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CR-SUBMANIFOLDS OF A NEARLY δ-LORENTZIAN TRANS SASAKIAN MANIFOLD

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Abstract. This paper considers the study of CR-submanifold of a nearly δ-Lorentzian trans Sasakian manifold, generalizing the results of a nearly δ-Lorentzian trans Sasakian manifold and thus those of Sasakian manifolds. We also obtain some results on parallel distribution relating to ξ-vertical CR-submanifold of a nearly δ-Lorentzian trans Sasakian manifold.

Keywords: CR-submanifold, nearly Lorentzian para-Sasakian manifold and ξ-vertical CR-submanifold.

AMS Subject Classification: 53C40, 53D12.

1. Introduction

The notion of CR-submanifolds of a Kaehler manifold was introduced by A. Bejancu in [2]. Since then several papers on CR-submanifolds of Sasakian manifolds have been studied by Kobayashi [8], Shahid et al. [9], Yano and Kon [6] and others. On the other hand, there is a class of almost para-contact metric manifolds, namely Lorentzian para-Sasakian manifolds. In 1989, K. Matsumoto [4] introduced the idea of Lorentzian para-Sasakian manifold. Then I. Mihai and R. Rosca [3] introduced the same notion independently and they obtained several results on this manifolds. Lorentzian para-Sasakian manifolds have also been studied by K. Matsumoto and I. Mihai [5], U.C. De and et al. [10] and others. In the present paper we study CR-submanifolds and CR-structure of a CR-submanifold of nearly δ-Lorentzian trans Sasakian manifold. CR-submanifolds have good interaction with other parts of mathematics and substantial applications to (pseudo)-conformal mapping and relativity ([1], [7]).

2. Preliminaries

A $(2n+1)$ dimensional manifold M, is said to be δ-almost contact metric manifold if it admits a 1-1 tensor fiels ϕ, a structure field ξ, a 1-form η ans an indefinite metric g such that

$$\phi^2X = X + \eta(X)\xi, \quad \eta(\xi) = -1 \tag{1}$$

$$g(\xi,\xi) = -\delta, \quad \eta(X) = \delta g(X,\xi) \tag{2}$$

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$$g(\phi X, \phi Y) = g(X, Y) + \delta\eta(X)\eta(Y), \tag{3}$$

for all vector fields X and Y on M, where δ is such that $\delta^2 = 1$ so that $\delta = \pm 1$. The above structure $(\phi, \xi, \eta, g, \delta)$ on M is called the δ Lorentzian structure on M. If $\delta = 1$ and this is the usual Lorentzian structure on M, the vector field ξ is the time like [19], that is M cotains a time like vector field.

From the above equations, one can deduce that

$$\phi\xi = 0, \quad \eta(\phi(X)) = 0$$

A δ-Lorentzian manifold with structure $(\phi, \xi, \eta, g, \delta)$ is said to be δ-Lorentzian trans Sasakian manifold M of type (α, β) if it satisfies the condition

$$(\bar{\nabla}_X \phi)Y = \alpha\{g(X, Y)\xi - \delta\eta(Y)X\} + \beta\{g(\phi X, Y) - \delta\eta(Y)\phi X\} \tag{4}$$

for any vector fields X and Y on M. If $\delta = 1$, then the δ-Lorentzian trans Sasakian is the usual Lorentzian trans Sasakian manifold of type (α, β).δ-Lorentzian trans Sasakian manifold of type $(0, 0), (0, \beta), (\alpha, 0)$ are the Lorentzian cosympletic, Lorentzian β-Kenmotsu and Lorentzian α-Sasakian manifolds respectively. In particular if $\alpha = 1, \beta = 0$ and $\alpha = 0, \beta = 1$, then δ-Lorentzian trans Sasakian manifold reduces to δ-Lorentzian Sasakian and δ-Lorentzian Kenmotsu manifolds respectively. On a δ-Lorentzian trans Sasakian manifold $\bar{M}$, we have

$$\bar{\nabla}_X \xi = -\delta\alpha\phi X - \beta\delta\phi^2 X \tag{5}$$

Further, δ-almost contact metric manifold M on $(\phi, \xi, \eta, g, \delta)$ is called nearly δ- Lorentzian trans-Sasakian manifold if

$$\begin{aligned}(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X &= \alpha\{2g(X, Y)\xi - \delta\eta(Y)X - \delta\eta(X)Y\} \\ &+\beta\{2g(\phi X, Y)\xi - \delta\eta(Y)\phi X - \delta\eta(X)\phi Y\}.\end{aligned} \tag{6}$$

Now, let M be a submanifold immersed in $\bar{M}$. The Riemannian metric induced on M is denoted by the same symbol g. Let TM and $T^{\perp}M$ be the Lie algebras of vector fields tangential to M and normal to M respectively and ∇ be the induced Levi-Civita connection on M. Then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{7}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{8}$$

for any X, $Y \epsilon TM$ and $N \epsilon T^{\perp}M$, where $\nabla^{\perp}$ is the connection on the normal bundle $T^{\perp}M$, h is the second fundamental form and A_N is the Weingarten map associated with N as

$$g(A_N X, Y) = g(h(X, Y), N). \tag{9}$$

For any $x \epsilon M$ and $X \epsilon T_x M$, we write

$$X = PX + QX, \tag{10}$$

where $PX \epsilon D$, $QX \epsilon D^{\perp}$ and $T_x M = D \bigcup D^{\perp}$. Similarly for N normal to M, we have

$$N = BN + CN, \tag{11}$$

where BN (respectively, CN) is the tangential component (respectively, normal component) of ϕN.

Now, let M be a submanifold immersed in $\bar{M}$. The Riemannian metric induced on M is denoted by the same symbol g. Let TM and $T^{\perp}M$ be the Lie algebras of vector fields tangential to M and normal to M respectively and ∇ be the induced Levi-Civita connection on M. Then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{12}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{13}$$

for any X, $Y \epsilon TM$ and $N \epsilon T^{\perp}M$, where $\nabla^{\perp}$ is the connection on the normal bundle $T^{\perp}M$, h is the second fundamental form and A_N is the Weingarten map associated with N as

$$g(A_N X, Y) = g(h(X,Y), N). \tag{14}$$

For any $x \epsilon M$ and $X \epsilon T_x M$, we write

$$X = PX + QX, \tag{15}$$

where $PX \epsilon D$, $QX \epsilon D^{\perp}$ and $T_x M = D \bigcup D^{\perp}$. Similarly for N normal to M, we have

$$\phi N = BN + CN, \tag{16}$$

where BN (respectively, CN) is the tangential component (respectively, normal component) of ϕN.

Definition 2.1. *An m-dimensional Riemannian submanifold M of $\bar{M}$ is called a CR-submanifold of M if there exists a differentiable distribution $D : x \to D_x$ on M satisfying the following conditions:*
(i) D is invariant under ϕ, that is, $\phi D_x \subset D_x$ for each $x \epsilon M$,
(ii) The complementary orthogonal distribution $D^{\perp} : x \to D^{\perp}_x \subset T_x M$ of D is anti-invariant, that is, $\phi D^{\perp}_x \subset T^{\perp}_x M$ for each $x \epsilon M$. If $dim D^{\perp}_x = 0$ (respectively, $dim D_x = 0$), then the CR-submanifold is called an invariant (respectively, anti-invariant) submanifold. The distribution D (respectively, $D^{\perp}$) is called the horizontal (respectively, vertical) distribution. Also the pair $(D, D^{\perp})$ is called ξ-horizontal (respectively, ξ-vertical) if $\xi_x \epsilon D_x$ (respectively, $\xi_x \epsilon D^{\perp}_x$) for $x \epsilon M$.

3. Some basic lemmas

Lemma 3.1. *Let M be a CR-submanifold of a nearly δ-Lorentzian trans Sasakian manifold $\bar{M}$. Then we have*

$$\begin{aligned} &P(\nabla_X \phi PY) + P(\nabla_Y \phi PX) - P(A_{\phi QY} X) - P(A_{\phi QX} Y) \\ &= 2\alpha g(X,Y)P\xi - \alpha\delta\eta(Y)PX - \alpha\delta\eta(X)PY - \beta\delta\eta(Y)\phi PX \\ &-\beta\delta\eta(X)\phi PY + \phi P\nabla_X Y + \phi P\nabla_Y X + 2\beta g(\phi PX, Y)P\xi \end{aligned} \tag{17}$$

$$\begin{aligned} &Q(\nabla_X \phi PY) + Q(\nabla_Y \phi PX) - Q(A_{\phi QY} X) - Q(A_{\phi QX} Y) = 2Bh(X,Y) \\ &+2\alpha g(X,Y)Q\xi - \alpha\delta\eta(Y)QX - \alpha\delta\eta(X)QY + 2\beta g(\phi QX, Y)Q\xi \end{aligned} \tag{18}$$

$$\begin{aligned} &h(X, \phi PY) + h(Y, \phi PX) + \nabla^{\perp}_X \phi QY + \nabla^{\perp}_Y \phi QX = \phi Q\nabla_Y X \\ &+\phi Q\nabla_X Y + 2Ch(X,Y) - \beta\delta\eta(Y)\phi QX - \beta\delta\eta(X)\phi QY \end{aligned} \tag{19}$$

for any $X, Y \epsilon TM$.

Proof. Using (8), (9) and (11) we get

$$\begin{aligned} (\bar{\nabla}_X \phi)Y + \phi(\nabla_X Y) + \phi h(X,Y) &= P\nabla_X(\phi PY) + Q\nabla_X(\phi PY) \\ &-PA_{\phi QY}X - QA_{\phi QY}X + h(X, \phi PY) + \nabla^{\perp}_X(\phi QY). \end{aligned}$$

Interchanging X and Y in the above equation and adding each other, using (5) and (12) we get

$$\begin{aligned} &P(\nabla_X \phi PY) + P(\nabla_Y \phi PX) - PA_{\phi QY}X - PA_{\phi QX}Y + Q(\nabla_X \phi PY) \\ &+Q(\nabla_Y \phi PX) - QA_{\phi QY}X - QA_{\phi QX}Y + h(X, \phi PY) + h(Y, \phi PX) \\ &+\nabla^{\perp}_X \phi QY + \nabla^{\perp}_Y \phi QX = 2Bh(X,Y) + 2Ch(X,Y) + 2\alpha g(X,Y)P\xi \\ &+2\alpha g(X,Y)Q\xi - \alpha\delta\eta(Y)PX - \alpha\delta\eta(Y)QX - \alpha\delta\eta(X)PY - \alpha\delta\eta(X)QY \end{aligned}$$

$$+2\beta g(\phi PX,Y)P\xi+2\beta g(\phi QX,Y)Q\xi-\beta\delta\eta(Y)\phi PX-\beta\delta\eta(Y)\phi QX-\beta\delta\eta(X)\phi PY$$
$$-\beta\delta\eta(X)\phi QY+\phi P\nabla_X Y+\phi Q\nabla_X Y+\phi P\nabla_Y X+\phi Q\nabla_Y X \tag{20}$$

Now equating horizontal, vertical and normal components in (15), we get the desired result.

□

Lemma 3.2. *Let M be a CR-submanifold of a nearly δ-Lorentzian trans Sasakian manifold $\bar{M}$. Then we have*

$$2(\bar{\nabla}_X\phi)Y=\nabla_X\phi Y-\nabla_Y\phi X+h(X,\phi Y)-h(Y,\phi X)-\phi[X,Y]+\alpha\{2g(X,Y)\xi$$
$$-\delta\eta(Y)X-\delta\eta(X)Y\}+\beta\{2g(\phi X,Y)\xi-\delta\eta(Y)\phi X-\delta\eta(X)\phi Y\} \tag{21}$$
$$2(\bar{\nabla}_Y\phi)X=\alpha\{2g(X,Y)\xi-\delta\eta(Y)X-\delta\eta(X)Y\}+\beta\{2g(\phi X,Y)\xi-\delta\eta(Y)\phi X$$
$$-\delta\eta(X)\phi Y\}-\nabla_X\phi Y+\nabla_Y\phi X-h(X,\phi Y)+h(Y,\phi X)+\phi[X,Y]. \tag{22}$$

Proof. From Gauss formula (7), we have

$$\bar{\nabla}_X\phi Y-\bar{\nabla}_Y\phi X=\nabla_X\phi Y+h(X,\phi Y)-\nabla_Y\phi X-h(Y,\phi X). \tag{23}$$

Also we have

$$\bar{\nabla}_X\phi Y-\bar{\nabla}_Y\phi X=(\bar{\nabla}_X\phi)Y-(\bar{\nabla}_Y\phi)X+\phi[X,Y]. \tag{24}$$

From (18) and (19), we get

$$(\bar{\nabla}_X\phi)Y-(\bar{\nabla}_Y\phi)X=\nabla_X\phi Y+h(X,\phi Y)-\nabla_Y\phi X-h(Y,\phi X)-\phi[X,Y]. \tag{25}$$

Also for nearly δ-Lorentzian trans Sasakian manifold, we have

$$(\bar{\nabla}_X\phi)Y+(\bar{\nabla}_Y\phi)X=\alpha\{2g(X,Y)\xi-\delta\eta(Y)X-\delta\eta(X)Y\}$$
$$+\beta\{2g(\phi X,Y)\xi-\delta\eta(Y)\phi X-\delta\eta(X)\phi Y\} \tag{26}$$

Adding (20) and (21), we get

$$2(\bar{\nabla}_X\phi)Y=\nabla_X\phi Y-\nabla_Y\phi X+h(X,\phi Y)-h(Y,\phi X)-\phi[X,Y]+\alpha\{2g(X,Y)\xi$$
$$-\delta\eta(Y)X-\delta\eta(X)Y\}+\beta\{2g(\phi X,Y)\xi-\delta\eta(Y)\phi X-\delta\eta(X)\phi Y\}$$

Subtracting (20) from (21) we get

$$2(\bar{\nabla}_Y\phi)X=\alpha\{2g(X,Y)\xi-\delta\eta(Y)X-\delta\eta(X)Y\}+\beta\{2g(\phi X,Y)\xi-\delta\eta(Y)\phi X$$
$$-\delta\eta(X)\phi Y\}-\nabla_X\phi Y+\nabla_Y\phi X-h(X,\phi Y)+h(Y,\phi X)+\phi[X,Y]$$

Hence Lemma is proved. □

Lemma 3.3. *Let M be a CR-submanifold of a nearly δ-Lorentzian trans Sasakian manifold $\bar{M}$. Then we have*

$$2(\bar{\nabla}_Y\phi)(Z)=A_{\phi Y}Z-A_{\phi Z}Y-\nabla^{\perp}_Z\phi Y+\nabla^{\perp}_Y\phi Z-\phi[Y,Z]+\alpha\{2g(Y,Z)\xi$$
$$-\delta\eta(Z)Y-\delta\eta(Y)Z\}+\beta\{2g(\phi Y,Z)\xi-\delta\eta(Z)\phi Y-\delta\eta(Y)\phi Z\},$$
$$2(\bar{\nabla}_Z\phi)(Y)=\alpha\{2g(Y,Z)\xi-\delta\eta(Z)Y-\delta\eta(Y)Z\}+\beta\{2g(\phi Y,Z)\xi-\delta\eta(Z)\phi Y$$
$$-\delta\eta(Y)\phi Z\}-A_{\phi Y}Z+A_{\phi Z}Y+\nabla^{\perp}_Z\phi Y-\nabla^{\perp}_Y\phi Z+\phi[Y,Z]$$

for any $Y,Z\epsilon D^{\perp}$.

Proof. From Weingarten formula (8), we have

$$\bar{\nabla}_Z\phi Y - \bar{\nabla}_Y\phi Z = A_{\phi Z}Y - A_{\phi Y}Z + \nabla^{\perp}_Z\phi Y - \nabla^{\perp}_Y\phi Z. \tag{27}$$

Also, we have

$$\bar{\nabla}_Z\phi Y - \bar{\nabla}_Y\phi Z = (\bar{\nabla}_Y\phi)(Z) - (\bar{\nabla}_Z\phi)(Y) + \phi[Y, Z]. \tag{28}$$

From (22) and (23), we get

$$(\bar{\nabla}_Y\phi)Z - (\bar{\nabla}_Z\phi)Y = A_{\phi Y}Z - A_{\phi Z}Y + \nabla^{\perp}_Y\phi Z - \nabla^{\perp}_Z\phi Y - \phi[Y, Z]. \tag{29}$$

Also for nearly δ-Lorentzian trans Sasakian manifold, we have

$$\begin{aligned}(\bar{\nabla}_Y\phi)Z + (\bar{\nabla}_Z\phi)Y &= \alpha\{2g(Y, Z)\xi - \delta\eta(Z)Y - \delta\eta(Y)Z\}\\ &+\beta\{2g(\phi Y, Z)\xi - \delta\eta(Z)\phi Y - \delta\eta(Y)\phi Z\}.\end{aligned} \tag{30}$$

Adding (24) and (25), we get

$$\begin{aligned}2(\bar{\nabla}_Y\phi)(Z) &= A_{\phi Y}Z - A_{\phi Z}Y - \nabla^{\perp}_Z\phi Y + \nabla^{\perp}_Y\phi Z - \phi[Y, Z] + \alpha\{2g(Y, Z)\xi\\ &-\delta\eta(Z)Y - \delta\eta(Y)Z\} + \beta\{2g(\phi Y, Z)\xi - \delta\eta(Z)\phi Y - \delta\eta(Y)\phi Z\}.\end{aligned}$$

Subtracting (24) from (25) we get

$$\begin{aligned}2(\bar{\nabla}_Z\phi)(Y) &= \alpha\{2g(Y, Z)\xi - \delta\eta(Z)Y - \delta\eta(Y)Z\} + \beta\{2g(\phi Y, Z)\xi - \delta\eta(Z)\phi Y\\ &-\delta\eta(Y)\phi Z\} - A_{\phi Y}Z + A_{\phi Z}Y + \nabla^{\perp}_Z\phi Y - \nabla^{\perp}_Y\phi Z + \phi[Y, Z]\end{aligned}$$

for any $Y, Z\epsilon D^{\perp}$ This proves our assertions. □

Lemma 3.4. *Let M be a CR-submanifold of a nearly δ-Lorentzian trans Sasakian manifold $\bar{M}$. Then we have*

$$\begin{aligned}2(\bar{\nabla}_X\phi)Y &= \alpha\{2g(X, Y)\xi - \delta\eta(Y)X - \delta\eta(X)Y\} + \beta\{2g(\phi X, Y)\xi - \delta\eta(Y)\phi X\\ &-\delta\eta(X)\phi Y\} - A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla_Y\phi X - h(Y, \phi X) - \phi[X, Y],\\ 2(\bar{\nabla}_Y\phi)X &= \alpha\{2g(X, Y)\xi - \delta\eta(Y)X - \delta\eta(X)Y\} + \beta\{2g(\phi X, Y)\xi - \delta\eta(Y)\phi X\\ &-\delta\eta(X)\phi Y\} + A_{\phi Y}X - \nabla^{\perp}_X\phi Y + \nabla_Y\phi X + h(Y, \phi X) + \phi[X, Y]\end{aligned}$$

for any $X\epsilon D$ and $Y\epsilon D^{\perp}$.

Proof. By using Gauss and Weingarten equation for $X\epsilon D$ and $Y\epsilon D^{\perp}$ respectively, we get

$$\bar{\nabla}_X\phi Y - \bar{\nabla}_Y\phi X = -A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla_Y\phi X - h(Y, \phi X). \tag{31}$$

Also, we have

$$\bar{\nabla}_X\phi Y - \bar{\nabla}_Y\phi X = (\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X + \phi[X, Y]. \tag{32}$$

From (26) and (27), we get

$$(\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X = -A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla_Y\phi X - h(Y, \phi X) - \phi[X, Y]. \tag{33}$$

Also for nearly δ-Lorentzian trans Sasakian manifold, we have

$$\begin{aligned}(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X &= \alpha\{2g(X, Y)\xi - \delta\eta(Y)X - \delta\eta(X)Y\}\\ &+\beta\{2g(\phi X, Y)\xi - \delta\eta(Y)\phi X - \delta\eta(X)\phi Y\}\end{aligned} \tag{34}$$

Adding (28) and (29), we get

$$\begin{aligned}2(\bar{\nabla}_X\phi)Y &= \alpha\{2g(X, Y)\xi - \delta\eta(Y)X - \delta\eta(X)Y\} + \beta\{2g(\phi X, Y)\xi - \delta\eta(Y)\phi X\\ &-\delta\eta(X)\phi Y\} - A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla_Y\phi X - h(Y, \phi X) - \phi[X, Y],\end{aligned}$$

Subtracting (20) from (21) we get

$$\begin{aligned}2(\bar{\nabla}_Y\phi)X &= \alpha\{2g(X, Y)\xi - \delta\eta(Y)X - \delta\eta(X)Y\} + \beta\{2g(\phi X, Y)\xi - \delta\eta(Y)\phi X\\ &-\delta\eta(X)\phi Y\} + A_{\phi Y}X - \nabla^{\perp}_X\phi Y + \nabla_Y\phi X + h(Y, \phi X) + \phi[X, Y]\end{aligned}$$

Hence Lemma is proved. □

4. Parallel distributions

Definition 4.1. *The horizontal (respectively, vertical) distribution D (respectively, $D^{\perp}$) is said to be parallel [1] with respect to the connection on M if $\nabla_X Y \epsilon D$ (respectively, $\nabla_Z W \epsilon D^{\perp}$) for any vector field $X, Y \epsilon D$ (respectively, $W, Z \epsilon D^{\perp}$).*

Proposition 4.1. *Let M be a ξ-vertical CR-submanifold of a nearly δ-Lorentzian trans Sasakian manifold $\bar{M}$. If the horizontal distribution D is parallel. Then we have*

$$h(X, \phi Y) = h(Y, \phi X) \tag{35}$$

for all $X, Y \epsilon D$.

Proof. Using parallelism of horizontal distribution D, we have

$$\nabla_X \phi Y \epsilon D, \quad \nabla_Y \phi X \epsilon D \tag{36}$$

for any $X, Y \epsilon D$. Thus using the fact that $X = QY = 0$ for $Y \epsilon D$, (13) gives

$$B(X, Y) = g(X, Y) Q\xi \tag{37}$$

for any $X, Y \epsilon D$. Also, since

$$\phi h(X, Y) = Bh(X, Y) + Ch(X, Y), \tag{38}$$

then

$$\phi h(X, Y) = g(X, Y) Q\xi + Ch(X, Y) \tag{39}$$

for any $X, Y \epsilon D$. Next from (14), we have

$$h(X, \phi Y) + h(Y, \phi X) = 2Ch(X, Y) = 2\phi h(X, Y) - 2g(X, Y) Q\xi \tag{40}$$

for any $X, Y \epsilon D$. Putting $X = \phi X \epsilon D$ in (35), we get

$$h(\phi X, \phi Y) + h(Y, \phi^2 X) = 2\phi h(\phi X, Y) - 2g(\phi X, Y) Q\xi \tag{41}$$

or

$$h(\phi X, \phi Y) + h(Y, X) = 2\phi h(\phi X, Y) - 2g(\phi X, Y) Q\xi. \tag{42}$$

Similarly, putting $Y = \phi Y \epsilon D$ in (35), we get

$$h(\phi Y, \phi X) + h(X, Y) = 2\phi h(X, \phi Y) - 2g(X, \phi Y) Q\xi. \tag{43}$$

Hence from (37) and (38), we have

$$\phi h(X, \phi Y) - \phi h(Y, \phi X) = g(X, \phi Y) Q\xi - g(\phi X, Y) Q\xi. \tag{44}$$

Operating ϕ on both sides of (39) and using $\phi\xi = 0$, we get

$$h(X, \phi Y) = h(Y, \phi X) \tag{45}$$

for all $X, Y \epsilon D$.

□

Now, for the distribution $D^{\perp}$, we prove the following proposition.

Proposition 4.2. *Let M be a ξ-vertical CR-submanifold of a nearly δ-Lorentzian trans Sasakian manifold $\bar{M}$. If the distribution $D^{\perp}$ is parallel with respect to the connection on M. Then we have*

$$A_{\phi Y} Z + A_{\phi Z} Y \epsilon D^{\perp} \tag{46}$$

for any $Y, Z \epsilon D^{\perp}$.

Proof. Let $Y, Z\epsilon D^{\perp}$. Then using Gauss and Weingarten formula, we obtain

$$-A_{\phi Z}Y + \nabla^{\perp}_{Y}\phi Z - A_{\phi Y}Z + \nabla^{\perp}_{Z}\phi Y = \phi\nabla_{Y}Z + \phi h(Y,Z) + \phi\nabla_{Z}Y$$
$$+\phi h(Z,Y) + 2g(Y,Z)\xi + \eta(Y)Z + \eta(Z)Y + 4\eta(Y)\eta(Z)\xi \tag{47}$$

for any $Y, Z\epsilon D^{\perp}$. Taking inner product with $X\epsilon D$ in (42), we get

$$g(A_{\phi Y}Z, X) + g(A_{\phi Z}Y, X) = g(\nabla_{Y}Z, \phi X) + g(\nabla_{Z}Y, \phi X). \tag{48}$$

If the distribution $D^{\perp}$ is parallel, then $\nabla_{Y}Z\epsilon D^{\perp}$ and $\nabla_{Z}Y\epsilon D^{\perp}$ for any $Y, Z\epsilon D^{\perp}$.
So from (43) we get

$$g(A_{\phi Y}Z, X) + g(A_{\phi Z}Y, X) = 0 \quad or \quad g(A_{\phi Y}Z + A_{\phi Z}Y, X) = 0, \tag{49}$$

which is equivalent to

$$A_{\phi Y}Z + A_{\phi Z}Y\epsilon D^{\perp} \tag{50}$$

for any $Y, Z\epsilon D^{\perp}$ and this completes the proof.

□

Definition 4.2. *A CR-submanifold is said to be mixed totally geodesic if $h(X, Z) = 0$ for all $X\epsilon D$ and $Y\epsilon D^{\perp}$.*

The following lemma is an easy consequence of (9).

Lemma 4.1. *Let M be a CR-submanifold of a nearly δ-Lorentzian trans Sasakian manifold $\bar{M}$. Then M is mixed totally geodesic if and only if $A_N X\epsilon D$ for all $X\epsilon D$.*

Definition 4.3. *A normal vector field $N \neq 0$ is called D-parallel normal section if $\nabla^{\perp}_{X}N = 0$ for all $X\epsilon D$.*

Proposition 4.3. *Let M be a mixed totally geodesic ξ-vertical CR-submanifold of a nearly δ-Lorentzian trans Sasakian manifold $\bar{M}$. Then the normal section $N\epsilon\phi D^{\perp}$ is D-parallel if and only if $\nabla_X\phi N\epsilon D$ for all $X\epsilon D$.*

Proof. Let $N\epsilon\phi D^{\perp}$. Then from (13) we have

$$Q(\nabla_{Y}\phi X) = 0 \tag{51}$$

for any $X\epsilon D, Y\epsilon D^{\perp}$. In particular, we have $Q(\nabla_{Y}X) = 0$. By using it in (3.3), we get

$$\nabla^{\perp}_{X}\phi QY = \phi Q\nabla_{X}Y \quad or \quad \nabla^{\perp}_{X}N = -\phi Q\nabla_{X}\phi N. \tag{52}$$

Thus, if the normal section $N \neq 0$ is D-parallel, then using Definition 4 and (4.18), we get

$$\phi Q(\nabla_{X}\phi N) = 0, \tag{53}$$

which is equivalent to $\nabla_X\phi N\epsilon D$ for all $X\epsilon D$. The converse part easily follows from (47). This completes the proof of the proposition. □

5. Integrability conditions of distributions

First we calculate the Nijenhuis tensor $N_{\phi}(X, Y)$ on a nearly δ-Lorentzian trans Sasakian manifold $\bar{M}$. For this, first we prove the following lemma.

Lemma 5.1. *Let M be a CR-submanifold of a nearly δ-Lorentzian trans Sasakian manifold $\bar{M}$, then*

$$(\bar{\nabla}_{\phi X}\phi)Y = \alpha\{2g(\phi X, Y)\xi - \delta\eta(Y)\phi X\} + \beta\{2g(X, Y)\xi - \delta\eta(Y)X$$
$$+(2-\delta)\eta(X)\eta(Y)\xi\} - \eta(X)\bar{\nabla}_{Y}\xi + \phi(\bar{\nabla}_{Y}\phi)(X) + \eta(\bar{\nabla}_{Y}X)\xi \tag{54}$$

for any $X, Y\epsilon T\bar{M}$.

Proof. From the definition of nearly δ-Lorentzian trans Sasakian manifold $\bar{M}$, we have

$$(\bar{\nabla}_{\phi X}\phi)Y = \alpha\{2g(\phi X, Y)\xi - \delta\eta(Y)\phi X\} + \beta\{2g(X, Y)\xi - \delta\eta(Y)X + (2-\delta)\eta(X)\eta(Y)\xi\} - (\bar{\nabla}_Y\phi)\phi X \tag{55}$$

Also, we have

$$(\bar{\nabla}_Y\phi)\phi X = \bar{\nabla}_Y\phi^2 X - \phi(\bar{\nabla}_Y\phi X) = \bar{\nabla}_Y\phi^2 X - \phi(\bar{\nabla}_Y\phi X) + \phi(\phi\bar{\nabla}_Y X) - \phi(\phi\bar{\nabla}_Y X) = \bar{\nabla}_Y X + \eta(X)\bar{\nabla}_Y\xi - \phi(\bar{\nabla}_Y\phi X - \phi\bar{\nabla}_Y X) - \phi(\phi\bar{\nabla}_Y X)$$

$$(\bar{\nabla}_Y\phi)\phi X = \eta(X)\bar{\nabla}_Y\xi - \phi(\bar{\nabla}_Y\phi)(X) - \eta(\bar{\nabla}_Y X)\xi \tag{56}$$

Using (51) in (52), we get

$$(\bar{\nabla}_{\phi X}\phi)Y = \alpha\{2g(\phi X, Y)\xi - \delta\eta(Y)\phi X\} + \beta\{2g(X, Y)\xi - \delta\eta(Y)X + (2-\delta)\eta(X)\eta(Y)\xi\} - \eta(X)\bar{\nabla}_Y\xi + \phi(\bar{\nabla}_Y\phi)(X) + \eta(\bar{\nabla}_Y X)\xi \tag{57}$$

for any $X, Y \epsilon T\bar{M}$, which completes the proof of the lemma. On a nearly δ-Lorentzian trans Sasakian manifold $\bar{M}$, Nijenhuis tensor is given by

$$N_\phi(X, Y) = (\bar{\nabla}_{\phi X}\phi)Y + \phi(\bar{\nabla}_Y\phi)X - (\bar{\nabla}_{\phi Y}\phi)X - \phi(\bar{\nabla}_X\phi)Y \tag{58}$$

for any $X, Y \epsilon T\bar{M}$.

From (49) and (53), we get

$$N_\phi(X, Y) = -\alpha\delta\eta(Y)\phi X - \beta\delta\eta(Y)X - \eta(X)\bar{\nabla}_Y\xi + 2\phi(\bar{\nabla}_Y\phi)(X) + \eta(\bar{\nabla}_Y X)\xi - \alpha\delta\eta(X)\phi Y + \beta\delta\eta(X)Y + \eta(Y)\bar{\nabla}_X\xi - 2\phi(\bar{\nabla}_X\phi)(Y) - \eta(\bar{\nabla}_X Y)\xi \tag{59}$$

Thus using (3) in the above equation and after some calculations, we obtain

$$N_\phi(X, Y) = \alpha\delta\eta(Y)\phi X + \alpha\delta\eta(X)\phi Y - \eta(X)\bar{\nabla}_Y\xi + \eta(Y)\bar{\nabla}_X\xi + \eta(\bar{\nabla}_Y X)\xi - \eta(\bar{\nabla}_X Y)\xi + 4\phi(\bar{\nabla}_Y\phi)X + \beta\delta\eta(X)\eta(Y)\xi \tag{60}$$

for any $X, Y \epsilon T\bar{M}$. Now we prove the following proposition. □

Proposition 5.1. *Let M be a ξ-vertical CR-submanifold of a nearly δ-Lorentzian trans Sasakian manifold $\bar{M}$. Then, the distribution D is integrable if the following conditions are satisfied:*

$$S(X, Z)\epsilon D, \quad h(X, \phi Z) = h(\phi X, Z) \tag{61}$$

for any $X, Z \epsilon D$.

Proof. The torsion tensor $S(X, Y)$ of the almost contact structure (ϕ, ξ, η, g) is given by

$$S(X, Y) = N_\phi(X, Y) + 2d\eta(X, Y)\xi = N_\phi(X, Y) + 2g(\phi X, Y)\xi \tag{62}$$

Thus, we have

$$S(X, Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + 2g(\phi X, Y)\xi \tag{63}$$

for any $X, Y \epsilon TM$. Suppose that the distribution D is integrable.
So for $X, Y \epsilon D$, $\quad Q[X, Y] = 0$ and $\eta([X, Y]) = 0$ as $\xi \epsilon D^\perp$. If $S(X, Y)\epsilon D$, then from (56) and (58) we have

$$[2g(\phi X, Y)\xi + \eta([X, Y])\xi + 4(\phi\nabla_Y\phi X + \phi h(Y, \phi X) + Q\nabla_Y X + h(X, Y))]\epsilon D \tag{64}$$

or

$$2g(\phi X, Y)Q\xi + \eta([X, Y])Q\xi + 4(\phi Q\nabla_Y\phi X + \phi h(Y, \phi X) + Q\nabla_Y X + h(X, Y)) = 0 \tag{65}$$

for any $X, Y \epsilon D$. Replacing Y by ϕZ for $Z \epsilon D$ in the above equation, we get

$$2g(\phi X, \phi Z)Q\xi + 4(\phi Q\nabla_{\phi Z}\phi X + \phi h(\phi Z, \phi X)$$

$$+Q\nabla_{\phi Z}X + h(X, \phi Z)) = 0 \tag{66}$$

Interchanging X and Z for $X, Z\epsilon D$ in (62) and subtracting these relations, we obtain

$$\phi Q[\phi X, \phi Z) + Q[X, \phi Z] + h(X, \phi Z) - h(Z, \phi X) = 0 \tag{67}$$

for any $X, Z\epsilon D$ and the assertion follows. □

Now, we prove the following proposition.

Proposition 5.2. *Let M be a CR-submanifold of a nearly δ-Lorentzian trans Sasakian manifold $\bar{M}$. Then*

$$3(A_{\phi Y}Z - A_{\phi Z}Y) = \phi P[Y, Z] + (2\alpha + \delta)\eta(Y)Z - (2\alpha + \delta)\eta(Z)Y$$
$$+\beta(2 + \delta)\eta(Y)\phi PZ - \beta(2 + \delta)\eta(Z)\phi PY - 2\beta\delta g(PZ, Y)P\xi \tag{68}$$

for any $Y, Z\epsilon D^{\perp}$.

Proof. For $Y, Z\epsilon D^{\perp}$ and $X\epsilon T(M)$, we get

$$2g(A_{\phi Z}Y, X) = 2g(h(X, Y), \phi Z) = g(h(X, Y), \phi Z) + g(h(X, Y), \phi Z)$$
$$= g(\bar{\nabla}_X Y, \phi Z) + g(\bar{\nabla}_Y X, \phi Z) = g(\bar{\nabla}_X Y + \bar{\nabla}_Y X, \phi Z)$$
$$= -g(\phi(\bar{\nabla}_X Y + \bar{\nabla}_Y X), Z) = -g(\bar{\nabla}_X \phi Y + \bar{\nabla}_Y \phi X$$
$$-(\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \phi)X, Z) = -g(\bar{\nabla}_X \phi Y, Z) - g(\bar{\nabla}_Y \phi X, Z)$$
$$+g[\alpha\{2g(X, Y)\xi - \delta\eta(Y)X - \delta\eta(X)Y\} + 2\beta g(\phi X, Y)\xi$$
$$-\delta\beta\{\eta(Y)\phi X + \eta(X)\phi Y\}, Z] = g(A_{\phi Y}Z, X) + g(\bar{\nabla}_Y Z, \phi X)$$
$$+2\alpha g(X, Y)g(\xi, Z) - \alpha\delta\eta(Y)g(X, Z) - \alpha\delta g(\xi, X)g(Y, Z) + 2\beta g(\phi X, Y)g(\xi, Z)$$
$$-\delta\beta\eta(Y)g(\phi X, Z) - \delta\beta g(X, \xi)g(\phi Y, Z) = g(A_{\phi Y}Z, X) - g(\phi(\bar{\nabla}_Y Z), X)$$
$$+2\alpha g(\eta(Z)Y, X) - \alpha\delta g(\eta(Y)Z, X) - \alpha\delta g(g(Y, Z)\xi, X)$$
$$+2\beta g(\eta(Z)\phi Y, X) - \delta\beta g(\eta(Y)\phi Z, X) - \delta\beta g(g(\phi Y, Z)\xi, X) \tag{69}$$

The above equation is true for all $X\epsilon T(M)$, therefore, transvecting the vector field X both sides, we obtain

$$2A_{\phi Z}Y = A_{\phi Y}Z - \phi\bar{\nabla}_Y Z + 2\alpha\eta(Z)Y - \alpha\delta\eta(Y)Z - \alpha\delta g(Y, Z)\xi$$
$$+2\beta\eta(Z)\phi Y - \beta\delta\eta(Y)\phi Z - \beta\delta g(\phi Y, Z)\xi \tag{70}$$

for any $Y, Z\epsilon D^{\perp}$. Interchanging the vector fields Y and Z, we get

$$2A_{\phi Y}Z = A_{\phi Z}Y - \phi\bar{\nabla}_Z Y + 2\alpha\eta(Y)Z - \alpha\delta\eta(Z)Y - \alpha\delta g(Z, Y)\xi$$
$$+2\beta\eta(Y)\phi Z - \beta\delta\eta(Z)\phi Y - \beta\delta g(\phi Z, Y)\xi \tag{71}$$

Subtracting (66) and (67), we get

$$3(A_{\phi Y}Z - A_{\phi Z}Y) = \phi P[Y, Z] + (2\alpha + \delta)\eta(Y)Z - (2\alpha + \delta)\eta(Z)Y$$
$$+\beta(2 + \delta)\eta(Y)\phi PZ - \beta(2 + \delta)\eta(Z)\phi PY - 2\beta\delta g(PZ, Y)P\xi \tag{72}$$

for any $Y, Z\epsilon D^{\perp}$, which completes the proof.

□

Theorem 5.1. *Let M be a CR-submanifold of a nearly δ-Lorentzian trans Sasakian manifold $\bar{M}$. Then, the distribution $D^{\perp}$ is integrable if and only if*

$$A_{\phi Y}Z - A_{\phi Z}Y = (2\alpha + \delta)\eta(Y)Z - (2\alpha + \delta)\eta(Z)Y \tag{73}$$

for any $Y, Z\epsilon D^{\perp}$.

Proof. First suppose that the distribution $D^{\perp}$ is integrable. Then $[Y, Z]\epsilon D^{\perp}$ for any $Y, Z\epsilon D^{\perp}$. Since P is a projection operator on D, so $P[Y, Z] = 0$. Thus from (64) we get (69). Conversely, we suppose that (69) holds. Then using (64), we have $\phi P[Y, Z] = 0$ for any $Y, Z\epsilon D^{\perp}$. Since rank $\phi = 2n$. Therefore, either $P[Y, Z] = 0$ or $P[Y, Z] = k\xi$. But $P[Y, Z] = k\xi$ is not possible as P is a projection operator on D. Thus, $P[Y, Z] = 0$, which is equivalent to $[Y, Z]\epsilon D^{\perp}$ for any$Y, Z\epsilon D^{\perp}$ and hence $D^{\perp}$ is integrable. □

Corollary 5.1. *Let M be a ξ-horizontal CR-submanifold of a nearly δ-Lorentzian trans Sasakian manifold $\bar{M}$. Then, the distribution $D^{\perp}$ is integrable if and only if*

$$A_{\phi Y}Z - A_{\phi Z}Y = 0 \tag{74}$$

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