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INVARIANT FILTERING RESULTS FOR WIDE BAND NOISE DRIVEN SIGNAL SYSTEMS

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Abstract. Filtering of wide band noise driven systems accounts the following problem. Given an autocovariance function, there are infinitely many wide band noise processes, which have this autocovariance function. Each of them produces its own best estimate. The problem is a selection of the best one of these best estimates. A similar problem arises in control theory as a selection of optimal one of the optimal controls. In this paper we investigate this problem for a wide class of wide band noises. It is proved that in the case of independent wide band and white noises corrupting, respectively, the signal and observations, the best estimates and the optimal controls in the linear filtering and LQG problems are independent of the respective wide band noises. We present a complete set of formulae for the best estimate and, respectively, for the optimal control in terms of the system parameters and autocovariance function of the wide band noise disturbing the signal system.

Keywords: Wide band noise, white noise, Kalman filtering, LQG problem.

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1. Introduction

Estimation and stochastic control theories employ the white noise model of disturbing noise processes. In such a way the most powerful estimation result, the Kalman filter, originated from Kalman [1] and Kalman and Bucy [2] and widely discussed in Bensoussan [3], Curtain and Prichard [4] etc., has been discovered for linear systems corrupted by independent or correlated white noises. The same can be said about optimal control law in LQG (linear quadratic Gaussian) control problem originated from Wonham [5] as well.

Although the results for systems with a white noise model of disturbing noises find wide applications in engineering (see, for example, Grassides and Junkins [6]), the real noises behave differently than the white noises. This was observed by engineers long ago. Fleming and Rishel [7] noticed that the real noises behave as a wide band noise in which white noises are an ideal case.

Perhaps, a well-developed stochastic calculus, that originates from the works of Ito [8, 9] and provides principles of working with white noises, is a reason for the wide use of white noise driven systems in estimation and control theory. The noises of non-white nature

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have already been considered in different works as well. Bucy and Joseph [10] introduced a coloured noise. In a series of works by Kushner [11], Kushner and Runggaldier [12, 13], Kushner and Ramachandran [14], Liptser et al [15] wide band noise driven systems are investigated by a method of approximation. Hu [16] and Wang et al [17] studied wide band noises in speech signals. In Bashirov et al [18, 19, 20, 21] an integral representation for wide band noises was initiated.

An important feature of wide band noises is that in applications they are observed by autocovariance functions though different wide band noises may have the same autocovariance function. Denote by $W(\Lambda)$ the collection of all wide band noises which have the autocovariance function Λ. Normally, different wide band noises from $W(\Lambda)$ should result different best estimates and different optimal controls in estimation and stochastic control problems. This creates a question: To what wide band noise from $W(\Lambda)$ should it be followed?

It may also happen that the filtering and control results are dependent on Λ but independent on $\varphi \in W(\Lambda)$. It is natural to call such results as invariant results. Invariant results have more applicable form than non-invariant results because they construct optimal filters on the basis of autocovariance functions. Previously, some invariant results were obtained in Bashirov et al [22, 23, 24, 25, 26, 27, 28]. The first paper is related to stochastic maximum principle. The next four papers to controllability of stochastic systems under wide band noises. The sixth paper discusses invariant results for wide band noise driven observation systems. The last paper presents invariant solutions for wide band noise driven signal systems. Being short conference presentations, it does not contain the proofs. In this paper we present complete proofs of these invariant results.

We prefer to write the arguments of functions in the subscripts, for example, f_t instead of $f(t)$. This allows to make shorter big expressions. $\mathbb{R}^n$ denotes an n-dimensional Euclidean space and $\mathbb{R}^{n\times k}$ the space of $(n \times k)$-matrices. As always, $\mathbb{R} = \mathbb{R}^1$. The norm and scalar product in all considered spaces are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, being clear from the context. I and 0 are the identity and zero matrices or operators independently on their dimensions. A^* is the adjoint of the linear closed operator A. In the case when $A \in \mathbb{R}^{n\times m}$, A^* becomes the transpose of A. For $F \in \mathbb{R}^{n\times n}$, we write $F \geq 0$ (respectively, $F > 0$) if $F^* = F$ and $\langle Fx, x\rangle \geq 0$ for all $x \in \mathbb{R}^n$ (respectively, $\langle Fx, x\rangle > 0$ for all nonzero $x \in \mathbb{R}^n$).

We assume that a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is given. $\mathbf{E}\,\eta$ is the expectation of η and $\operatorname{cov}(\xi, \eta)$ is the covariance of ξ and η, noticing that $\operatorname{cov}\eta = \operatorname{cov}(\eta, \eta)$. The conditional expectation is denoted by $\mathbf{E}(\,\cdot\,|\,\cdot\,)$. We say that a Wiener process w is standard if it satisfies $w_0 = 0$, $\mathbf{E}w_t = 0$ and $\operatorname{cov}(w_t, w_s) = I\min(t, s)$.

By $L_2(a, b; H)$ we denote the space of all square integrable H-valued functions on $[a, b]$. $C(a, b; H)$ is the space of all H-valued continuous functions on $[a, b]$. $W^{1,2}(a, b; H)$ denotes the space of H-valued functions f on $[a, b]$ which admit the representation $f_t = f_a + \int_a^t g_s\, ds$ with $g \in L_2(a, b; H)$. In particular, this implies that f is continuous, a.e. differentiable, and $f' = g$ a.e.

2. Wide band noises

A random process $\varphi : [0, \infty) \times \Omega \to \mathbb{R}^n$ is said to be an n-dimensional wide band noise or, simply, a wide band noise if

$$\operatorname{cov}(\varphi_{t+\sigma}, \varphi_t) = \begin{cases} 0, & \sigma \geq \varepsilon, \\ \Lambda_{t,\sigma}, & 0 \leq \sigma < \varepsilon, \end{cases}$$

where $\varepsilon > 0$ and Λ is an $\mathbb{R}^{n\times n}$-valued nonzero function. In the case when $\mathbf{E}\,\varphi_t = 0$ and $\Lambda_{t,\sigma} \equiv \Lambda_\sigma$ the wide band noise φ is said to be stationary (in the wide sense).

One can verify that the random process φ defined by

$$\varphi_t = \int_{\max(0,t-\varepsilon)}^{t} \Phi_{t,s-t}\, dw_s, t \geq 0, \tag{1}$$

where Φ is an $\mathbb{R}^{n\times k}$-valued relaxing function on $[0,\infty)\times[-\varepsilon,0]$ and w is a k-dimensional standard Wiener process, is an n-dimensional wide band noise with

$$\operatorname{cov}(\varphi_{t+\sigma},\varphi_t) = \int_{\max(0,t+\sigma-\varepsilon)}^{t} \mathbf{E}(\Phi_{t+\sigma,s-t-\sigma}\Phi^*_{t,s-t})\, ds$$

if $0 \leq \sigma < \varepsilon$. If Φ is nonrandom and depends only on its second argument, then

$$\operatorname{cov}(\varphi_{t+\sigma},\varphi_t) = \int_{\max(-t,\sigma-\varepsilon)}^{0} \Phi_{s-\sigma}\Phi^*_s\, ds \tag{2}$$

if $0 \leq \sigma < \varepsilon$, that is, φ becomes stationary for $t \geq \varepsilon$. Thus, the formula in (1) presents a wide band noise as a distributed delay of a white noise. This issue is studied in Bashirov et al [29, 30].

For a moment, consider a simplest case when φ is a one-dimensional wide band noise being stationary since the instant ε and having the autocovariance function $\Lambda : [0,\varepsilon] \to \mathbb{R}$. Then, by (2), in order to be represented as

$$\varphi_t = \int_{\max(0,t-\varepsilon)}^{t} \Phi_{s-t}\, dw_s,$$

where w is also one-dimensional, the function $\Phi : [-\varepsilon, 0] \to \mathbb{R}$ should satisfy the equation

$$\Lambda_\sigma = \int_{\sigma-\varepsilon}^{0} \Phi_{s-\sigma}\Phi_s\, ds.$$

This is a convolution equation. In Bashirov and Uğural [31, 32], it is proved that if Λ is a positive definite function and some very general conditions hold, then this equation has an infinite number of solutions $\Phi \in L_2(-\varepsilon, 0; \mathbb{R})$, noticing that the positive definiteness is a defining property of autocovariance functions. This result seemingly extends to multidimensional, non-stationary, and random cases. Therefore, given an autocovariance function Λ, there are infinitely many relaxing functions Φ and, respectively, infinitely many wide band noise processes in the form of (1) that have the same autocovariance function Λ.

Let us fix the autocovariance function Λ and denote the collection of all wide band noise processes having the autocovariance function Λ by $W(\Lambda)$. This is too wide class. According to Section 2, we are interested in those $\varphi \in W(\Lambda)$ which have an integral representation. Depending on selections of Φ in (1), we can define the following subclasses of $W(\Lambda)$:

- Denote by $W_{L_2^{\mathcal{F}}}(\Lambda)$ the collection of all $\varphi \in W(\Lambda)$ such that φ has the representation in (1) with $\Phi \in C(0,\infty; L_2([-\varepsilon,0]\times\Omega; \mathbb{R}^{n\times k}))$ such that for all $t \geq 0$ and $\max(-t,-\varepsilon) \leq \theta \leq 0$, $\Phi_{t,\theta}$ is $\mathcal{F}_{t+\theta}$-measurable, where $\{\mathcal{F}_t\}$ is a complete and continuous filtration generated by w. Here the measurability condition surves the existence of stochastic integral in (1). This class is suitable for a study of control and estimation problems for stochastic systems disturbed by wide band noises that are dependent on state or control. In such a way, in Bashirov [22] a stochastic maximum principle is proved for wide band noise driven nonlinear systems.
- Denote by $W_{W^{1,2}}(\Lambda)$ the collection of all $\varphi \in W(\Lambda)$ such that φ has the representation in (1) with $\Phi \in C(0,\infty; W^{1,2}(-\varepsilon, 0; \mathbb{R}^{n\times k}))$. One can also define its

subclass $W_{W_0^{1,2}}(\Lambda)$ of all $\varphi \in W_{W^{1,2}}(\Lambda)$ with the integral representation in (1) where $\Phi_{t,-\varepsilon} = 0$. This class was employed in Bashirov [33].

- Denote by $W_{L_2}(\Lambda)$ the collection of all $\varphi \in W(\Lambda)$ such that φ has the representation in (1) with $\Phi \in C(0,\infty; L_2(-\varepsilon, 0; \mathbb{R}^{n\times k}))$. This class is our concern in this paper.
- Denote by $W_\delta(\Lambda)$ the collection of all $\varphi \in W(\Lambda)$ such that φ has the representation in (1) with the relaxing function Φ in the form

$$\Phi_{t,\theta} = \sum_{i=1}^{m} F_i \delta_{\theta+t-\lambda_{i,t}},$$

where δ is Dirac's delta-function, $0 \le \varepsilon_1 < \cdots < \varepsilon_m \le \varepsilon$, λ_i satisfies the inequalities $t - \varepsilon_i \le \lambda_{i,t} \le t$, and $F_i \in \mathbb{R}^{n\times k}$ for all $i = 1, \ldots, m$. Then

$$\varphi_t = \int_{\max(0,t-\varepsilon)}^{t} \sum_{i=1}^{m} F_i \delta_{s-\lambda_{i,t}}\, dw_s = \sum_{i=1}^{m} F_i w'_{\max(0,\lambda_{i,t})}.$$

Thus φ becomes a delayed (multiply and time-dependent) white noise. This kind of relaxing functions has been studied in Bashirov et al [34, 35, 36, 37] by approximation of them with relaxing functions from $C(0,\infty; W_0^{1,2}(-\varepsilon, 0; \mathbb{R}^{n\times k}))$.

3. Setting of basic filtering problem.

Just for simplicity, below we consider filtering and LQG problems for a partially observable stationary linear system in finite-dimensional Euclidean spaces, assuming that the signal noise is wide band and the observation noise is white. A more general case when the signal process takes values in a Hilbert space and the system is non-stationary can be handled with minor changes. The wide band noise will be assumed to be non-stationary in general because the main object of discussion in this paper is the wide band nature of the signal noise. We will mainly concentrate on linear filtering problem. LQG problem will be considered as an application of the filtering result.

Throughout this paper we assume:

(F): $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{m\times n}$, w and v are $\mathbb{R}^k$- and $\mathbb{R}^m$-valued standard Wiener processes, ξ is an $\mathbb{R}^n$-valued Gaussian random variable with $\mathbf{E}\,\xi = 0$, (w, v) and ξ are independent, w and v are correlated with $\operatorname{cov}(w_t, v_s) = E \min(t, s)$.

Note that w and v are assumed to be correlated just for generality. The contributions of this paper to these problems are in the case $E = 0$, that is, when w and v are uncorrelated. This is equivalent to their independence because of Gaussian nature of the noises.

Consider the partially observable linear system

$$\begin{cases} x'_t = Ax_t + \varphi_t,\ x_0 = \xi,\ t > 0, \\ dz_t = Cx_t\,dt + dv_t,\ z_0 = 0,\ t > 0, \end{cases} \tag{3}$$

where x and z are vector-valued signal and observation systems. We also assume:

(W): $\varepsilon > 0$ and φ is an n-dimensional wide band noise with the autocovariance function $\operatorname{cov}(\varphi_{t+\sigma}, \varphi_t) = \Lambda_{t,\sigma}$ for $t \ge 0$ and $0 \le \sigma \le \varepsilon$, so that it has the integral representation in (1) for some $\Phi \in C(0,\infty; L_2(-\varepsilon, 0; \mathbb{R}^{n\times k}))$, that is, $\varphi \in W_{L_2}(\Lambda)$.

The filtering problem for the system in (3) consists of finding equations for the best estimate $\hat{x}_t$ of x_t based on the observations z_s, $0 \le s \le t$, that is, for the conditional expectation $\hat{x}_t = \mathbf{E}(x_t | z_s, 0 \le s \le t)$.

Note that the signal system in (3) is given in terms of derivative while the observation system in terms of differential. By this, we stress on the fact that unlike white noises, which are generalized derivatives of Wiener processes and do not exist in the ordinary sense, wide band noises are well-defined random processes.

In condition (W), the continuity of Φ in the first variable is not an essential restriction. It can be replaced by measurability and local boundedness. But in this paper being $L_2(-\varepsilon, 0; \mathbb{R}^{n\times k})$-valued relaxing function is essential.

4. Filtering: correlated noises

In this section we obtain an optimal filter in the filtering problem for the system in (3) assuming that the wide band noise φ is given by its relaxing function Φ.

Theorem 4.1. *Under the conditions (F) and (W), the best estimate process $\hat{x}$ in the filtering problem for the system in (3) is uniquely determined as a solution of the system of equations*

$$\begin{cases} d\hat{x}_t = (A\hat{x}_t + \psi_{t,0})\,dt + P_tC^*(dz_t - C\hat{x}_t\,dt), \\ (\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta})\psi_{t,\theta}\,dt = (Q_{t,\theta}C^* + \Phi_{t-\theta,\theta}E)(dz_t - C\hat{x}_t\,dt), \\ \hat{x}_0 = 0,\ \psi_{0,\theta} = \psi_{t,-\varepsilon} = 0,\ -\varepsilon \le \theta \le 0,\ t > 0, \end{cases} \tag{4}$$

where P, Q and G are solutions of

$$\begin{cases} P_t' = AP_t + P_tA^* + Q_{t,0} + Q_{t,0}^* - P_tC^*CP_t, \\ P_0 = \operatorname{cov}\xi,\ t > 0, \end{cases} \tag{5}$$

$$\begin{cases} (\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta})Q_{t,\theta} = Q_{t,\theta}A^* + G_{t,\theta,0} - (Q_{t,\theta}C^* + \Phi_{t-\theta,\theta}E)CP_t, \\ Q_{0,\theta} = Q_{t,-\varepsilon} = 0,\ -\varepsilon \le \theta \le 0,\ t > 0, \end{cases} \tag{6}$$

and

$$\begin{cases} (\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \tau})G_{t,\theta,\tau} = \Phi_{t-\theta,\theta}\Phi_{t-\tau,\tau}^* - (Q_{t,\theta}C^* + \Phi_{t-\theta,\theta}E)(CQ_{t,\tau}^* + E^*\Phi_{t-\tau,\tau}^*), \\ G_{0,\theta,\tau} = G_{t,-\varepsilon,\tau} = G_{t,\theta,-\varepsilon} = 0,\ -\varepsilon \le \theta \le 0,\ -\varepsilon \le \tau \le 0,\ t > 0. \end{cases} \tag{7}$$

Moreover, the mean square error is equal to

$$e_t = \mathbf{E}\|\hat{x}_t - x_t\|^2 = \operatorname{tr} P_t.$$

Proof. The idea of the proof is as follows. Define the $L_2(-\varepsilon, 0; \mathbb{R}^n)$-valued random process ϕ by

$$[\phi_t]_\theta = \int_{\max(0,t-\varepsilon-\theta)}^t \Psi_{s,s-t+\theta}\,dw_s,\ -\varepsilon \le \theta \le 0,\ t \ge 0, \tag{8}$$

where

$$\Psi_{t,\theta} = \Phi_{t-\theta,\theta},\ -\varepsilon \le \theta \le 0,\ t \ge 0. \tag{9}$$

One can verify the equality

$$\Gamma\phi_t = [\phi_t]_0 = \varphi_t, \tag{10}$$

for φ defined by (1), where Γ is a linear operator from $W^{1,2}(-\varepsilon, 0; \mathbb{R}^n)$ to $\mathbb{R}^n$, assigning to $h \in W^{1,2}(-\varepsilon, 0; \mathbb{R}^n)$ its value h_0. Let $-d/d\theta$ be a differential operator on $L_2(-\varepsilon, 0; \mathbb{R}^n)$ with the domain

$$D(-d/d\theta) = \{h \in W^{1,2}(-\varepsilon, 0; \mathbb{R}^n) : h_{-\varepsilon} = 0\},$$

noticing that $(-d/d\theta)^* = d/d\theta$ and

$$D(d/d\theta) = \{h \in W^{1,2}(-\varepsilon, 0; \mathbb{R}^n) : h_0 = 0\}.$$

One can verify that ϕ is a mild solution of the linear stochastic differential equation

$$d\phi_t = (-d/d\theta)\phi_t\,dt + \Psi_t\,dw_t,\ \phi_0 = 0,\ t > 0. \tag{11}$$

Equations (9)–(11) lead to the reduction of the linear system in (3), driven by the wide band noise φ, to a linear system, driven by a white noise, with an enlarged $\mathbb{R}^n \times L_2(-\varepsilon, 0; \mathbb{R}^n)$-valued signal process. Indeed, letting

$$\tilde{x}_t = \begin{bmatrix} x_t \\ \phi_t \end{bmatrix}, \ \tilde{\xi} = \begin{bmatrix} \xi \\ 0 \end{bmatrix},$$

and

$$\tilde{A} = \begin{bmatrix} A & \Gamma \\ 0 & -d/d\theta \end{bmatrix}, \ \tilde{\Phi}_t = \begin{bmatrix} 0 \\ \Psi_t \end{bmatrix}, \ \tilde{C} = \begin{bmatrix} C & 0 \end{bmatrix},$$

we obtain that

$$\begin{cases} d\tilde{x}_t = \tilde{A}\tilde{x}_t\,dt + \tilde{\Phi}_t\,dw_t, \ \tilde{x}_0 = \tilde{\xi}, \ t > 0, \\ dz_t = \tilde{C}\tilde{x}_t\,dt + dv_t, \ z_0 = 0, \ t > 0. \end{cases} \tag{12}$$

Obviously, the first component of $\hat{\tilde{x}}_t = \mathbf{E}(\tilde{x}_t | z_s, 0 \le s \le t)$ is the best estimate $\hat{x}_t$ for the system in (3). Therefore, it remains to find the equations for $\hat{\tilde{x}}$. This will be done by methods of functional analysis.

In (12), $\tilde{A}$ is a densely defined closed linear operator on $\mathbb{R}^n \times L_2(-\varepsilon, 0; \mathbb{R}^n)$ with

$$D(\tilde{A}) = \mathbb{R}^n \times D(-d/d\theta),$$

generating a strongly continuous semigroup. According to linear filtering theory in Hilbert spaces (see, for example, Curtain and Prichard [4]), the best estimate process $\hat{\tilde{x}}$ is a unique mild solution of the equation

$$\begin{cases} d\hat{\tilde{x}}_t = \tilde{A}\hat{\tilde{x}}_t\,dt + (\tilde{P}_t\tilde{C}^* + \tilde{\Phi}_t E)(dz_t - \tilde{C}\hat{\tilde{x}}_t\,dt), \\ \hat{\tilde{x}}_0 = 0, \ t > 0, \end{cases} \tag{13}$$

where $\tilde{P}$ is a scalar product solution of the operator Riccati equation

$$\begin{cases} \tilde{P}_t' = \tilde{A}\tilde{P}_t + \tilde{P}_t\tilde{A}^* + \tilde{\Phi}_t\tilde{\Phi}_t^* - (\tilde{P}_t\tilde{C}^* + \tilde{\Phi}_t E)(\tilde{C}\tilde{P}_t + E^*\tilde{\Phi}_t^*), \\ \tilde{P}_0 = \operatorname{cov}\tilde{\xi}, \ t > 0, \end{cases} \tag{14}$$

and

$$\mathbf{E}\|\tilde{x}_t - \hat{\tilde{x}}_t\|^2 = \operatorname{tr}\tilde{P}_t. \tag{15}$$

Here, the values of $\tilde{P}$ are self-adjoint Hilbert–Schmidt operators on the Hilbert space $\mathbb{R}^n \times L_2(-\varepsilon, 0; \mathbb{R}^n)$. Therefore, we can decompose $\tilde{P}_t$ as

$$\tilde{P}_t = \begin{bmatrix} P_t & \tilde{Q}_t^* \\ \tilde{Q}_t & \tilde{G}_t \end{bmatrix},$$

assuming that $\tilde{Q}_t$ and $\tilde{G}_t$ are linear integral operators from $\mathbb{R}^n$ and $L_2(-\varepsilon, 0; \mathbb{R}^n)$ to $L_2(-\varepsilon, 0; \mathbb{R}^n)$, respectively. Let $Q_{t,\theta}$ and $G_{t,\theta,\tau}$ be respective kernels, that is,

$$[\tilde{Q}_t x]_\theta = Q_{t,\theta}x, \ -\varepsilon \le \theta \le 0, \ t \ge 0, \ x \in \mathbb{R}^n,$$

and

$$[\tilde{G}_t h]_\theta = \int_{-\varepsilon}^0 G_{t,\theta,\tau}h_\tau\,d\tau, \ -\varepsilon \le \theta \le 0, \ t \ge 0, \ h \in L_2(-\varepsilon, 0; \mathbb{R}^n).$$

We will deduce the equations for P, Q and G from (14) in the following way.

At first, note that

$$\tilde{A}^* = \begin{bmatrix} A^* & 0 \\ \Gamma^* & d/d\theta \end{bmatrix},$$

where Γ^* is understood as

$$\int_{-\varepsilon}^0 \langle \Gamma^* x, h_\theta \rangle\,d\theta = \langle x, h_0 \rangle, \ x \in \mathbb{R}^n, \ h \in D(-d/d\theta).$$

Take arbitrary $(x,g),(y,h) \in \mathbb{R}^n \times D(d/d\theta)$, noticing that $g_0 = h_0 = 0$. Writing (14) for the component $\tilde{G}$ of $\tilde{P}$, we obtain

$$\tilde{G}_t' = (-d/d\theta)\tilde{G}_t + \tilde{G}_t(d/d\theta) + \Psi_t\Psi_t^* - (\tilde{Q}_tC^* + \Psi_tE)(C\tilde{Q}_t^* + E^*\Psi_t^*),$$

or in scalar product

$$\begin{aligned}\langle \tilde{G}_t' g, h\rangle =& \langle \tilde{G}_t g, (d/d\theta)h\rangle + \langle \tilde{G}_t(d/d\theta)g, h\rangle + \langle \Psi_t\Psi_t^* g, h\rangle \\ &- \langle (\tilde{Q}_tC^* + \Psi_tE)(C\tilde{Q}_t^* + E^*\Psi_t^*)g, h\rangle.\end{aligned}$$

Here, the terms can be evaluated in the following way:

$$\begin{aligned}\langle \tilde{G}_t' g, h\rangle &= \int_{-\varepsilon}^0\int_{-\varepsilon}^0 \left\langle \frac{\partial}{\partial t} G_{t,\theta,\tau} g_\tau, h_\theta \right\rangle d\tau d\theta, \\ \langle \tilde{G}_t g, (d/d\theta)h\rangle &= \int_{-\varepsilon}^0\int_{-\varepsilon}^0 \langle G_{t,\theta,\tau} g_\tau, h_\theta'\rangle\, d\tau d\theta \\ &= -\int_{-\varepsilon}^0 \langle G_{t,-\varepsilon,\tau} g_\tau, h_{-\varepsilon}\rangle\, d\tau - \int_{-\varepsilon}^0\int_{-\varepsilon}^0 \left\langle \frac{\partial}{\partial \theta} G_{t,\theta,\tau} g_\tau, h_\theta \right\rangle d\tau d\theta, \\ \langle \tilde{G}_t(d/d\theta)g, h\rangle &= \int_{-\varepsilon}^0\int_{-\varepsilon}^0 \langle G_{t,\theta,\tau} g_\tau', h_\theta\rangle\, d\tau d\theta \\ &= -\int_{-\varepsilon}^0 \langle G_{t,\theta,-\varepsilon} g_{-\varepsilon}, h_\theta\rangle\, d\theta - \int_{-\varepsilon}^0\int_{-\varepsilon}^0 \left\langle \frac{\partial}{\partial \tau} G_{t,\theta,\tau} g_\tau, h_\theta \right\rangle d\tau d\theta, \\ \langle \Psi_t\Psi_t^* g, h\rangle &= \int_{-\varepsilon}^0\int_{-\varepsilon}^0 \langle \Phi_{t-\theta,\theta}\Phi_{t-\tau,\tau}^* g_\tau, h_\theta\rangle\, d\tau d\theta, \\ \langle \tilde{W}_t\tilde{W}_t^* g, h\rangle &= \int_{-\varepsilon}^0\int_{-\varepsilon}^0 \langle W_{t,\theta}W_{t,\tau}^* g_\tau, h_\theta\rangle\, d\tau d\theta,\end{aligned}$$

where for brevity we denote

$$\tilde{W}_t = \tilde{Q}_tC^* + \Psi_tE \text{ and } W_{t,\theta} = Q_{t,\theta}C^* + \Phi_{t-\theta,\theta}E.$$

Hence,

$$\begin{aligned}0 =& \int_{-\varepsilon}^0\int_{-\varepsilon}^0 \left\langle \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \tau}\right) G_{t,\theta,\tau} g_\theta, h_\tau \right\rangle d\tau d\theta \\ &+ \int_{-\varepsilon}^0\int_{-\varepsilon}^0 \langle (W_{t,\theta}W_{t,\tau}^* - \Phi_{t-\theta,\theta}\Phi_{t-\tau,\tau}^*) g_\theta, h_\tau\rangle\, d\tau d\theta \\ &+ \int_{-\varepsilon}^0 \langle G_{t,-\varepsilon,\tau} g_\tau, h_{-\varepsilon}\rangle\, d\tau + \int_{-\varepsilon}^0 \langle G_{t,\theta,-\varepsilon} g_{-\varepsilon}, h_\theta\rangle\, d\theta.\end{aligned}$$

Since $g, h \in D(d/d\theta)$, where $D(d/d\theta)$ is dense in $L_2(-\varepsilon, 0; \mathbb{R}^n)$, we can extend the last equality to all four-tuples $(g_{-\varepsilon}, g, h_{-\varepsilon}, h) \in \mathbb{R}^n \times L_2(-\varepsilon,0;\mathbb{R}^n) \times \mathbb{R}^n \times L_2(-\varepsilon,0;\mathbb{R}^n)$, treating $g_{-\varepsilon}$ and $h_{-\varepsilon}$ independently on g and h. This implies that G satisfies (7) with the zero initial and boundary conditions. Additionally, we also obtain that $G_{t,\cdot,\theta}, G_{t,\theta,\cdot} \in D(-d/d\theta)$.

In the same way, from (14), we derive the equation for $\tilde{Q}$ as

$$\tilde{Q}_t' = (-d/d\theta)\tilde{Q}_t + \tilde{Q}_tA^* + \tilde{G}_t^*\Gamma^* - (\tilde{Q}_tC^* + \Psi_tE)CP_t,$$

or in scalar product

$$\langle \tilde{Q}_t' x, h\rangle = \langle \tilde{Q}_t x, (d/d\theta)h\rangle + \langle \tilde{Q}_tA^* x, h\rangle + \langle \Gamma^* x, \tilde{G}_t h\rangle - \langle (\tilde{Q}_tC^* + \Psi_tE)CP_t x, h\rangle.$$

Here,

$$\langle \tilde{Q}_t' x, h\rangle = \int_{-\varepsilon}^{0} \left\langle \frac{\partial}{\partial t} Q_{t,\theta} x, h_\theta \right\rangle d\theta,$$

$$\langle \tilde{Q}_t x, (d/d\theta) h\rangle = \int_{-\varepsilon}^{0} \langle Q_{t,\theta} x, h_\theta' \rangle \, d\theta$$

$$= - \langle Q_{t,-\varepsilon} x, h_{-\varepsilon}\rangle - \int_{-\varepsilon}^{0} \left\langle \frac{\partial}{\partial \theta} Q_{t,\theta} x, h_\theta \right\rangle d\theta,$$

$$\langle \tilde{Q}_t A^* x, h\rangle = \int_{-\varepsilon}^{0} \langle Q_{t,\theta} A^* x, h_\theta \rangle \, d\theta,$$

$$\langle \Gamma^* x, \tilde{G}_t h\rangle = \langle x, \Gamma \tilde{G}_t h\rangle = \int_{-\varepsilon}^{0} \langle x, G_{t,0,\tau} h_\tau \rangle \, d\tau = \int_{-\varepsilon}^{0} \langle G_{t,0,\theta}^* x, h_\theta \rangle \, d\theta$$

$$= \int_{-\varepsilon}^{0} \langle G_{t,\theta,0} x, h_\theta \rangle \, d\theta,$$

$$\langle \tilde{W}_t C P_t x, h\rangle = \int_{-\varepsilon}^{0} \langle W_{t,\theta} C P_t x, h_\theta \rangle \, d\theta.$$

Hence,

$$0 = \int_{-\varepsilon}^{0} \left\langle \left(\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} \right) Q_{t,\theta} - Q_{t,\theta} A^* - G_{t,\theta,0} \right) x, h_\theta \right\rangle d\theta$$
$$+ \int_{-\varepsilon}^{0} \langle W_{t,\theta} C P_t x, h_\theta \rangle \, d\theta + \langle Q_{t,-\varepsilon} x, h_{-\varepsilon}\rangle.$$

In a similar way we can extend the last equality to all triples $(x, h_{-\varepsilon}, h) \in \mathbb{R}^n \times \mathbb{R}^n \times L_2(-\varepsilon, 0; \mathbb{R}^n)$, treating $h_{-\varepsilon}$ independently on h. This implies that Q satisfies (6) with the zero initial and boundary conditions. Additionally, we obtain $Q_{t,\cdot}, Q_{t,\cdot}^* \in D(-d/d\theta)$.

Next, we concentrate on the equation for P. From Eq. (14), we deduce

$$P_t' = A P_t + P_t A^* + \tilde{Q}_t^* \Gamma^* + \Gamma \tilde{Q}_t - P_t C^* C P_t,$$

or in scalar product

$$\langle P_t' x, y\rangle = \langle P_t x, A^* y\rangle + \langle P_t A^* x, y\rangle + \langle \tilde{Q}_t^* \Gamma^* x, y\rangle + \langle \tilde{Q}_t x, \Gamma^* y\rangle - \langle P_t C^* C P_t x, y\rangle.$$

Here,

$$[\tilde{Q}_t x]_\theta = Q_{t,\theta} x, \ -\varepsilon \le \theta \le 0,$$

implying

$$\langle \tilde{Q}_t x, \Gamma^* y\rangle = \langle \Gamma \tilde{Q}_t x, y\rangle = \langle Q_{t,0} x, y\rangle.$$

Similarly,

$$\langle \tilde{Q}_t^* \Gamma^* x, y\rangle = \langle Q_{t,0}^* x, y\rangle.$$

Then

$$\langle (P_t' - A P_t - P_t A^* - Q_{t,0}^* - Q_{t,0} + P_t C^* C P_t) x, y\rangle = 0.$$

Since $x, y \in \mathbb{R}$ are arbitrary, we obtain the equation in (5) for P.

Now we consider (13). It produces two equations

$$d\hat{x}_t = A \hat{x}_t \, dt + \Gamma \psi_t \, dt + P_t C^* (dz_t - C \hat{x}_t \, dt)$$

and

$$d\psi_t = (-d/d\theta) \psi_t \, dt + (\tilde{Q}_t C^* + \Psi_t E)(dz_t - C \hat{x}_t \, dt),$$

where we let $\psi = \hat{\phi}$. It is not difficult to see that they produce the system in (4). Finally, the formula for the error e_t of estimation follows from (15). This completes the proof. □

5. Filtering: independent noises

In the case of independent noises, i.e., $E = 0$, Theorem 4.1 produces an exceptional result: the filter from Theorem 4.1 becomes independent on relaxing function Φ, depends just on the autocovariance function Λ.

Theorem 5.1. *Under the conditions (F), (W) and $E = 0$, the best estimate process $\hat{x}$ in the filtering problem for the system in (3) is uniquely determined as a solution of the system of equations*

$$\begin{cases} d\hat{x}_t = (A\hat{x}_t + \psi_{t,0})\,dt + P_tC^*(dz_t - C\hat{x}_t\,dt), \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}\right)\psi_{t,\theta}\,dt = Q_{t,\theta}C^*(dz_t - C\hat{x}_t\,dt), \\ \hat{x}_0 = 0,\ \psi_{0,\theta} = \psi_{t,-\varepsilon} = 0,\ -\varepsilon \le \theta \le 0,\ t > 0, \end{cases} \tag{16}$$

where P, Q and R are solutions of

$$\begin{cases} P_t' = AP_t + P_tA^* + Q_{t,0} + Q_{t,0}^* - P_tC^*CP_t, \\ P_0 = \operatorname{cov}\xi,\ t > 0, \end{cases} \tag{17}$$

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}\right)Q_{t,\theta} = Q_{t,\theta}A^* + \Lambda_{t,-\theta} - R_{t,\theta,0} - Q_{t,\theta}C^*CP_t, \\ Q_{0,\theta} = Q_{t,-\varepsilon} = 0,\ -\varepsilon \le \theta \le 0,\ t > 0, \end{cases} \tag{18}$$

and

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \tau}\right)R_{t,\theta,\tau} = Q_{t,\theta}C^*CQ_{t,\tau}^*, \\ R_{0,\theta,\tau} = R_{t,-\varepsilon,\tau} = R_{t,\theta,-\varepsilon} = 0,\ -\varepsilon \le \theta \le 0,\ -\varepsilon \le \tau \le 0,\ t > 0. \end{cases} \tag{19}$$

Proof. Letting $E = 0$ in (4)–(7), we obtain (16) and (17) exactly, but the equations for Q and G become

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}\right)Q_{t,\theta} = Q_{t,\theta}A^* + G_{t,\theta,0} - Q_{t,\theta}C^*CP_t, \\ Q_{0,\theta} = Q_{t,-\varepsilon} = 0,\ -\varepsilon \le \theta \le 0,\ t > 0, \end{cases} \tag{20}$$

and

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \tau}\right)G_{t,\theta,\tau} = \Phi_{t-\theta,\theta}\Phi_{t-\tau,\tau}^* - Q_{t,\theta}C^*CQ_{t,\tau}^*, \\ G_{0,\theta,\tau} = G_{t,-\varepsilon,\tau} = G_{t,\theta,-\varepsilon} = 0,\ -\varepsilon \le \theta \le 0,\ -\varepsilon \le \tau \le 0,\ t > 0. \end{cases} \tag{21}$$

The solution of Eq. (21) has the representation

$$G_{t,\theta,\tau} = \int_{\max(0,t-\theta-\varepsilon,t-\tau-\varepsilon)}^{t} (\Phi_{t-\theta,s-t+\theta}\Phi_{t-\tau,s-t+\tau}^* - Q_{s,s-t+\theta}C^*CQ_{s,s-t+\tau}^*)\,ds.$$

Then

$$G_{t,\theta,0} = \int_{\max(0,t-\theta-\varepsilon)}^{t} (\Phi_{t-\theta,s-t+\theta}\Phi_{t,s-t}^* - Q_{s,s-t+\theta}C^*CQ_{s,s-t}^*)\,ds.$$

Using $\Lambda_{t,-\theta} = \operatorname{cov}(\varphi_{t-\theta}, \varphi_t)$, one can derive

$$\Lambda_{t,-\theta} = \int_{\max(0,t-\theta-\varepsilon)}^{t} \Phi_{t-\theta,s-t+\theta}\Phi_{t,s-t}^*\,ds.$$

This implies

$$G_{t,\theta,0} = \Lambda_{t,-\theta} - \int_{\max(0,t-\theta-\varepsilon)}^{t} Q_{s,s-t+\theta}C^*CQ_{s,s-t}^*\,ds.$$

Therefore, we can introduce a function R as a solution of (19) and write (20) in the form of (18). This proves the theorem. □

6. Application to LQG problem.

Application of Theorem 5.1 to LQG problem gives an immediate result. Add to the signal system in (3) a control action and consider LQG problem of minimizing the cost functional

$$J(u) = \mathbf{E}\left(\langle x_T, Hx_T\rangle + \int_0^T (\langle x_t, Mx_t\rangle + \langle u_t, Nu_t\rangle)\, dt\right) \tag{22}$$

over the partially observable system

$$\begin{cases} x_t' = Ax_t + Bu_t + \varphi_t,\ x_0 = \xi,\ t > 0, \\ dz_t = Cx_t\, dt + dv_t,\ z_0 = 0,\ t > 0, \end{cases} \tag{23}$$

where additionally to the conditions of Theorem 5.1 we assume:

(C): $T > 0$, $B \in \mathbb{R}^{n\times l}$, $H, M \in \mathbb{R}^{n\times n}$, $N \in \mathbb{R}^{l\times l}$, $H \geq 0$, $M \geq 0$ and $N > 0$.

Theorem 6.1. *Under the conditions (F), (W), (C) and $E = 0$, the optimal control u^* in the LQG problem (22)–(23) is uniquely determined by*

$$u_t^* = -G^{-1}B^*\left(V_t\hat{x}_t^* + \int_t^{\min(T,t+\varepsilon)} \mathcal{Y}_{s,t}^* V_s \psi_{t,t-s}\, ds\right), \tag{24}$$

where $\hat{x}_t^*$ is the best estimate of the state x_t^*, defined by (22) and corresponding to the optimal control $u = u^*$, ψ is the associated process, both satisfying

$$\begin{cases} d\hat{x}_t^* = (A\hat{x}_t^* + \psi_{t,0} + Bu_t^*)\, dt + P_tC^*(dz_t^* - C\hat{x}_t^*\, dt), \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}\right)\psi_{t,\theta}\, dt = Q_{t,\theta}C^*(dz_t^* - C\hat{x}_t^*\, dt), \\ \hat{x}_0^* = 0,\ \psi_{0,\theta} = \psi_{t,-\varepsilon} = 0,\ -\varepsilon \leq \theta \leq 0,\ 0 < t \leq T, \end{cases} \tag{25}$$

z^* is the observation process, defined by (23) and corresponding to the optimal control $u = u^*$, V is a solution of the Riccati equation

$$\begin{cases} V_t' + V_tA + A^*V_t + M - V_tBN^{-1}B^*V_t = 0, \\ V_T = H,\ 0 \leq t < T, \end{cases} \tag{26}$$

P, Q and R are solutions of (17)–(19), and $\mathcal{Y}$ is a bounded perturbation of the transition matrix e^{At} of A by $-BN^{-1}B^*V_t$.

Proof. This theorem is proved in Bashirov [33], pp. 224–225, for relaxing functions Φ with values in $W^{1,2}(-\varepsilon, 0; \mathbb{R}^{n\times k})$ and satisfying $\Phi_{t,-\varepsilon} = 0$. Equations (20) and (21) were derived for Q and G. Taking into consideration Theorem 5.1 and transformation of equations (20)–(21) into (18)–(19), given in the proof of Theorem 5.1, we obtain this result valid in the form of Theorem 6.1. □

Similar to Theorem 5.1, this theorem presents the optimal control law in the LQG problem independently on $\varphi \in W_{L_2}(\Lambda)$, just dependent on Λ. Another notable feature of this theorem is that it does not fall into the frame of classical separation principle since the observations z_s, $0 \leq s \leq t$, are dependent on x_τ for $t \leq \tau \leq t+\varepsilon$, that is, the system in (23) is a noncausal system. Indeed, equation (24) falls into extended separation principle.

7. Conclusion

In this paper linear filtering and optimal control problems are handled in the case when the signal is corrupted by a wide band noise, the observations by a white noise, and the cost functional is quadratic. Three theorems are proved.

Theorems 5.1 and 6.1 define how the optimal filter and optimal control can be designed on the basis of the system and cost functional parameters A, B, C, M, N, H and the autocovariance function Λ of the wide band noise φ. They provide a complete set of equations for them. Just ordinary differential equations in the case of classic theory are modified to systems of equations which include ordinary and partial differential equations.

To point out another implicit advantage from theorems of this paper, assume that a study of some real process requires estimation of a linear system disturbed by wide band or delayed white noise. To make the model simpler replace the noise by a white noise without any delay, which is more or less close to the noise of the system. Then the error of estimation by white noise Kalman filter will deviate from the real error, and one of the reasons for this deviation is the replacement of the noise processes. Therefore, the error of estimation of the filter from Theorems 5.1 and 6.1 is more adequate (precise) than the one of classic Kalman filter. This does not means that the error from these theorems is always smaller than the error of the classic Kalman filter. If it is smaller, this is a consequence from the improvement of adequacy of the model. On the contrary, if it is greater, then this can be explained as an inappropriate replacement of the wide band noise by white noise. This issue should be of great importance for tracking of satellites, in particular, for getting preciseness of GPS. In this way, it is remarkable numerical calculations from Bashirov et al [35], where it was detected that a replacement of wide band noise (in the form of pointwise delayed white noise) by a white noise produces a lost of preciseness which is asymptotically (as time increases) nonrecoverable.

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