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FRACTIONAL PLUS FRACTIONAL CAPACITATED TRANSPORTATION PROBLEM WITH ENHANCED FLOW

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Abstract. This paper presents an algorithm to solve a fractional plus fractional capacitated transportation problem with enhanced flow (EP). A related transportation problem (RTP) is formed and it is shown that to each corner feasible solution to (RTP),there is a corresponding feasible solution to enhanced flow problem (EP). An optimal solution to (EP) is shown to be determined from an optimal solution to (RTP). A numerical example is given in support of the theory and is verified by using a computing software Excel Solver.

Keywords: capacitated, transportation problem, optimal solution, feasible solution, enhanced flow, related transportation problem.

AMS Subject Classification: 90C08; 90B06

1. Introduction

The standard transportation problem is concerned with transporting at a minimum cost, a homogeneous commodity from each of the factories (or origins) to a number of markets (or destinations). Quite frequently, it may so happen that there is an extra demand in the markets for the commodity. In order to meet this extra demand, the factories have to increase their production. The total flow from the factories to the markets is now enhanced by the amount of extra demand. The standard transportation problem has now no longer transportation structure because of this flow structure. Enhanced flow problems have been studied by many researchers in the past years. Khurana and Arora [9] have studied enhanced flow and restricted flow in a sum of linear and linear fractional transportation problem. Khanna [8] discussed impact of extra flow in a linear transportation problem in 1982. In 2011, Khurana and Arora [10] presented an algorithm to solve a fixed charge bi-criterion indefinite quadratic transportation problem with enhanced flow. Gupta and Arora [6] studied enhanced flow constraint in a capacitated fixed charge indefinite quadratic transportation problem.

Another important class of transportation problems consist of capacitated transportation problems where the decision variables are bounded. Many researchers like Dahiya and Verma [1] , Das et.al. [3], Gupta and Arora [4, 5] have contributed a lot in this field. Dan et.al. [2] discussed paradox in sum of a linear and linear fractional transportation problem. Joshi and Gupta [7] have studied linear fractional transportation problem with varying demand and supply. Xie et. al. [11] studied both duration and cost optimization

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for transportation problem. This extensive literature on capacitated transportation problem and flow constraint motivated us to solve capacitated transportation problem with enhanced flow when the objective function is the sum of two linear fractional functions.

This paper is organized as : In section 2, fractional plus fractional capacitated transportation problem with enhanced flow is formulated. To solve this enhanced problem, a related transportation problem is also formed. In section 3, optimality criterion for the solution of fractional plus fractional capacitated transportation problem is developed . In section 4, it is shown that the enhanced problem and related problem are equivalent. In section 5, an algorithm is presented to solve a fractional plus fractional capacitated transportation problem with enhanced flow . In section 6, a numerical illustration is included in support of theory.

2. Problem Formulation

Let $I = \{1, 2, ..., m\}$ be the index set of m origins.
$J = \{1, 2, ..., n\}$ is the index set of n destinations.
x_{ij} = the number of units transported from i^{th} origin to j^{th} destination .
c_{ij} = the actual cost of transporting one unit of a commodity from i^{th} origin to the j^{th} destination.
d_{ij} = the standard cost of transporting one unit of a commodity from i^{th} origin to the j^{th} destination.
e_{ij} = the actual cost of purchasing one unit of a commodity from i^{th} origin by the j^{th} destination.
f_{ij} = the standard cost of purchasing one unit of a commodity from i^{th} origin by the j^{th} destination.
l_{ij} and u_{ij} are the bounds on number of units to be transported from i^{th} origin to j^{th} destination.
a_i = the number of units available at the origin i
b_j = the number of units demanded by the destination j
P = Total flow
Consider a fractional plus fractional capacitated transportation problem with enhanced flow constraint given by :

$$(EP) : \min\{\frac{\sum\limits_{i\in I}\sum\limits_{j\in J} c_{ij}x_{ij}}{\sum\limits_{i\in I}\sum\limits_{j\in J} d_{ij}x_{ij}} + \frac{\sum\limits_{i\in I}\sum\limits_{j\in J} e_{ij}x_{ij}}{\sum\limits_{i\in I}\sum\limits_{j\in J} f_{ij}x_{ij}}\}$$

subject to

$$\sum_{j\in J} x_{ij} \geq a_i, \forall i \in I$$

$$\sum_{i\in I} x_{ij} \geq b_j, \forall j \in J$$

$$\sum_{i\in I}\sum_{j\in J} x_{ij} = P \text{ where } P > \max(\sum_{i\in I} a_i, \sum_{j\in J} b_j)$$

$$l_{ij} \leq x_{ij} \leq u_{ij} \text{ and integers}, \forall i \in I, \forall j \in J$$

In order to solve the problem (EP), we consider the following related problem (RTP) with an additional origin and an additional destination. Let $I' = \{1, 2, ..., m, m + 1\}$

and $J^{'} = \{1, 2, ..., n, n+1\}$.

$$(RTP) : \min\{\frac{\sum\limits_{i\in I^{'}}\sum\limits_{j\in J^{'}} c^{'}_{ij}y_{ij}}{\sum\limits_{i\in I^{'}}\sum\limits_{j\in J^{'}} d^{'}_{ij}y_{ij}} + \frac{\sum\limits_{i\in I^{'}}\sum\limits_{j\in J^{'}} e^{'}_{ij}y_{ij}}{\sum\limits_{i\in I^{'}}\sum\limits_{j\in J^{'}} f^{'}_{ij}y_{ij}}\}$$

subject to

$\sum\limits_{j\in J^{'}} y_{ij} = a^{'}_i, \forall i \in I^{'}$

$\sum\limits_{i\in I^{'}} y_{ij} = b^{'}_j, \forall j \in J^{'}$

$l_{ij} \le y_{ij} \le u_{ij}$ and integers, $\forall i \in I, \forall j \in J$

$0 \le y_{m+1,j} \le \sum\limits_{i\in I} u_{ij} - b_j, \forall j \in J$

$0 \le y_{i,n+1} \le \sum\limits_{j\in J} u_{ij} - a_i, \forall i \in I$

$y_{m+1,n+1} \ge 0$ and integers

$a^{'}_i = \sum\limits_{j\in J} u_{ij}, \forall i \in I,\ a^{'}_{m+1} = \sum\limits_{i\in I}\sum\limits_{j\in J} u_{ij} - P = b^{'}_{n+1}; b^{'}_j = \sum\limits_{i\in I} u_{ij},\ \forall j \in J$

$c^{'}_{ij} = c_{ij}, c^{'}_{m+1,j} = c^{'}_{i,n+1} = 0; \forall i \in I, \forall j \in J,\ c^{'}_{m+1,n+1} = M$

$d^{'}_{ij} = d_{ij}, d^{'}_{m+1,j} = d^{'}_{i,n+1} = 0; \forall i \in I, \forall j \in J,\ d^{'}_{m+1,n+1} = M$

$e^{'}_{ij} = e_{ij}, e^{'}_{m+1,j} = e^{'}_{i,n+1} = 0; \forall i \in I, \forall j \in J,\ e^{'}_{m+1,n+1} = M$

$f^{'}_{ij} = f_{ij}, f^{'}_{m+1,j} = f^{'}_{i,n+1} = 0; \forall i \in I, \forall j \in J,\ f^{'}_{m+1,n+1} = M$

where M is a large positive number.

3. Optimality criteria for a fractional plus fractional capacitated transportation problem

Theorem 3.1. *Let $X^0 = \{x^0_{ij}\}_{I^{'}\times J^{'}}$ be the feasible solution of problem (RTP). Let $C^0 = \sum\limits_{i\in I^{'}}\sum\limits_{j\in J^{'}} c^{'}_{ij}x^0_{ij};\ D^0 = \sum\limits_{i\in I^{'}}\sum\limits_{j\in J^{'}} d^{'}_{ij}x^0_{ij};\ E^0 = \sum\limits_{i\in I^{'}}\sum\limits_{j\in J^{'}} e^{'}_{ij}x^0_{ij};\ F^0 = \sum\limits_{i\in I^{'}}\sum\limits_{j\in J^{'}} f^{'}_{ij}x^0_{ij}$. Let B be the set of cells (i, j) which are basic and N_1 and N_2 denotes the set of non-basic cells (i, j) which are at their lower bounds and upper bounds respectively. Let $u^1_i, u^2_i, u^3_i, u^4_i, v^1_j, v^2_j, v^3_j, v^4_j; i \in I^{'}, j \in J^{'}$ be the dual variables such that $u^1_i + v^1_j = c^{'}_{ij}, \forall (i, j) \in B; u^2_i + v^2_j = d^{'}_{ij}, \forall (i, j) \in B; u^3_i + v^3_j = e^{'}_{ij}, \forall (i, j) \in B; u^4_i + v^4_j = f^{'}_{ij}, \forall (i, j) \in B; u^1_i + v^1_j = z^1_{ij}, \forall (i, j) \notin B; u^2_i + v^2_j = z^2_{ij}, \forall (i, j) \notin B; u^3_i + v^3_j = z^3_{ij}, \forall (i, j) \notin B; u^4_i + v^4_j = z^4_{ij}, \forall (i, j) \notin B$. Then a feasible solution $X^0 = \{x^0_{ij}\}_{I^{'}\times J^{'}}$ of problem (RTP) with objective function value $\frac{C^0}{D^0} + \frac{E^0}{F^0}$ will be an optimal solution if and only if the following conditions holds.*

$$\delta^1_{ij} = \frac{\theta_{ij}[D^0(c^{'}_{ij} - z^1_{ij}) - C^0(d^{'}_{ij} - z^2_{ij})]}{D^0[D^0 + \theta_{ij}(d^{'}_{ij} - z^2_{ij})]} + \frac{\theta_{ij}[F^0(e^{'}_{ij} - z^3_{ij}) - E^0(f^{'}_{ij} - z^4_{ij})]}{F^0[F^0 + \theta_{ij}(f^{'}_{ij} - z^4_{ij})]} \ge 0; \forall (i, j) \in N_1 \quad (1)$$

$$\delta^2_{ij} = \frac{-\theta_{ij}[D^0(c^{'}_{ij} - z^1_{ij}) - C^0(d^{'}_{ij} - z^2_{ij})]}{D^0[D^0 - \theta_{ij}(d^{'}_{ij} - z^2_{ij})]} - \frac{\theta_{ij}[F^0(e^{'}_{ij} - z^3_{ij}) - E^0(f^{'}_{ij} - z^4_{ij})]}{F^0[F^0 - \theta_{ij}(f^{'}_{ij} - z^4_{ij})]} \ge 0; \forall (i, j) \in N_2 \quad (2)$$

Proof. Let $X^0 = \{x^0_{ij}\}_{I'\times J'}$ be a feasible solution of problem (EP) with equality constraints. Let z^0 be the corresponding value of objective function. Then

$$z^0 = [\frac{\sum\limits_{i\in I'}\sum\limits_{j\in J'} c'_{ij}x^0_{ij}}{\sum\limits_{i\in I'}\sum\limits_{j\in J'} d'_{ij}x^0_{ij}} + \frac{\sum\limits_{i\in I'}\sum\limits_{j\in J'} e'_{ij}x^0_{ij}}{\sum\limits_{i\in I'}\sum\limits_{j\in J'} f'_{ij}x^0_{ij}}] = \frac{C^0}{D^0} + \frac{E^0}{F^0}$$

$$= \frac{\sum\limits_{i\in I'}\sum\limits_{j\in J'}(c'_{ij} - u^1_i - v^1_j)x^0_{ij} + \sum\limits_{i\in I'}\sum\limits_{j\in J'}(u^1_i + v^1_j)x^0_{ij}}{\sum\limits_{i\in I'}\sum\limits_{j\in J'}(d'_{ij} - u^2_i - v^2_j)x^0_{ij} + \sum\limits_{i\in I'}\sum\limits_{j\in J'}(u^2_i + v^2_j)x^0_{ij}}$$

$$+ \frac{\sum\limits_{i\in I'}\sum\limits_{j\in J'}(e'_{ij} - u^3_i - v^3_j)x^0_{ij} + \sum\limits_{i\in I'}\sum\limits_{j\in J'}(u^3_i + v^3_j)x^0_{ij}}{\sum\limits_{i\in I'}\sum\limits_{j\in J'}(f'_{ij} - u^4_i - v^4_j)x^0_{ij} + \sum\limits_{i\in I'}\sum\limits_{j\in J'}(u^4_i + v^4_j)x^0_{ij}}$$

$$= \frac{\sum\sum\limits_{(i,j)\in N_1}(c'_{ij} - z^1_{ij})l_{ij} + \sum\sum\limits_{(i,j)\in N_2}(c'_{ij} - z^1_{ij})u_{ij} + \sum\limits_{i\in I'}\sum\limits_{j\in J'}(u^1_i + v^1_j)x^0_{ij}}{\sum\sum\limits_{(i,j)\in N_1}(d'_{ij} - z^2_{ij})l_{ij} + \sum\sum\limits_{(i,j)\in N_2}(d'_{ij} - z^2_{ij})u_{ij} + \sum\limits_{i\in I'}\sum\limits_{j\in J'}(u^2_i + v^2_j)x^0_{ij}}$$

$$+ \frac{\sum\sum\limits_{(i,j)\in N_1}(e'_{ij} - z^3_{ij})l_{ij} + \sum\sum\limits_{(i,j)\in N_2}(e'_{ij} - z^3_{ij})u_{ij} + \sum\limits_{i\in I'}\sum\limits_{j\in J'}(u^3_i + v^3_j)x^0_{ij}}{\sum\sum\limits_{(i,j)\in N_1}(f'_{ij} - z^4_{ij})l_{ij} + \sum\sum\limits_{(i,j)\in N_2}(f'_{ij} - z^4_{ij})u_{ij} + \sum\limits_{i\in I'}\sum\limits_{j\in J'}(u^4_i + v^4_j)x^0_{ij}}$$

$$= \frac{\sum\sum\limits_{(i,j)\in N_1}(c'_{ij} - z^1_{ij})l_{ij} + \sum\sum\limits_{(i,j)\in N_2}(c'_{ij} - z^1_{ij})u_{ij} + \sum\limits_{i\in I'} a_i u^1_i + \sum\limits_{j\in J'} b_j v^1_j}{\sum\sum\limits_{(i,j)\in N_1}(d'_{ij} - z^2_{ij})l_{ij} + \sum\sum\limits_{(i,j)\in N_2}(d'_{ij} - z^2_{ij})u_{ij} + \sum\limits_{i\in I'} a_i u^2_i + \sum\limits_{j\in J'} b_j v^2_j}$$

$$+ \frac{\sum\sum\limits_{(i,j)\in N_1}(e'_{ij} - z^3_{ij})l_{ij} + \sum\sum\limits_{(i,j)\in N_2}(e'_{ij} - z^3_{ij})u_{ij} + \sum\limits_{i\in I'} a_i u^3_i + \sum\limits_{j\in J'} b_j v^3_j}{\sum\sum\limits_{(i,j)\in N_1}(f'_{ij} - z^4_{ij})l_{ij} + \sum\sum\limits_{(i,j)\in N_2}(f'_{ij} - z^4_{ij})u_{ij} + \sum\limits_{i\in I'} a_i u^4_i + \sum\limits_{j\in J'} b_j v^4_j}$$

Let some non-basic variable $x_{ij} \in N_1$ undergoes change by an amount θ_{rs} where θ_{rs} is given by $\min\{u_{rs} - l_{rs}; x^0_{ij} - l_{ij}$ for all basic cells (i,j) with a $(-\theta)$ entry in $\theta-$ loop;$u_{ij} - x^0_{ij}$ for all basic cells (i,j) with a $(+\theta)$ entry in $\theta-$ loop $\}$. Then new value of the objective function $\widehat{z}$ will be given by

$$\widehat{z} = \frac{C^0 + \theta_{rs}(c'_{rs} - z^1_{rs})}{D^0 + \theta_{rs}(d'_{rs} - z^2_{rs})} + \frac{E^0 + \theta_{rs}(e'_{rs} - z^3_{rs})}{F^0 + \theta_{rs}(f'_{rs} - z^4_{rs})}$$

$$\widehat{z} - z^0 = \frac{C^0 + \theta_{rs}(c'_{rs} - z^1_{rs})}{D^0 + \theta_{rs}(d'_{rs} - z^2_{rs})} - \frac{C^0}{D^0} + \frac{E^0 + \theta_{rs}(e'_{rs} - z^3_{rs})}{F^0 + \theta_{rs}(f'_{rs} - z^4_{rs})} - \frac{E^0}{F^0}$$

$$= \frac{\theta_{rs}[D^0(c'_{rs} - z^1_{rs}) - C^0(d'_{rs} - z^2_{rs})]}{D^0[D^0 + \theta_{rs}(d'_{rs} - z^2_{rs})]} + \frac{\theta_{rs}[F^0(e'_{rs} - z^3_{rs}) - E^0(f'_{rs} - z^4_{rs})]}{F^0[F^0 + \theta_{ij}(f'_{rs} - z^4_{rs})]} = \delta^1_{rs}(say)$$

Similarly, when some non- basic variable $x_{pq} \in N_2$ undergoes change by an amount θ_{pq} then

$$\widehat{z} - z^0 = \frac{-\theta_{pq}[D^0(c'_{pq} - z^1_{pq}) - C^0(d'_{pq} - z^2_{pq})]}{D^0[D^0 - \theta_{pq}(d'_{pq} - z^2_{pq})]} - \frac{\theta_{pq}[F^0(e'_{pq} - z^3_{pq}) - E^0(f'_{pq} - z^4_{pq})]}{F^0[F^0 - \theta_{pq}(f'_{pq} - z^4_{pq})]} = \delta^2_{pq}(say)$$

Hence X^0 will be local optimal solution if $\delta^1_{ij} \geq 0; \forall(i,j) \in N_1$ and $\delta^2_{ij} \geq 0; \forall(i,j) \in N_2$. Conversely, if $\delta^1_{ij} \geq 0; \forall(i,j) \in N_1$ and $\delta^2_{ij} \geq 0; \forall(i,j) \in N_2$, then $\widehat{z} - z^0 \geq 0$ which simply means $\widehat{z} \geq z^0$. This shows that the minimum value of the objective function is z^0. This proves the theorem. □

4. Theoretical Development

Definition 4.1. ***Corner feasible solution:*** *A basic feasible solution $\{y_{ij}\}_{I' \times J'}$ to problem (RTP) is called a corner feasible solution (cfs) if $y_{m+1,n+1} = 0$.*

Theorem 4.1. *A non- corner feasible solution of (RTP) cannot provide a basic feasible solution to (EP).*

Proof. Let $\{y_{ij}\}_{I' \times J'}$ be a non -corner feasible solution to (RTP).Then $y_{m+1,n+1} = \lambda(> 0)$.

Therefore, $\sum_{i \in I'} y_{i,n+1} = \sum_{i \in I} y_{i,n+1} + y_{m+1,n+1} = \sum_{i \in I} y_{i,n+1} + \lambda = \sum_{i \in I}\sum_{j \in J} u_{ij} - P$

$$\Rightarrow \sum_{i \in I} y_{i,n+1} = \sum_{i \in I}\sum_{j \in J} u_{ij} - (P + \lambda)$$

Now, for $i \in I$,

$$\sum_{j \in J'} y'_{ij} = a'_i = \sum_{j \in J} u_{ij}$$

$$\Rightarrow \sum_{i \in I}\sum_{j \in J'} y_{ij} = \sum_{i \in I}\sum_{j \in J} u_{ij}$$

$$\Rightarrow \sum_{i \in I}\sum_{j \in J} y_{ij} + \sum_{i \in I} y_{i,n+1} = \sum_{i \in I}\sum_{j \in J} u_{ij}$$

$$\Rightarrow \sum_{i \in I}\sum_{j \in J} y_{ij} + \sum_{i \in I}\sum_{j \in J} u_{ij} - (P + \lambda) = \sum_{i \in I}\sum_{j \in J} u_{ij}$$

$$\Rightarrow \sum_{i \in I}\sum_{j \in J} y_{ij} = P + \lambda$$

This implies that total quantity transported from the sources in I to the destinations in J is $(P + \lambda) > P$, a contradiction to assumption that total flow is P and hence $\{y_{ij}\}_{I' \times J'}$ can not provide a feasible solution to (RTP). □

Theorem 4.2. *There is a one -to-one correspondence between the feasible solution to problem (EP) and the corner feasible solution to problem (RTP).*

Proof. Let$\{x_{ij}\}_{I\times J}$ be a feasible solution of (EP).
Define $\{y_{ij}\}_{I'\times J'}$ by the following transformation

$$y_{ij} = x_{ij}; \forall i \in I, \forall j \in J$$

$$y_{i,n+1} = \sum_{j\in J} u_{ij} - \sum_{j\in J} x_{ij}; \forall i \in I$$

$$y_{m+1,j} = \sum_{i\in I} u_{ij} - \sum_{i\in I} x_{ij}; \forall j \in J$$

$$y_{m+1,n+1} = 0$$

It can be shown that$\{y_{ij}\}_{I'\times J'}$ so defined is a cfs to (RTP).

Since, $l_{ij} \le x_{ij} \le u_{ij}$ and $y_{ij} = x_{ij}; \forall i \in I, \forall j \in J$

Therefore, $l_{ij} \le y_{ij} \le u_{ij}; \forall i \in I, \forall j \in J$

Since, $\sum_{j\in J} x_{ij} \ge a_i; \forall i \in I$ and $y_{i,n+1} = \sum_{j\in J} u_{ij} - \sum_{j\in J} x_{ij}; \forall i \in I,$

Therefore, $0 \le y_{i,n+1} \le \sum_{j\in J} u_{ij} - a_i; \forall i \in I$

Since, $\sum_{i\in I} x_{ij} \ge b_j; \forall j \in J$ and by definition of $y_{m+1,j}$, we have

$$0 \le y_{m+1,j} \le \sum_{i\in I} u_{ij} - b_j; \forall j \in J$$

Clearly, $y_{m+1,n+1} \ge 0$

Now, $y_{ij} = x_{ij}; \forall i \in I, \forall j \in J$ and definition of $y_{i,n+1}$ implies that

$$\sum_{j\in J'} y_{ij} = \sum_{j\in J} y_{ij} + y_{i,n+1} = \sum_{j\in J} x_{ij} + \sum_{j\in J} u_{ij} - \sum_{j\in J} x_{ij} = \sum_{j\in J} u_{ij} = a_i$$

For $i = m+1$,

$$\sum_{j\in J'} y_{m+1,j} = \sum_{j\in J} y_{ij} + y_{m+1,n+1}$$

$$= \sum_{j\in J}(\sum_{i\in I} u_{ij} - \sum_{i\in I} x_{ij})$$

$$= \sum_{i\in I}\sum_{j\in J} u_{ij} - \sum_{i\in I}\sum_{j\in J} x_{ij}$$

$$= \sum_{i\in I}\sum_{j\in J} u_{ij} - P = a'_{m+1}$$

Therefore, $\sum_{j\in J'} y_{ij} = a'_i, i \in I'$

Similarly, it can be shown that $\sum_{i\in I'} y_{ij} = b'_j, j \in J'$

Therefore, $\{y_{ij}\}_{I'\times J'}$is a cfs to problem (RTP) .

Conversely, let $\{y_{ij}\}_{I'\times J'}$ be a cfs to problem (RTP). Define $x_{ij}, i \in I, j \in j$ by the following transformation.

$$x_{ij} = y_{ij}; \forall i \in I, \forall j \in J$$

It implies that $l_{ij} \leq x_{ij} \leq u_{ij}$ and integers,$\forall i \in I, \forall j \in J$
Now, for $i \in I$, the source constraints in problem (RTP) implies

$$\sum_{j\in J'} y_{ij} = a'_i = \sum_{j\in J} u_{ij}$$

$$\sum_{j\in J} y_{ij} + y_{i,n+1} = \sum_{j\in J} u_{ij}$$

$$\Rightarrow a_i \leq \sum_{j\in J} y_{ij} \leq \sum_{j\in J} u_{ij} \text{ because } (0 \leq y_{i,n+1} \leq \sum_{j\in J} u_{ij} - a_i; \forall i \in I)$$

Hence, $\sum_{j\in J} y_{ij} \geq a_i; \forall i \in I$ and subsequently $\sum_{j\in J} x_{ij} \geq a_i; \forall i \in I)$

Similarly, $\sum_{i\in I} y_{ij} \geq b_j; \forall j \in J$ and subsequently $\sum_{i\in I} x_{ij} \geq b_j; \forall j \in J)$

For $i = m + 1$

$$\sum_{j\in J'} y_{m+1,j} = a'_{m+1} = \sum_{i\in I}\sum_{j\in J} u_{ij} - P$$

$$\Rightarrow \sum_{j\in J} y_{m+1,j} = \sum_{i\in I}\sum_{j\in J} u_{ij} - P \text{ because } y_{m+1,n+1} = 0$$

Now, for $j \in J$ the destination constraints in (RTP) give $\sum_{i\in I} y_{ij} + y_{m+1,j} = \sum_{i\in I} u_{ij}$

Therefore, $\sum_{i\in I}\sum_{j\in J} y_{ij} + \sum_{j\in J} y_{m+1,j} = \sum_{i\in I}\sum_{j\in J} u_{ij}$

$$\Rightarrow \sum_{i\in I}\sum_{j\in J} y_{ij} = \sum_{i\in I}\sum_{j\in J} u_{ij} - \sum_{j\in J} y_{m+1,j} = P$$

$$\Rightarrow \sum_{i\in I}\sum_{j\in J} x_{ij} = P$$

Therefore, $\{x_{ij}\}_{I\times J}$ is a feasible solution of (EP).

□

Remark 4.1. *If problem (RTP) has a cfs ,then since $c'_{m+1,n+1} = M = d'_{m+1,n+1} = e'_{m+1,n+1} = f'_{m+1,n+1}$, it follows that non-corner feasible solution cannot be an optimal solution of problem (EP) .*

Theorem 4.3. *The value of the objective function of problem (EP) at a feasible solution $\{x_{ij}\}_{I\times J}$ is equal to the value of the objective function of problem (RTP) at its corresponding cfs $\{y_{ij}\}_{I'\times J'}$ and conversely.*

Proof. The value of the objective function of (RTP) at a feasible solution $\{y_{ij}\}_{I'\times J'}$ is

$$z = [\frac{\sum\limits_{i\in I'}\sum\limits_{j\in J'} c'_{ij}y_{ij}}{\sum\limits_{i\in I'}\sum\limits_{j\in J'} d'_{ij}y_{ij}} + \frac{\sum\limits_{i\in I'}\sum\limits_{j\in J'} e'_{ij}y_{ij}}{\sum\limits_{i\in I'}\sum\limits_{j\in J'} f'_{ij}y_{ij}}]$$

$$= [\frac{\sum\limits_{i\in I}\sum\limits_{j\in J} c'_{ij}y_{ij} + \sum\limits_{j\in J} c'_{m+1,j}y_{m+1,j} + \sum\limits_{j\in J} c'_{i,n+1}y_{i,n+1} + c'_{m+1,n+1}y_{m+1,n+1}}{\sum\limits_{i\in I}\sum\limits_{j\in J} d'_{ij}y_{ij} + \sum\limits_{j\in J} d'_{m+1,j}y_{m+1,j} + \sum\limits_{j\in J} d'_{i,n+1}y_{i,n+1} + d'_{m+1,n+1}y_{m+1,n+1}}]$$

$$+[\frac{\sum\limits_{i\in I}\sum\limits_{j\in J} e'_{ij}y_{ij} + \sum\limits_{j\in J} e'_{m+1,j}y_{m+1,j} + \sum\limits_{j\in J} e'_{i,n+1}y_{i,n+1} + e'_{m+1,n+1}y_{m+1,n+1}}{\sum\limits_{i\in I}\sum\limits_{j\in J} f'_{ij}y_{ij} + \sum\limits_{j\in J} f'_{m+1,j}y_{m+1,j} + \sum\limits_{j\in J} f'_{i,n+1}y_{i,n+1} + f'_{m+1,n+1}y_{m+1,n+1}}]$$

$$= [\frac{\sum\limits_{i\in I}\sum\limits_{j\in J} c_{ij}x_{ij}}{\sum\limits_{i\in I}\sum\limits_{j\in J} d_{ij}x_{ij}} + \frac{\sum\limits_{i\in I}\sum\limits_{j\in J} e_{ij}x_{ij}}{\sum\limits_{i\in I}\sum\limits_{j\in J} f_{ij}x_{ij}}]$$

= value of objective function of problem (EP)at its corresponding feasible solution$\{x_{ij}\}_{I\times J}$

because $\forall i \in I, j \in J, c'_{ij} = c_{ij}, d'_{ij} = d_{ij}, e'_{ij} = e_{ij}, f'_{ij} = f_{ij}; x_{ij} = y_{ij}; y_{m+1,n+1} = 0$

$c'_{i,n+1} = c'_{m+1,j} = d'_{i,n+1} = d'_{m+1,j} = e'_{i,n+1} = e'_{m+1,j} = f'_{i,n+1} = f'_{m+1,j} = 0$

The converse can be proved in a similar way. □

Theorem 4.4. *There is a one -to-one correspondence between the optimal solution to problem (EP) and optimal solution among the corner feasible solution to problem (RTP).*

Proof. Let $\{x^0_{ij}\}_{I\times J}$ be an optimal solution to problem (EP) yielding objective function value z^0 and $\{y^0_{ij}\}_{I'\times J'}$ be the corresponding feasible solution to (RTP). Then by Theorem (4.3), the value yielded by $\{y^0_{ij}\}_{I'\times J'}$ is z^0. If possible, let $\{y^0_{ij}\}_{I'\times J'}$ be not an optimal solution to (RTP). Therefore, there exists a corner feasible solution $\{y^{'}_{ij}\}$ to (RTP) with the value $z^1 < z^0$. Let $\{x^{'}_{ij}\}$ be the corresponding feasible solution to (EP).Then by theorem (4.3), $z^1 = [\frac{\sum\limits_{i\in I}\sum\limits_{j\in J} c_{ij}x^{'}_{ij}}{\sum\limits_{i\in I}\sum\limits_{j\in J} d_{ij}x^{'}_{ij}} + \frac{\sum\limits_{i\in I}\sum\limits_{j\in J} e_{ij}x^{'}_{ij}}{\sum\limits_{i\in I}\sum\limits_{j\in J} f_{ij}x^{'}_{ij}}]$ which is less than z^0, a contradiction to the assumption that $\{x^0_{ij}\}_{I\times J}$ is an optimal solution to (EP). Hence,$\{y^0_{ij}\}_{I'\times J'}$ must be an optimal solution to (RTP). Similarly, it can be proved that an optimal feasible solution to (RTP) will give an optimal solution to (EP). □

Theorem 4.5. *Optimizing problem (RTP) is equivalent to optimizing problem (EP) provided problem (EP) has a feasible solution.*

Proof. As problem (EP) has a feasible solution , by theorem (4.2), there exists a cfs to problem (RTP). Thus, by remark (4.1),an optimal solution to problem (RTP) will be a cfs. Hence, by theorem (4.5), an optimal solution to problem (EP) can be obtained. □

5. Algorithm

Step 1: Given a fractional capacitated transportation problem (EP) with enhanced flow, form a related transportation problem (RTP) by introducing a dummy source and a dummy destination.

Step 2: Find an initial basic feasible solution to (RTP). Let B be its corresponding basis.

Step 3: Calculate dual variables $u^1_i, u^2_i, u^3_i, u^4_i, v^1_j, v^2_j, v^3_j, v^4_j; i \in I', j \in J'$ by using the equations given below and taking one of the $u^{'}_i s$ *or* $v^{'}_j s$ as zero.

$u^1_i + v^1_j = c^{'}_{ij}; u^2_i + v^2_j = d^{'}_{ij}; u^3_i + v^3_j = e^{'}_{ij}; u^4_i + v^4_j = f^{'}_{ij}, \forall(i,j) \in B$

$u^1_i + v^1_j = z^1_{ij}; u^2_i + v^2_j = z^2_{ij}; u^3_i + v^3_j = z^3_{ij}; u^4_i + v^4_j = z^4_{ij}, \forall(i,j) \in N_1$ and N_2.

N_1 and N_2 denotes the set of non- basic cells (i,j) which are at their lower bounds and upper bounds respectively.

Step 4: Calculate $\theta_{ij}, c^{'}_{ij} - z^1_{ij}; d^{'}_{ij} - z^2_{ij}; e^{'}_{ij} - z^3_{ij}; f^{'}_{ij} - z^4_{ij}; \forall i \in I', j \in J'$ for all non- basic cells and also calculate $C^0 = \sum\limits_{i\in I'}\sum\limits_{j\in J'} c^{'}_{ij}y_{ij}; D^0 = \sum\limits_{i\in I'}\sum\limits_{j\in J'} d^{'}_{ij}y_{ij}; E^0 = \sum\limits_{i\in I'}\sum\limits_{j\in J'} e^{'}_{ij}y_{ij}; F^0 = \sum\limits_{i\in I'}\sum\limits_{j\in J'} f^{'}_{ij}y_{ij}$.

Step 5: Calculate δ^1_{ij} and δ^2_{ij} given by equation (1) and (2). If $\delta^1_{ij} \geq 0; \forall(i,j) \in N_1$ and $\delta^2_{ij} \geq 0; \forall(i,j) \in N_2$, then the current solution so obtained is the optimal solution to (RTP) and subsequently to (EP).Then go to step 6. Otherwise some $(i,j) \in N_1$ for which $\delta^1_{ij} \leq 0$ or some $(i,j) \in N_2$ for which $\delta^2_{ij} \leq 0$ will enter the basis. Go to step 3.

Step 6: Find the optimal cost $z^0 = \frac{C^0}{D^0} + \frac{E^0}{F^0}$

6. Numerical Illustration

Consider the following 3×3 fractional plus fractional capacitated transportation problem.

$$(EP): \min\{\frac{\sum_{i=1}^{3}\sum_{j=1}^{3} c_{ij}x_{ij}}{\sum_{i=1}^{3}\sum_{j=1}^{3} d_{ij}x_{ij}} + \frac{\sum_{i=1}^{3}\sum_{j=1}^{3} e_{ij}x_{ij}}{\sum_{i=1}^{3}\sum_{j=1}^{3} f_{ij}x_{ij}}\}$$

subject to

$$\sum_{j=1}^{3} x_{1j} \geq 30; \sum_{j=1}^{3} x_{2j} \geq 60; \sum_{j=1}^{3} x_{3j} \geq 90$$

$$\sum_{i=1}^{3} x_{i1} \geq 60; \sum_{i=1}^{3} x_{i2} \geq 50; \sum_{i=1}^{3} x_{i3} \geq 40$$

$$\sum_{i=1}^{3}\sum_{j=1}^{3} x_{ij} = P = 200 > \max(\sum_{i=1}^{3} a_i = 180, \sum_{j=1}^{3} b_j = 150)$$

$1 \leq x_{11} \leq 20; 2 \leq x_{12} \leq 10; 0 \leq x_{13} \leq 20$

$0 \leq x_{21} \leq 20; 2 \leq x_{22} \leq 20; 1 \leq x_{23} \leq 50$

$1 \leq x_{31} \leq 50; 4 \leq x_{32} \leq 40; 3 \leq x_{33} \leq 30$

In order to solve (EP), we first convert it into related transportation problem (RTP)

Table 1. Cost table

origin↓	destinations →	D1	D2	D3
O1	$(c_{ij}, d_{ij}) \rightarrow$	(2,3)	(3,4)	(4,5)
	$(e_{ij}, f_{ij}) \rightarrow$	(5,4)	(9,6)	(9,2)
O2	$(c_{ij}, d_{ij}) \rightarrow$	(1,4)	(2,4)	(2,6)
	$(e_{ij}, f_{ij}) \rightarrow$	(4,3)	(2,3)	(1,7)
O3	$(c_{ij}, d_{ij}) \rightarrow$	(1,3)	(5,1)	(6,4)
	$(e_{ij}, f_{ij}) \rightarrow$	(2,4)	(2,3)	(8,2)

with $c_{i4} = 0 = d_{i4} = e_{i4} = f_{i4}$ for $i = 1,2,3$ and $c_{4j} = 0 = d_{4j} = e_{4j} = f_{4j}$ for $j = 1,2,3$ and $c_{44} = d_{44} = e_{44} = f_{44} = M$. Also, $0 \leq x_{14} \leq 20; 0 \leq x_{24} \leq 30; 0 \leq x_{34} \leq 30; 0 \leq x_{41} \leq 30; 0 \leq x_{42} \leq 20; 0 \leq x_{43} \leq 60; x_{44} \geq 0; a_1^{'} = \sum_{j=1}^{3} u_{1j} = 50; a_2^{'} = \sum_{j=1}^{3} u_{2j} = 90; a_3^{'} = \sum_{j=1}^{3} u_{3j} = 120; a_4^{'} = \sum_{i=1}^{3}\sum_{j=1}^{3} u_{ij} - P = 260 - 200 = 60; b_1^{'} = \sum_{i=1}^{3} u_{i1} = 90; b_2^{'} = \sum_{i=1}^{3} u_{i2} = 70; b_3^{'} = \sum_{i=1}^{3} u_{i3} = 100$. Find an initial basic feasible solution to the (RTP) so formed. This solution is shown in table 2 and is tested for optimality. In table 2, $C^0 = 473; D^0 = 719; E^0 = 518; F^0 = 897$

Since in table 3, $\delta_{ij}^1 \geq 0; \forall(i,j) \in N_1$ and $\delta_{ij}^2 \geq 0; \forall(i,j) \in N_2$, therefore the solution in table 2 is an optimal solution of problem (RTP) and hence yields an optimal solution of

TABLE 2. Corner feasible solution of problem (RTP)

x_{ij}	D1	D2	D3	D4	u_i^1	u_i^2	u_i^3	u_i4
$O1$	**20**	$\overline{10}$	$\underline{0}$	$\overline{20}$	2	3	5	4
$O2$	**10**	$\overline{20}$	$\overline{50}$	**10**	1	4	4	3
$O3$	$\overline{50}$	**37**	$\underline{3}$	$\overline{30}$	5	1	2	3
$O4$	**10**	**3**	**47**	$\underline{0}$	0	0	0	0
v_j^1	0	0	0	-1				
v_j^2	0	0	0	-4				
v_j^3	0	0	0	-4				
v_j^4	0	0	0	-3				

Notes.Entries of the form $\underline{a}$ and $\overline{b}$ represent non- basic cells which are at their lower and upper bounds respectively. Entries in bold are basic cells.

TABLE 3. Computation of δ_{ij}^1 and δ_{ij}^2

NB	$O1D2$	$O1D3$	$O1D4$	$O2D2$	$O2D3$	$O3D1$	$O3D3$	$O3D4$
θ_{ij}	0	19	0	10	10	3	17	3
$c_{ij}^{'} - z_{ij}^1$	1	2	-1	1	1	-4	1	-4
$d_{ij}^{'} - z_{ij}^2$	1	2	1	0	2	2	3	3
$e_{ij}^{'} - z_{ij}^3$	4	4	-1	-2	-3	0	6	2
$f_{ij}^{'} - z_{ij}^4$	2	-2	-1	0	4	1	-1	0
δ_{ij}^1 and δ_{ij}^2	0	0.131	0	0.0083	0.0665	0.0243	0.1055	0.0185

(EP) with minimum cost $= z^0 = \frac{473}{719} + \frac{518}{897} = 1.2353$.We also verified this optimal solution by using a computing software Excel Solver.

7. CONCLUSION

In order to solve a capacitated transportation problem where the objective function is a sum of two fractional functions and the total flow constraint is enhanced to a specified level, a related transportation problem is formulated which possesses a corner feasible solution. Optimal solution to enhanced flow problem can be determined from optimal corner feasible solution to related transportation problem. As future work, it is intended to apply proposed algorithm to a sum of n fractional functions when the decision variables are bounded. Moreover, the developed algorithm can also be applied in a solid fixed charge capacitated transportation problem, indefinite quadratic transportation problem, fuzzy transportation problem , with or without flow constraint.

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