Phi(f)dm.$$

If $\max\{x,y\}=1$*, then*

$$\begin{aligned}\int_{T,A} f dm &= ⨍ f dm \\ &= \bigvee_{\alpha\in[0,1]} (\alpha \wedge m(F_\alpha)) \\ &= \bigvee_{\alpha\in[0,1]} (\alpha \wedge (1-\alpha)) = \frac{1}{2}.\end{aligned}$$

Consequently,

$$\Phi\left(\int_{T,A} f dm\right) = \Phi\left(\frac{1}{2}\right) = \frac{1}{2}. \tag{4}$$

In addition, because $\Phi(f)(x)=\Phi(x)\chi_{[0,1]}(x)$*, then:*

$$m\left(\{\Phi(f) \geqslant \frac{1}{2}\}\right) = m\left(\{\frac{1}{2}\}\right) = 0. \tag{5}$$

But, from (4) *and* (5) *we have:*

$$\int_{T,A} \Phi(f) dm \geq \frac{1}{2} \quad \Rightarrow \quad m\left(\{\Phi(f) \geq \frac{1}{2}\}\right) \geq \frac{1}{2}$$

in contradiction with (5)*, which implies that*

$$\int_{T,A} \Phi(f) dm < \frac{1}{2}.$$

Consequently, inequality (3) *is not verified.*

Example 3.2. *The condition "*$\Phi(x) \leq x$ *for all* $0 \leq x \leq \int_{T,A} f d\mu$*" in Theorem (2.9) cannot be avoided. In fact, consider* $f(x) = x\chi_{[0,\frac{1}{2}]}(x)$ *and define* $\Phi(x)=\sqrt{x}$*. Also* T *be defined as*

$$T_D(x,y) = \begin{cases} 0 & if \quad \max\{x,y\} < 1, \\ min\{x,y\} & if \quad \max\{x,y\} = 1. \end{cases}$$

Thus: If $\max\{x,y\}<1$*, then* $T_D(x,y)=0$*. So we have* $\int_{T,A} f dm = 0$*. Consequently,*

$$\Phi\left(\int_{T,A} f dm\right) = 0.$$

Hence

$$\Phi\left(\int_{T,A} f dm\right) = 0 \leq \int_{T,A} \Phi(f) dm.$$

If $\max\{x,y\}=1$*, then*

$$\begin{aligned}\int_{T,A} f d\mu &= ⨍ f d\mu \\ &= \bigvee_{\alpha\in[0,\frac{1}{2}]} [\alpha \wedge \mu(F_\alpha)) \\ &= \bigvee_{\alpha\in[0,\frac{1}{2}]} [\alpha \wedge (\frac{1}{2}-\alpha)] = \frac{1}{4}.\end{aligned}$$

Consequently,

$$\Phi\left(\int_{T,A} f d\mu\right) = \Phi\left(\frac{1}{4}\right) = \frac{1}{2}.$$

Furthermore, because $\Phi(f)(x) = \sqrt{x}\chi_{[0,\frac{1}{2}]}(x)$*, then:*

$$\begin{aligned}
\int_{T,A} \Phi(f) d\mu &= \fint \Phi(f) d\mu \\
&= \bigvee_{\alpha\in[0,\sqrt{\frac{1}{2}}]} [\alpha \wedge \mu(F_\alpha)] \\
&= \bigvee_{\alpha\in[0,\sqrt{\frac{1}{2}}]} [\alpha \wedge (\frac{1}{2} - \alpha^2)] \\
&= \frac{-1+\sqrt{3}}{2} < \frac{1}{2} \\
&= \Phi\left(\int_{T,A} f d\mu\right).
\end{aligned}$$

Which implies that inequality (3) *is not verified.*

Theorem 3.1. *Let* (X, Σ) *be a fuzzy measurable space and* $f : X \to [0, 1]$ *be a measurable function. Let* T *be a seminorm such that* $F(x) = \int_{T,[0,x]} f(t)dt$*. Let* $\Phi : [0, 1] \to [0, 1]$ *be a continuous and strictly increasing function such that* $\Phi(x) \le x$*, for every* $x \in [0, F(x)]$*, then:*

$$\bigvee_{\alpha\in[0,1]} T\left(\alpha, \left(\frac{\int_{T,A} \Phi(f)dx}{\alpha}\right)\right) \ge \int_{T,A} \frac{\Phi(F(x))}{x} dx \tag{6}$$

holds for any $A \in \Sigma$.

Proof. By Theorem 2.2 we have

$$K = \int_{T,A} \Phi(f)dx \ge \Phi\left(\int_{T,A} f d\mu\right).$$

Since $[0, x] \subseteq A$, then we have

$$\Phi\left(\int_{T,A} f d\mu\right) \ge \Phi\left(\int_{T,[0,x]} f d\mu\right)$$

then

$$K = \int_{T,A} \Phi(f)dx \ge \Phi\left(\int_{T,[0,x]} f d\mu\right) = \Phi(F(x))$$

thus

$$\bigvee_{\alpha\in[0,1]} T\left(\alpha, \left(\frac{\int_{T,A} \Phi(f)dx}{\alpha}\right)\right) = \int_{T,A} \frac{K}{x} dx \ge \int_{T,A} \frac{\Phi((F(x))}{x} dx.$$

This completes the proof. $\square$

3.2. **A related inequality.** In this section, we prove a related inequality for semiconormed fuzzy integrals. The main result of this section is as follows.

Theorem 3.2. *Let* (X, Σ, μ) *be a fuzzy measure space and* $f : X \to [0,1]$ *measurable function. Let* S *be a semiconorm such that* $\int_{S,A} f d\mu = a$ *for any* $A \in \Sigma$*. Let* $\Phi : [0,1] \to [0,1]$ *be a continuous and strictly increasing function such that* $\Phi(x) \leq x$*, for every* $x \in [0,a]$*, then:*

$$\Phi\left(\int_{S,\ A} f d\mu\right) \geq \int_{S,\ A} \Phi(f) d\mu. \tag{7}$$

Proof. Let $\int_{S,A} f d\mu = a$. Then, for any $\varepsilon > 0$, there exist a_ε such that $\mu(A \cap F_{\bar{a}_\varepsilon}) = a_0$, where $S(a_\varepsilon, a_0) \leq a + \varepsilon$, (Thus $a_\varepsilon \leq a + \varepsilon$).

Denote $\varphi_{\bar{\alpha}} = \{x | \Phi(f(x)) > \alpha\}$. Since $\Phi : [0,1] \to [0,1]$ is a continuous and strictly increasing function, there holds

$$\varphi_{\Phi(\bar{a}_\varepsilon)} \subset \{x | f(x) > \bar{a}_\varepsilon\}.$$

Then

$$\mu\left(A \cap \varphi_{\Phi(\bar{a}_\varepsilon)}\right) \leq \mu(A \cap \{x | f(x) > \bar{a}_\varepsilon\}) = a_0.$$

Hence

$$\begin{aligned}
\int_{S,\ A} \Phi(f) d\mu &= \bigwedge_{a \in [0,1]} S(a, \mu(A \cap \varphi_{\bar{a}})) \\
&\leq S\left(\Phi(a_\varepsilon), \mu(A \cap \varphi_{\Phi(\bar{a}_\varepsilon)})\right) \\
&\leq S\left(\Phi(a_\varepsilon), a_0\right) \\
&\leq (\Phi(S(a_\varepsilon, a_0))) \\
&\leq \Phi(a + \varepsilon).
\end{aligned}$$

Thus

$$\int_{S,A} \Phi(f) d\mu \leq \Phi(a), \tag{8}$$

follows from the arbitrariness of ε. Consequently from 8 we conclude:

$$\begin{aligned}
\int_{S,\ A} \Phi(f) d\mu &= \bigwedge_{a \in [0,1]} S(a, \mu(A \cap \varphi_{\bar{a}})) \\
&\leq \Phi(a) = \Phi\left(\int_{S,\ A} f d\mu\right).
\end{aligned}$$

This completes the proof. ☐

As a direct consequence of Theorem (3.2.1) we have

Corollary 3.1. *Let* (X, Σ, μ) *be a fuzzy measure space and* $f : X \to [0,1]$ *be a measurable function. If* S *be a semiconorm. Let* $\Phi : [0,1] \to [0,1]$ *be a continuous and strictly increasing function such that* $\Phi(x) \leq x$*, for every* $x \in [0, \mu(x)]$*, then:*

$$\Phi\left(\int_{S,\ A} f d\mu\right) \geq \int_{S,\ A} \Phi(f) d\mu.$$

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