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INTUITIONISTIC FUZZY LABELING GRAPHS

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Abstract. In this paper, some new connectivity concepts in intuitionistic fuzzy labeling graphs are defined. The concepts of strong arc, partial cut node, bridge and block are introduced. Also, intuitionistic fuzzy labeling tree is defined and investigated many interesting properties. Finally, partial intuitionistic fuzzy labeling tree is defined and established many interesting properties on it, which plays a major role in many areas of science and technology.

Keywords: Intuitionistic fuzzy graphs, intuitionistic fuzzy labeling graphs, intuitionistic fuzzy labeling tree, partial intuitionistic fuzzy labeling tree.

AMS Subject Classification: 05C72

1. Introduction

Concept of graph theory have applications in many areas of computer science, including data mining, image segmentation, clustering, image capturing, networking, etc. An intuitionistic fuzzy set is a generalization of the notion of a fuzzy set. Intuitionistic fuzzy models gives more precision, flexibility and compatibility to the system as compared to the fuzzy models.

In 1983, Atanassov [8, 9] introduced the concept of intuitionistic fuzzy set as a generalization of fuzzy sets. Atanassov added a new components which determines the degree of non-membership in the definition of fuzzy set. The fuzzy sets give the degree of membership, while intuitionistic fuzzy sets give both the degree of membership and the degree of non-membership, which are more or less independent from each other, the only requirement is that the sum of these two degrees is not greater than one. Intuitionistic fuzzy sets have been applied in a wide variety of fields including computer science, engineering, mathematics, medicine, chemistry, economics, etc .

In 1975, Rosenfeld [20] discussed the concept of fuzzy graph whose basic idea was introduced by Kauffman [11] in 1973. The fuzzy relations between fuzzy sets were also considered by Rosenfeld and he developed the structure of fuzzy graphs, obtained analogs of several graphs theoretical concepts. Atanassov introduced the concept of intuitionistic

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fuzzy relation. Different types of intuitionistic fuzzy graphs and their applications can be found in several papers. Sahoo and Pal [22] discussed the concept of intuitionistic fuzzy competition graph.

1.1. **Motivation.** Here, we have presented new connectivity concepts in intuitionistic fuzzy labeling graphs and we have defined strong arc, partial cut node, bridge and block by connectivity concepts of intuitionistic fuzzy graph. Also, intuitionistic fuzzy labeling tree and partial intuitionistic fuzzy labeling tree were defined and established many interesting properties on them, which plays a major role in many areas of science and technology.

1.2. **Review of literature.** After Rosenfeld [20] the fuzzy graph theory increases with its various types of branches, such as - fuzzy tolerance graph [28], fuzzy threshold graph [27], bipolar fuzzy graphs [18, 19, 33], highly irregular interval valued fuzzy graphs [14, 15], isometry on interval-valued fuzzy graphs [17], balanced interval-valued fuzzy graphs [12, 16], fuzzy k-competition graphs and p-competition fuzzy graphs [31], fuzzy planar graphs [26, 34], bipolar fuzzy hypergraphs [29, 30], $m-$step fuzzy compitition graphs [25], etc. Fuzzy graph is used in telecommunication system [32]. A new concept of fuzzy colouring of fuzzy graph is given [35]. Sarwar and Akram discussed novel concepts of bipolar fuzzy competition graphs [36].

Akram and Davvaz [1] defined strong intuitionistic fuzzy graphs. They also discuss intuitionistic fuzzy hypergraphs with applications [3]. An novel application of intuitionistic fuzzy digraphs is given by Akram et al. [2]. Also, Akram and Al-Shehrie [4] defined intuitionistic fuzzy cycles and intuitionistic fuzzy trees and intuitionistic fuzzy planar graphs [7]. Balanced intuitionistic fuzzy graphs is discuss by Karunambigai et al. [10]. Also, Parvathi, Karunambigai [13] defined intuitionistic fuzzy graphs. Sahoo and Pal [22] discussed the concept of intuitionistic fuzzy competition graph. They also discussed intuitionistic fuzzy tolerance graph with application [23], different types of products on intuitionistic fuzzy graphs [21] and product of intuitionistic fuzzy graphs and their degree [24]. Akram and Akmal defined intuitionistic fuzzy graph structures [5] and operations on intuitionistic fuzzy graph structures [6].

2. Preliminaries

A graph is an ordered pair $G = (V, E)$, where V is the set of all vertices of G, which is non empty and E is the set of all edges of G. Two vertices x, y in a graph G are said to be adjacent in G if (x, y) is an edge of G. A simple graph is a graph without loops and multiple edges. A complete graph is a simple graph in which every pair of distinct vertices is connected by an edge. The complete graph on n vertices has $\frac{n(n-1)}{2}$ edges.

An isomorphism of graphs G_1 and G_2 is a bijection between the vertex sets of G_1 and G_2 such that any two vertices v_1 and v_2 of G_1 are adjacent in G_1 if and only if $f(v_1)$ and $f(v_2)$ are adjacent in G_2. An isomorphic graphs are denoted by $G_1 \cong G_2$.

2.1. **Intuitionistic fuzzy graphs.** An intuitionistic fuzzy set A on the set X is characterized by a mapping $m : X \to [0, 1]$, which is called as a membership function and $n : X \to [0, 1]$, which is called as a non-membership function. An intuitionistic fuzzy set is denoted by $A = (X, m_A, n_A)$. The membership function of the intersection of two intuitionistic fuzzy sets $A = (X, m_A, n_A)$ and $B = (X, m_B, n_B)$ is defined as $m_{A\cap B} = \min\{m_A, m_B\}$ and the non-membership function $n_{A\cap B} = \max\{n_A, n_B\}$. We write $A = (X, m_A, n_A) \subseteq B = (X, m_B, n_B)$ (intuitionistic fuzzy subset) if $m_A(x) \leq m_B(x)$ and $n_A(x) \geq n_A(x)$ for all $x \in X$.

Here, an intuitionistic fuzzy graph is defined below:

Definition 2.1. [13] *An intuitionistic fuzzy graph is of the form* $G = (V, \sigma, \mu)$ *where* $\sigma = (\sigma_1, \sigma_2)$, $\mu = (\mu_1, \mu_2)$ *and*

(i) $V = \{v_0, v_1, \ldots, v_n\}$ *such that* $\sigma_1 : V \to [0,1]$ *and* $\sigma_2 : V \to [0,1]$, *denote the degree of membership and non-membership of the vertex* $v_i \in V$ *respectively and* $0 \leq \sigma_1(v_i) + \sigma_2(v_i) \leq 1$ *for every* $v_i \in V$ $(i = 1, 2, \ldots, n)$.

(ii) $\mu_1 : V \times V \to [0,1]$ *and* $\mu_2 : V \times V \to [0,1]$, *where* $\mu_1(v_i, v_j)$ *and* $\mu_2(v_i, v_j)$ *denote the degree of membership and non-membership value of the edge* (v_i, v_j) *respectively such that* $\mu_1(v_i, v_j) \leq \min\{\sigma_1(v_i), \sigma_1(v_j)\}$ *and* $\mu_2(v_i, v_j) \leq \max\{\sigma_2(v_i), \sigma_2(v_j)\}$, $0 \leq \mu_1(v_i, v_j) + \mu_2(v_i, v_j) \leq 1$ *for every* (v_i, v_j).

Now, we give an example of intuitionistic fuzzy graph:

Example 2.1. *Let* $G = (V, \sigma, \mu)$ *be an intuitionistic fuzzy graph, where* $\sigma(v) = (\sigma_1(v), \sigma_2(v))$, $\mu(u, v) = (\mu_1(u, v), \mu_2(u, v))$. *Let the vertex set be* $V = \{v_1, v_2, v_3, v_4\}$ *and* $\sigma(v_1) = (0.3, 0.6)$, $\sigma(v_2) = (0.8, 0.2)$, $\sigma(v_3) = (0.2, 0.8)$, $\sigma(v_4) = (0.5, 0.4)$; $\mu(v_1, v_2) = (0.25, 0.45)$, $\mu(v_2, v_3) = (0.18, 0.75)$, $\mu(v_3, v_4) = (0.15, 0.52)$, $\mu(v_4, v_1) = (0.3, 0.25)$, $\mu(v_1, v_3) = (0.2, 0.8)$, $\mu(v_2, v_4) = (0.4, 0.35)$. *The corresponding intuitionistic fuzzy graph is shown in Figure 1.*

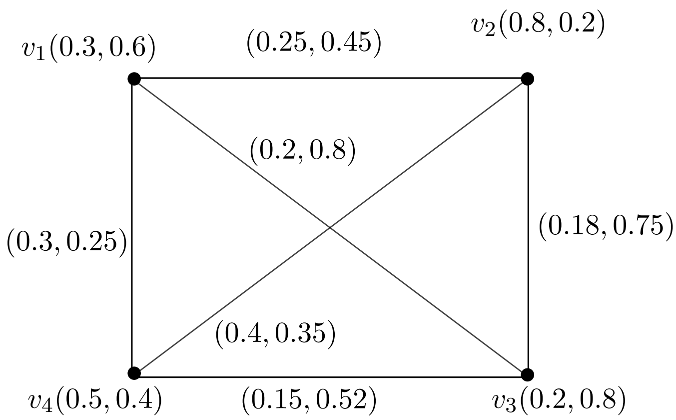

Figure 1. An intuitionistic fuzzy graph

3. Intuitionistic fuzzy labeling graphs

Now, we define intuitionistic fuzzy labeling graphs as follows.

Definition 3.1. *An intuitionistic fuzzy graph* $G = (V, \sigma, \mu)$ *is said to be an intuitionistic fuzzy labeling graph if* $\sigma_1 : V \to [0,1]$, $\sigma_2 : V \to [0,1]$ *and* $\mu_1 : V \times V \to [0,1]$, $\mu_2 : V \times V \to [0,1]$ *is bijective such that the membership and non-membership values of the vertices and edges are distinct and* $\mu_1(v_i, v_j) < \min\{\sigma_1(v_i), \sigma_1(v_j)\}$, $\mu_2(v_i, v_j) < \max\{\sigma_2(v_i), \sigma_2(v_j)\}$, $0 \leq \mu_1(v_i, v_j) + \mu_2(v_i, v_j) \leq 1$ *for every edges* (v_i, v_j).

Now, we give an example of intuitionistic fuzzy labeling graph as follows:

Example 3.1. *In Figure 2,* σ_1, σ_2 *and* μ_1, μ_2 *are bijective such that no vertices and edges have the membership and non-membership values.*

Definition 3.2. *Intuitionistic fuzzy labeling graph* $H = (V, \tau, \rho)$ *is called an intuitionistic fuzzy labeling subgraph of* $G = (V, \sigma, \mu)$ *if* $\tau_1(u) \leq \sigma_1(u)$, $\tau_2(u) \geq \sigma_2(u)$ *for all* $u \in V$ *and* $\rho_1(u, v) \leq \mu_1(u, v)$, $\rho_2(u, v) \geq \mu_2(u, v)$ *for all edges* (u, v).

Theorem 3.1. *If* $H = (V, \tau, \rho)$ *is an intuitionistic fuzzy labeling subgraph of* $G = (V, \sigma, \mu)$, *then* $\rho_1^\infty(u, v) \leq \mu_1^\infty(u, v)$ *and* $\rho_2^\infty(u, v) \geq \mu_2^\infty(u, v)$ *for all* $u, v \in V$.

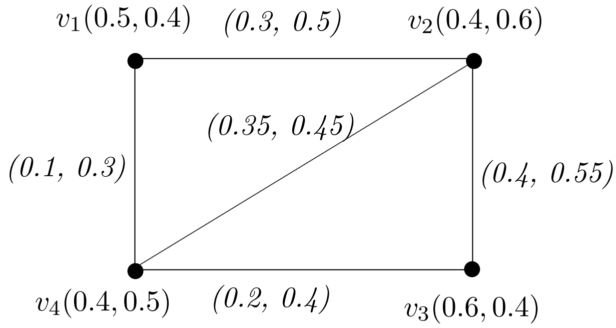

FIGURE 2. An intuitionistic fuzzy labeling graph

Proof. Let $G = (V, \sigma, \mu)$ be any intuitionistic fuzzy labeling graph and $H = (V, \tau, \rho)$ be its subgraph. Let (u, v) be any path in G then its strength be $(\mu_1^\infty(u, v), \mu_2^\infty(u, v))$. Since H is a subgraph of G, then $\tau_1(u) \leq \sigma_1(u)$, $\rho_1(u, v) \leq \mu_1(u, v)$ and $\tau_2(u) \geq \sigma_2(u)$, $\rho_2(u, v) \geq \mu_2(u, v)$, which implies that $\rho_1^\infty(u, v) \leq \mu_1^\infty(u, v)$ and $\rho_2^\infty(u, v) \geq \mu_2^\infty(u, v)$ for all $u, v \in V$.

We denote $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$, throughout the paper.

Theorem 3.2. *The union of any two intuitionistic fuzzy labeling graphs $G' = (V', \sigma', \mu')$ and $G'' = (V'', \sigma'', \mu'')$ is also an intuitionistic fuzzy labeling graph, if the membership and non-membership values of the edges between G' and G'' are distinct.*

Proof. Let $G' = (V', \sigma', \mu')$ and $G'' = (V'', \sigma'', \mu'')$ be any two intuitionistic fuzzy labeling graphs such that the membership and non-membership values of the edges between G' and G'' are distinct. Let $G = (V, \sigma, \mu)$ be the union of two intuitionistic fuzzy labeling graphs G' and G''. Now, we shall prove that G is an intuitionistic fuzzy labeling graph. Now,

$$\sigma_1(u) = \begin{cases} \sigma_1'(u), & if \quad u \in V' - V'' \\ \sigma_1''(u), & if \quad u \in V'' - V' \\ \sigma_1'(u) \vee \sigma_1''(u), & if \quad u \in V' \cap V'' \end{cases}$$

$$\sigma_2(u) = \begin{cases} \sigma_2'(u), & if \quad u \in V' - V'' \\ \sigma_2''(u), & if \quad u \in V'' - V' \\ \sigma_2'(u) \wedge \sigma_2''(u), & if \quad u \in V' \cap V'' \end{cases}$$

and

$$\mu_1(u, v) = \begin{cases} \mu_1'(u, v), & if \quad (u, v) \in E' - E'' \\ \mu_1''(u, v), & if \quad (u, v) \in E'' - E' \\ \mu_1'(u, v) \vee \mu_1''(u, v), & if \quad (u, v) \in E' \cap E'' \end{cases}$$

$$\mu_2(u, v) = \begin{cases} \mu_2'(u, v), & if \quad (u, v) \in E' - E'' \\ \mu_2''(u, v), & if \quad (u, v) \in E'' - E' \\ \mu_2'(u, v) \wedge \mu_2''(u, v), & if \quad (u, v) \in E' \cap E'' \end{cases}$$

Then membership and non-membership values of the vertices and edges are distinct. Hence, $G = (V, \sigma, \mu)$ is an intuitionistic fuzzy labeling graph.

Definition 3.3. *Let $G = (V, \sigma, \mu)$ be an intuitionistic fuzzy labeling graph. The strength of the path P of n edges e_i for $i = 1, 2, \ldots, n$ is denoted by $S(P) = (S_1(P), S_2(P))$ and defined by $S_1(P) = \min_{1 \leq i \leq n} \mu_1(e_i)$ and $S_2(P) = \max_{1 \leq i \leq n} \mu_2(e_i)$.*

Definition 3.4. *Let $G = (V, \sigma, \mu)$ be an intuitionistic fuzzy labeling graph. The strength of connectedness of a pair of vertices $u, v \in V$, denoted by $CONN_G(u, v) = (CONN_{1G}(u, v), CONN_{2G}(u, v))$ and is defined by $CONN_{1G}(u, v) = \max\{S_1(P) |\, P\, is\, a\, u\text{-}v path\, in\, G\}$ and $CONN_{2G}(u, v) = \min\{S_2(P) |\, P\, is\, a\, u\text{-}v path\, in\, G\}$.*

If u and v are isolated vertices of G, then $CONN_G(u, v) = (0, 0)$.

Example 3.2. *Consider the following intuitionistic fuzzy labeling graph G. Here $CONN_G(v_1, v_2) = (0.3, 0.5)$, $CONN_G(v_1, v_3) = (0.3, 0.4)$, $CONN_G(v_1, v_4) = (0.2, 0.3)$ and so on, which is shown in Figure 3.*

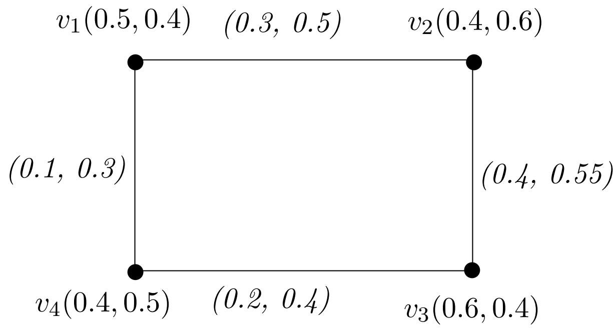

FIGURE 3. Connectedness in intuitionistic fuzzy labeling graph

Proposition 3.1. *Let G be an intuitionistic fuzzy labeling graph and H is an intuitionistic fuzzy labeling subgraph of G. Then for every pair of vertices $u, v \in V$, we have $CONN_{1H}(u, v) \leq CONN_{1G}(u, v)$ and $CONN_{2H}(u, v) \geq CONN_{2G}(u, v)$.*

Definition 3.5. *A $u-v$ path in an intuitionistic fuzzy labeling graph G is called a strongest $u - v$ path if $S_1(P) = CONN_{1G}(u, v)$ and $S_2(P) = CONN_{2G}(u, v)$.*

Definition 3.6. *Let G be an intuitionistic fuzzy labeling graph. A node w is called a partial cut node ($p-$cut node) of G if there exists a pair of nodes $u, v \in G$ such that $u \neq v \neq w$ and $CONN_{1(G-w)}(u, v) < CONN_{1G}(u, v)$, $CONN_{2(G-w)}(u, v) > CONN_{2G}(u, v)$.*

A connected intuitionistic fuzzy labeling graph having no p-cut nodes is called a partial block ($p-$block).

Example 3.3. *Let G be an intuitionistic fuzzy labeling graph, which is shown in Figure 4. Node v_2 is a partial cut node, since $CONN_{1(G-v_2)}(v_1, v_3) = 0.02 < 0.03 = CONN_{1G}(v_1, v_3)$ and $CONN_{2(G-v_2)}(v_1, v_3) = 0.6 > 0.3 = CONN_{2G}(v_1, v_3)$.*

Definition 3.7. *Let G be an intuitionistic fuzzy labeling graph. An arc $e = (u, v)$ is called partial bridge (p- bridge) if $CONN_{1(G-e)}(u, v) < CONN_{1G}(u, v)$, $CONN_{2(G-e)}(u, v) > CONN_{2G}(u, v)$.*

A p-bridge is said to be a partial bond (p-bond) if $CONN_{1(G-e)}(x, y) < CONN_{1G}(x, y)$, $CONN_{2(G-e)}(x, y) > CONN_{2G}(x, y)$ with at least one of x or y different from both u and v and is said to be a partial cut bond (p-cut bond) if both x or y are different from u and v.

Example 3.4. *In Figure 5, all arcs except the arc (v_3, v_4) are partial bridge. In particular, arc (v_1, v_2) is a partial cut bond, since $CONN_{1(G-(v_1,v_2))}(v_3, v_4) = 0.02 < 0.03 = CONN_{1G}(v_3, v_4)$ and $CONN_{2(G-(v_1,v_2))}(v_3, v_4) = 0.5 > 0.4 = CONN_{2G}(v_3, v_4)$.*

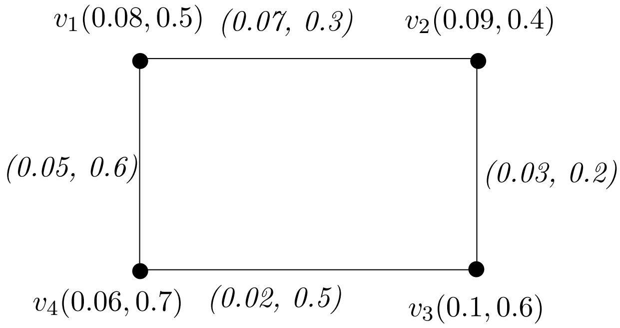

FIGURE 4. An intuitionistic fuzzy labeling graph in which v_2 is a partial cut node

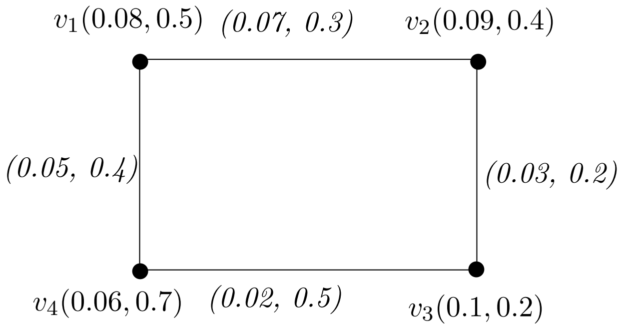

FIGURE 5. An intuitionistic fuzzy labeling graph in which all arcs except (v_3, v_4) are partial bridge

Definition 3.8. *Let G be an intuitionistic fuzzy labeling graph and C, a cycle in G. Then,*
(i) C is called a strong cycle if all arcs in C are strong.
(ii) An arc $e = (x, y) \in E$ is called $\alpha-$strong if $CONN_{1(G-e)}(x, y) < \mu_1(u, v)$ and $CONN_{2(G-e)}$
$(x, y) > \mu_2(u, v)$; a $\delta-$arc if $CONN_{1(G-e)}(x, y) > \mu_1(u, v)$ and $CONN_{2(G-e)}(x, y) < \mu_2(u, v)$.
(iii) A $u - v$ path P in G is called a strong $u - v$ path if all the arcs of P are strong. In particular, if all the arcs of P are $\alpha-$strong, then P is called $\alpha-$strong path.

Clearly, an arc $e = (x, y)$ is strong if it is $\alpha-$strong. If (x, y) is strong arc, then x and y are said to be strong neighbors of each other.

Example 3.5. *In Figure 6, the arcs (v_1, v_2), (v_1, v_3), (v_1, v_4) are $\alpha-$strong and the arc (v_3, v_4) is a $\delta-$arc. Also $P = v_4v_1v_2$ is an $\alpha-$strong path. But, there is no any strong cycle.*

Theorem 3.3. *Let G be a connected intuitionistic fuzzy labeling graph and let u and v be any two nodes in G. Then there exists a strong path from u to v.*

Proof. Suppose that $G = (V, \sigma, \mu)$ is a connected intuitionistic fuzzy labeling graph. Let u and v be any two nodes of G. If the arc (u, v) is strong, then there is nothing to prove. Otherwise, either (u, v) is a δ-arc or there exist a path of length more then one from u to v.

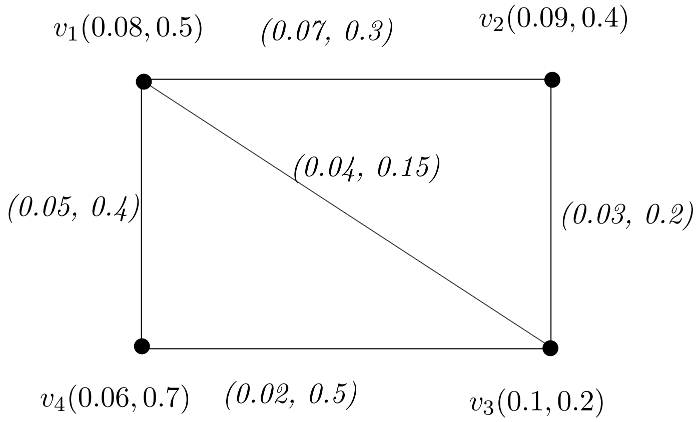

FIGURE 6. $v_4v_1v_2$ is an α-strong path

In the first case, we can find a path P (say) such that $S_1(P) > \mu_1(u,v)$ and $S_2(P) < \mu_2(u,v)$. In either case, the path from u to v of length more then one. If some arc on this path is not strong, replace it by a path having more strength. This argument can not repeated arbitrary often; hence we can find a path from u to v on which all the arcs are strong. Hence, there exists a strong path from u to v.

Theorem 3.4. *A connected intuitionistic fuzzy labeling graph G is a partial block if and only if any two nodes $x, y \in V$ such that (x, y) is not $\alpha-$ strong are joined by two internally disjoint strongest path.*

Proof. Suppose that G is a partial block. Let $x, y \in V$ such that (x, y) is not $\alpha-$ strong arc. Now, we shall prove that there exist two internally disjoint strongest $x - y$ paths. If not, i.e there exist exactly one internally disjoint strongest $x-y$ path in G. Since (x, y) is not $\alpha-$ strong, length of all strongest $x - y$ path must be at least two. Also for all strongest $x - y$ paths in G, there must be a common vertex. Let z be a such node in G. Then $CONN_{1(G-z)}(x,y) > CONN_{1G}(x,y)$ and $CONN_{2(G-z)}(x,y) < CONN_{2G}(x,y)$, which contradict the fact that G has no P-cut nodes. Hence there exist two internally disjoint strongest $x - y$ paths.

Conversely, let any two nodes of G are joined by two internally disjoint strongest paths. Let w be a node in G. For any pair of nodes $u, v \in V$ such that $u \neq v \neq w$, there always exists a strongest path not containing w. So, w can not be a $p-$cut node. Hence G is a partial block.

4. Intuitionistic fuzzy labeling tree

Next, we define intuitionistic fuzzy labeling tree as follows.

Definition 4.1. *A graph $G = (V, \sigma, \mu)$ is said to be intuitionistic fuzzy labeling tree, if it has intuitionistic fuzzy labeling and an intuitionistic fuzzy spanning subgraph $F = (V, \sigma, \rho)$ which is a tree, where for all arcs (u, v) not in F, $\mu_1(u,v) < \rho_1^\infty(u,v)$ and $\mu_2(u,v) > \rho_2^\infty(u,v)$.*

Theorem 4.1. *If G is an intuitionistic fuzzy labeling tree, then the arcs of intuitionistic fuzzy spanning subgraph F are intuitionistic fuzzy bridges of G.*

Proof. Let $G = (V, \sigma, \mu)$ be an intuitionistic fuzzy labeling tree and $F = (V, \sigma, \rho)$ be its spanning subgraph. Let (x, y) be an arc in F. Then $\rho_1^\infty(x,y) < \mu_1(x,y) \leq \mu_1^\infty(x,y)$ and $\rho_2^\infty(x,y) > \mu_2(x,y) \geq \mu_2^\infty(x,y)$, which implies that the arc (x, y) is an intuitionistic fuzzy bridge of G. Since the arc (x, y) is an arbitrary, then the arcs of F are intuitionistic fuzzy bridges of G.

Theorem 4.2. *Every intuitionistic fuzzy labeling graph is an intuitionistic fuzzy labeling tree.*

Proof. Let $G = (V, \sigma, \mu)$ be an intuitionistic fuzzy labeling graph. Since μ is bijective, each and every vertex of G will have at least one arc as intuitionistic fuzzy bridge. Therefore, the spanning subgraph F will exist, such that whose arcs are intuitionistic fuzzy bridges. Hence, by Theorem 5, every intuitionistic fuzzy labeling graph is an intuitionistic fuzzy labeling tree.

4.1. **Partial intuitionistic fuzzy labeling tree.** Finally, we define intuitionistic fuzzy labeling tree as follows.

Definition 4.2. *A connected intuitionistic fuzzy labeling graph $G = (V, \sigma, \mu)$ is called a partial intuitionistic fuzzy labeling tree if G has a spanning subgraph $F = (V, \sigma, \mu')$ which is a tree, where for all arc (x, y) of G which are not in F, $CONN_{1G}(x, y) > \mu_1(x, y)$ and $CONN_{2G}(x, y) < \mu_2(x, y)$.*

When the graph G is not connected and the above condition is satisfied by all components of G, then G is called a partial forest.

Example 4.1. *In Figure 7, $CONN_G(v_1, v_2) = (0.07, 0.3)$, $CONN_G(v_2, v_3) = (0.03, 0.2)$, $CONN_G(v_3, v_4) = (0.03, 0.4)$ and $CONN_G(v_1, v_4) = (0.05, 0.4)$. The following is an example of a partial intuitionistic fuzzy labeling tree. By removing the arc (v_3, v_4) we will get a spanning tree F. Note that for the arc (v_3, v_4), $CONN_{1G}(v_3, v_4) = 0.03 > 0.02 = \mu_1(v_3, v_4)$ and $CONN_{2G}(v_3, v_4) = 0.4 < 0.5 = \mu_2(v_3, v_4)$.*

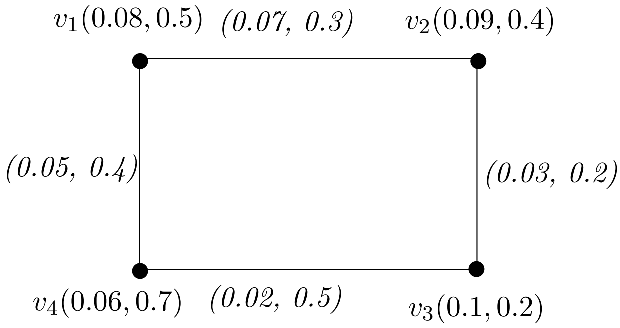

FIGURE 7. Partial intuitionistic fuzzy labeling tree

Next we have a characterization of a partial intuitionistic fuzzy labeling tree.

Theorem 4.3. *A connected intuitionistic fuzzy labeling graph G is a partial tree if and only if any cycle C of G, there exists an arc $e = (u, v)$ such that $\mu_1(e) < CONN_{1(G-e)}(u, v)$ and $\mu_2(e) > CONN_{2(G-e)}(u, v)$, where $G - e$ is the subgraph of G obtained by deleting the arc e from G.*

Proof. Let G be a connected intuitionistic fuzzy labeling graph. If there is no cycle, it is clearly a tree and hence it is a partial tree. If there exists cycles in G, let (u, v) be an arc belonging to the cycle C with mimimum membership weight and maximum non-membership weight in G, then delete the arc (u, v) from G. If there are still cycles in the graph, we can repeat the process. Not at each stage no previously deleted arc is strongest the arc being currently deleted. When no cycle remain, the resulting subgraph is a tree F. Let (u, v) not be an arc in F. Then (u, v) is one of the arc deleted in the process of constructing F. Since F is a tree and the arc (u, v) having minimum membership value and maximum non-membership value from the arcs of a cycle in G, it follows that there exists a path from u to v whose membership value grater than $\mu_1(u, v)$ and non-membership value less than $\mu_2(u, v)$ and that does not involve (u, v) or any arcs deleted prior to it. If this path involves arcs that were deleted later, the path can be futher diverted and so on. This process stabilizer with a path consisting entirely the arcs of F. Thus G is a partial intuitionistic fuzzy labeling tree.

Conversely, if G is a partial intuitionistic fuzzy labeling tree and P is cycle, then some arc $e = (u, v)$ of P does not belong to F. Thus by definition we have $\mu_1(e) < CONN_{1(G-e)}(u, v) < CONN_{1G}(u, v)$ and $\mu_2(e) > CONN_{2(G-e)}(u, v) > CONN_{2G}(u, v)$.

Theorem 4.4. *If there exist at most one strongest path between any two nodes of G, then G must be a partial forest.*

Proof. Suppose G is not a partial forest. Then there is a cycle C in G such that $\mu_1(u, v) \geq CONN_{1G}(u, v)$ and $\mu_2(u, v) \leq CONN_{2G}(u, v)$, for all arcs (u, v) of the cycle C. Thus (u, v) is the strongest path from u to v. If we choose (u, v) to be a weakest arc of C, it follows that the rest of the cycle C is also a strongest path from u to v, a contradiction. Hence, G must be a partial forest.

Theorem 4.5. *If G is a partial tree and not a tree, then there exists at least one arc (u, v) for which $\mu_1(u, v) < CONN_{1G}(u, v)$ and $\mu_2(u, v) > CONN_{2G}(u, v)$.*

Proof. If G is a partial tree, then by definition there exists a spanning tree F such that $\mu_1(u, v) < CONN_{1G}(u, v)$ and $\mu_2(u, v) > CONN_{2G}(u, v)$, for all arcs (u, v) not in F. By hypothesis, there exists at least one such arc (since G is not a tree) and hence the result follows.

Theorem 4.6. *If G is a partial tree and F, the spanning tree, then the arcs of F are the partial bridges of G.*

Proof. Let (u, v) be an arc in F. Then this arc is the unique path between u and v in F. If there is no other paths in G from u and v, then clearly (u, v) is a bridge of G and hence a partial bridge of G. If there exists a path say P from u to v in G, then P will definitely contain an arc (x, y) which is not in F such that $\mu_1(u, v) < CONN_{1G}(u, v)$ and $\mu_2(u, v) > CONN_{2G}(u, v)$. Then (u, v) is not a weakest arc of any cycle in G and hence (u, v) is a partial bridge.

5. Conclusion

Connectivity concepts are the major key in intuitionistic fuzzy graph problems. Here, we have presented new connectivity concepts in intuitionistic fuzzy labeling graphs. Also, we defined strong arc, partial cut node, bridge and block by connectivity concepts of intuitionistic fuzzy graph. Also, intuitionistic fuzzy labeling tree and partial intuitionistic fuzzy labeling tree were defined and established many interesting properties on them, which plays a major role in many areas of science and technology. We are extending our research work to (1) Covering problem on intuitionistic fuzzy graphs, (2) Chromatic number in intuitionistic fuzzy graphs and (3) Colouring of intuitionistic fuzzy graphs.

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