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DIVISOR CORDIAL LABELING IN THE CONTEXT OF JOIN AND BARYCENTRIC SUBDIVISION

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Abstract. A divisor cordial labeling of a graph G with vertex set $V(G)$ is a bijection f from $V(G)$ to $\{1, 2, \ldots, |V(G)|\}$ such that an edge $e = uv$ is assigned the label 1 if $f(u)|f(v)$ or $f(v)|f(u)$ and the label 0 otherwise, then $|e_f(0) - e_f(1)| \leq 1$. A graph which admits divisor cordial labeling is called a divisor cordial graph. In this paper we prove that the graphs $AC_n + K_1$, $\left(\bigcup_{i=1}^{n} C_{m_i}\right) + K_1$, $\left(P_m \cup \bigcup_{i=1}^{n} C_{m_i}\right) + K_1$ and $\left(K_{1,m} \cup \bigcup_{i=1}^{n} C_{m_i}\right) + K_1$ are divisor cordial graphs. In addition to this we prove that the barycentric subdivision of complete bipartite graphs $K_{2,n}$ and $K_{3,n}$ admit divisor cordial labeling.

Keywords: Divisor Cordial Labeling; Join; Barycentric Subdivision.

AMS Subject Classification: 05C38,05C69,05C76.

1. Introduction

Throughout this work, by a graph we mean finite, undirected, simple graph $G = (V(G), E(G))$ of order $|V(G)|$ and size $|E(G)|$. For any undefined notations and terminology we follow Gross and Yellen[5] while for number theory we follow Burton[2].

Definition 1.1. If the vertices are assigned values subject to certain condition(s) then it is known as *graph labeling.*

A dynamic survey on graph labeling is regularly updated by Gallian[4].

Definition 1.2. A mapping $f : V(G) \rightarrow \{0, 1\}$ is called *binary vertex labeling* of G and $f(v)$ is called the label of the vertex v of G under f.

Notation 1.1. If for an edge $e = uv$, the induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ is given by $f^*(e = uv) = |f(u) - f(v)|$. Then
$v_f(i)$=number of vertices of G having label i under f,
$e_f(i)$=number of edges of G having label i under f^*.

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Definition 1.3. A binary vertex labeling f of a graph G is called a *cordial labeling* if $|v_f(0)-v_f(1)| \leq 1$ and $|e_f(0)-e_f(1)| \leq 1$. A graph which admits cordial labeling is called a *cordial graph.*

The concept of cordial labeling was introduced by Cahit[3] in which he investigated several results on this concept. After this many labeling techniques are also introduced with minor changes in cordial labeling. The product cordial labeling, total product cordial labeling, prime cordial labeling and divisor cordial labeling are some of them. The present work is focused on divisor cordial labeling.

Definition 1.4. Let $G = (V(G), E(G))$ be a simple graph and $f : V(G) \rightarrow \{1, 2, \ldots, |V(G)|\}$ be a bijection. For each edge $e = uv$, assign the label 1 if either $f(u)|f(v)$ or $f(v)|f(u)$ and the label 0 otherwise. The function f is called a *divisor cordial labeling* if $|e_f(0)-e_f(1)| \leq 1$.
A graph which admits divisor cordial labeling is called a *divisor cordial graph.*

The concept of divisor cordial labeling was introduced by Varatharajan et al.[7] and they proved the following results:

- The star graph $K_{1,n}$ is divisor cordial.
- The complete bipartite graph $K_{2,n}$ is divisor cordial.
- The complete bipartite graph $K_{3,n}$ is divisor cordial.

Vaidya and Shah[6] proved that

- $S'(B_{n,n})$ is a divisor cordial graph.
- $DS(B_{n,n})$ is a divisor cordial graph.

Kanani and Bosmia[1] proved that

- The bistar $B_{m,n}$ is a divisor cordial graph.
- $S'(B_{m,n})$ is a divisor cordial graph.
- $DS(B_{m,n})$ is a divisor cordial graph.
- $D_2(B_{m,n})$ is a divisor cordial graph.
- Restricted $B^2_{m,n}$ is a divisor cordial graph.
- The barycentric subdivision $S(B_{m,n})$ of the bistar $B_{m,n}$ is a divisor cordial graph.
- $B_{m,n} \odot K_1$ is a divisor cordial graph.

Definition 1.5. *Armed crown* is the graph obtained by attaching a path P_2 at each vertex of cycle C_n. It is denoted by AC_n, where n is the number of vertices in cycle C_n.

Definition 1.6. Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs with no vertex in common. The join of G_1 and G_2, denoted by G_1+G_2 is the graph with vertex set $V(G_1+G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1+G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$.

Definition 1.7. Let $G = (V(G), E(G))$ be a graph. Let $e = uv$ be an edge of G and w is not a vertex of G. The edge e is *subdivided* when it is replaced by the edges $e^{'} = uw$ and $e^{''} = wv$.

Definition 1.8. Let $G = (V(G), E(G))$ be a graph. If every edge of graph G is subdivided, then the resulting graph is called *barycentric subdivision* of graph G.
In other words barycentric subdivision is the graph obtained by inserting a vertex of degree 2 into every edge of original graph.
The barycentric subdivision of any graph G is denoted by $S(G)$.

2. Main Results

Theorem 2.1. *$AC_n + K_1$ is a divisor cordial graph.*

Proof. Let AC_n be the arm crown with the vertex set $\{u_i, w_i, v_i : 1 \leq i \leq n\}$, where each v_i is a pendant vertex, each w_i is a vertex of degree two and each u_i is a vertex of degree three for all $i = 1, 2, \ldots, n$. Let u_0 be the vertex of K_1. If $G = AC_n + K_1$ then $V(G) = V(AC_n) \cup \{u_0\}$ and $E(G) = E(AC_n) \cup \{u_0u_i, u_0w_i, u_0v_i : 1 \leq i \leq n\}$. We note that $|V(G)| = 3n + 1$ and $|E(G)| = 6n$.

Define vertex labeling $f : V(G) \to \{1, 2, \ldots, 3n + 1\}$ as follows:
$f(u_0) = 1$.
$f(w_i) = 3i; 1 \leq i \leq n$.

$$f(u_i) = \begin{cases} 2; & i = 1 \\ f(w_i) - 1; & i \equiv 0(mod\ 2) \text{ and } 2 \leq i \leq n \\ f(w_i) - 2; & i \equiv 1(mod\ 2) \text{ and } 2 \leq i \leq n \end{cases}$$

$$f(v_i) = \begin{cases} f(w_i) + 2; & i \equiv 0(mod\ 2) \text{ and } 1 \leq i \leq n-1 \\ f(w_i) + 1; & i \equiv 1(mod\ 2) \text{ and } 1 \leq i \leq n-1 \\ 3n + 1; & i = n \end{cases}$$

In view of the above defined labeling pattern we have $e_f(0) = e_f(1) = 3n$.
Thus, $|e_f(0) - e_f(1)| \leq 1$.
Hence, $AC_n + K_1$ is a divisor cordial graph. □

Example 2.1. Divisor cordial labeling of the graph $AC_6 + K_1$ is shown in the FIGURE 1.

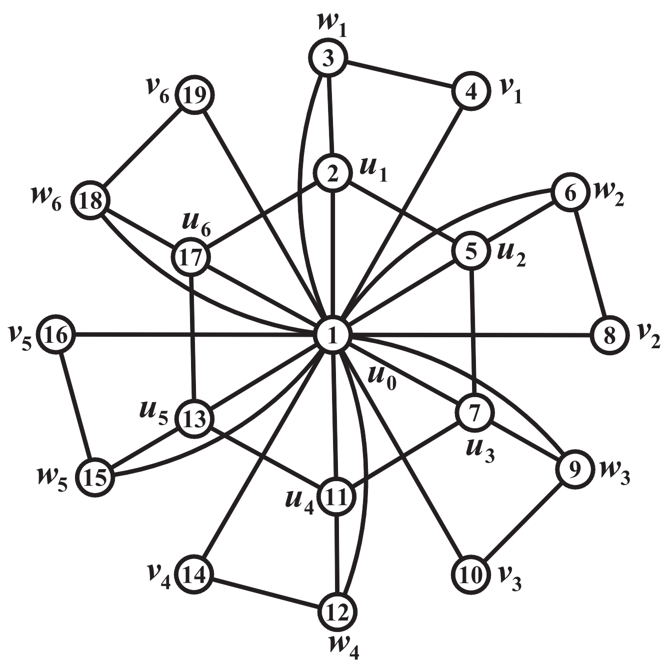

Figure 1. Divisor cordial labeling of $AC_6 + K_1$.

Theorem 2.2. $\left(\bigcup_{i=1}^{n} C_{m_i}\right) + K_1$ *is a divisor cordial graph.*

Proof. Let C_{m_i} be the i^{th} cycle with the consecutive vertices $v_{i1}, v_{i2}, \ldots, v_{im_i}$ for $1 \leq i \leq n$ and v_0 be the vertex of K_1. If $G = \left(\bigcup_{i=1}^{n} C_{m_i}\right) + K_1$ then $V(G) = \bigcup_{i=1}^{n} V(C_{m_i}) \cup \{v_0\}$ and $E(G) = \bigcup_{i=1}^{n} E(C_{m_i}) \cup \{v_0v_{ij} : 1 \leq j \leq m_i, 1 \leq i \leq n\}$. We note that $|V(G)| = 1 + m_1 + m_2 + \ldots + m_n$ and $|E(G)| = 2(m_1 + m_2 + \ldots + m_n)$. The proof is divided into following two cases.

Case 1: $n = 1$.
Here, $G = C_{m_1} + K_1$.
Therefore, G is wheel W_{m_1} and is a divisor cordial graph as proved in [7].

Case 2: $n > 1$.
Define vertex labeling $f : V(G) \to \{1, 2, \ldots, 1 + m_1 + m_2 + \ldots + m_n\}$ as follows:
Let p_1 be the largest prime number such that $p_1 \leq 1 + m_1 + m_2$ and p_2 be the second largest prime number such that $p_2 < p_1 \leq 1 + m_1 + m_2$.
$f(v_0) = 1$.
$f(v_{1m_1}) = p_1$, $f(v_{2m_2}) = p_2$.
Now, label the vertices $v_{11}, v_{12}, \ldots, v_{1(m_1-1)}, v_{21}, v_{22}, \ldots, v_{2(m_2-1)}$ successively from the ordered set $\{2, 3, 4, \ldots, 1 + m_1 + m_2\} - \{p_1, p_2\}$.

$$f(v_{ij}) = \begin{cases} 1 + m_1 + m_2 + \ldots + m_{i-1} + 2j; & 1 \leq j \leq \left\lfloor \frac{m_i}{2} \right\rfloor \text{ and } 3 \leq i \leq n \\ 2 + m_1 + m_2 + \ldots + m_{i-1} + 2(m_i - j); & \left\lfloor \frac{m_i}{2} \right\rfloor < j \leq m_i \text{ and } 3 \leq i \leq n \end{cases}$$

In view of the above defined labeling pattern we have $e_f(0) = e_f(1) = m_1 + m_2 + \ldots + m_n$.
Thus, $|e_f(0) - e_f(1)| \leq 1$.
Hence, $\left(\bigcup_{i=1}^{n} C_{m_i}\right) + K_1$ is a divisor cordial graph. $\square$

Example 2.2. Divisor cordial labeling of the graph $(C_3 \cup C_5 \cup C_4) + K_1$ is shown in the FIGURE 2.

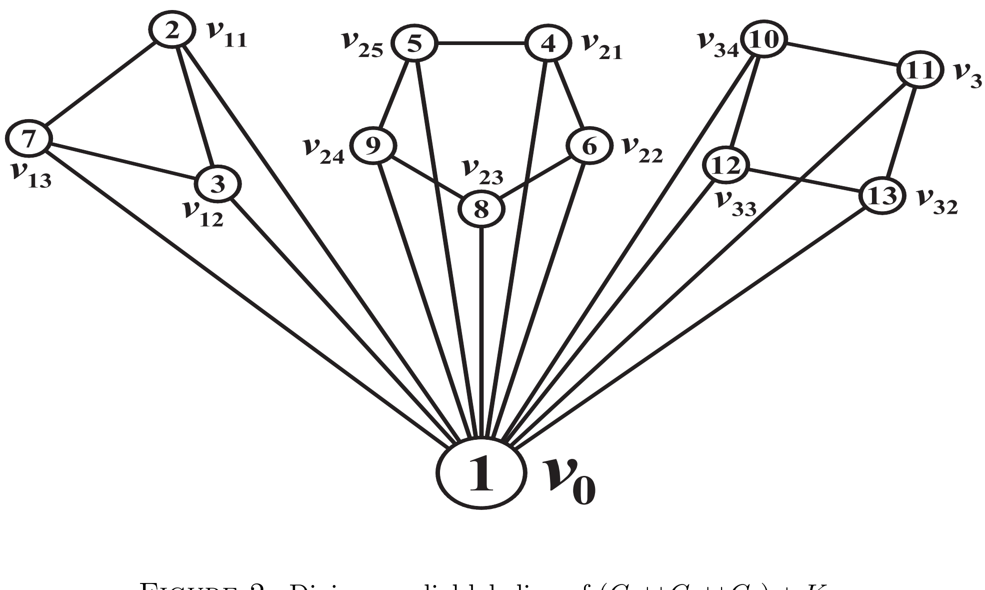

FIGURE 2. Divisor cordial labeling of $(C_3 \cup C_5 \cup C_4) + K_1$.

Theorem 2.3. $\left(P_m \cup \bigcup_{i=1}^{n} C_{m_i}\right) + K_1$ *is a divisor cordial graph.*

Proof. Let P_m be the path with consecutive vertices $u_1, u_2, \ldots, u_m$ and C_{m_i} be the i^{th} cycle with the consecutive vertices $v_{i1}, v_{i2}, \ldots, v_{im_i}$ for $1 \leq i \leq n$. v_0 be the vertex of K_1. If $G = \left(P_m \cup \bigcup_{i=1}^{n} C_{m_i}\right) + K_1$ then $V(G) = V(P_m) \cup \bigcup_{i=1}^{n} V(C_{m_i}) \cup \{v_0\}$ and $E(G) = E(P_m) \cup \bigcup_{i=1}^{n} E(C_{m_i}) \cup \{v_0 v_{ij}, v_0 u_k : 1 \leq j \leq m_i, 1 \leq i \leq n, 1 \leq k \leq m\}$. We note that $|V(G)| = 1+m+m_1+m_2+\ldots+m_n$ and $|E(G)| = 2(m+m_1+m_2+\ldots+m_n)-1$.

Define vertex labeling $f : V(G) \to \{1, 2, \ldots, 1+m+m_1+m_2+\ldots+m_n\}$ as follows:
$f(v_0) = 1$.
$f(u_k) = 1+k; 1 \leq k \leq m$.

$$f(v_{1j}) = \begin{cases} 1+m+2j; & 1 \leq j \leq \left\lfloor \frac{m_1}{2} \right\rfloor \\ 2+m+2(m_1-j); & \left\lfloor \frac{m_1}{2} \right\rfloor < j \leq m_1 \end{cases}$$

$$f(v_{ij}) = \begin{cases} 1+m+m_1+m_2+\ldots+m_{i-1}+2j; & 1 \leq j \leq \left\lfloor \frac{m_i}{2} \right\rfloor \text{ and } 2 \leq i \leq n \\ 2+m+m_1+m_2+\ldots+m_{i-1}+2(m_i-j); & \left\lfloor \frac{m_i}{2} \right\rfloor < j \leq m_i \text{ and } 2 \leq i \leq n \end{cases}$$

In view of the above defined labeling pattern we have $e_f(0) = m+m_1+m_2+\ldots+m_n-1$ and $e_f(1) = m+m_1+m_2+\ldots+m_n$.
Thus, $|e_f(0) - e_f(1)| \leq 1$.
Hence, $\left(P_m \cup \bigcup_{i=1}^{n} C_{m_i}\right) + K_1$ is a divisor cordial graph. □

Example 2.3. Divisor cordial labeling of the graph $(P_7 \cup C_4 \cup C_3 \cup C_5 \cup C_4 \cup C_5) + K_1$ is shown in the FIGURE 3.

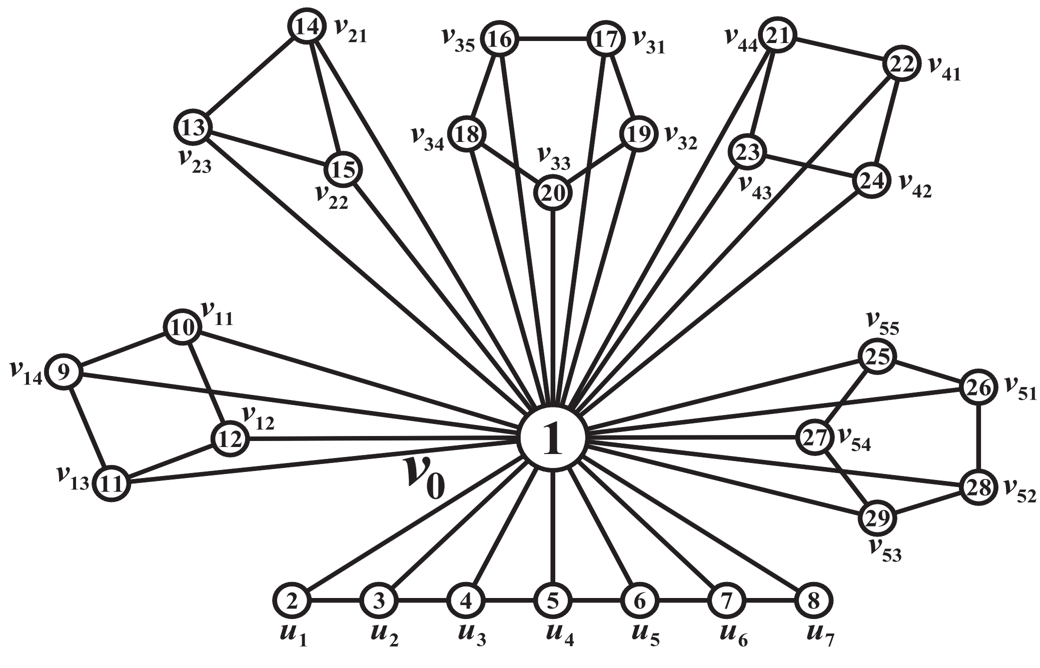

FIGURE 3. Divisor cordial labeling of $(P_7 \cup C_4 \cup C_3 \cup C_5 \cup C_4 \cup C_5) + K_1$.

Theorem 2.4. $\left(K_{1,m} \cup \bigcup_{i=1}^{n} C_{m_i}\right) + K_1$ *is a divisor cordial graph.*

Proof. Let $K_{1,m}$ be the star with vertices $u_0, u_1, u_2, \ldots, u_m$, where u_0 is the apex vertex of star $K_{1,m}$ and C_{m_i} be the i^{th} cycle with the consecutive vertices $v_{i1}, v_{i2}, \ldots, v_{im_i}$ for $1 \leq i \leq n$. v_0 be the vertex of K_1. If $G = \left(K_{1,m} \cup \bigcup_{i=1}^{n} C_{m_i}\right) + K_1$ then $V(G) = V(K_{1,m}) \cup \bigcup_{i=1}^{n} V(C_{m_i}) \cup \{v_0\}$ and $E(G) = E(K_{1,m}) \cup \bigcup_{i=1}^{n} E(C_{m_i}) \cup \{v_0 v_{ij}, v_0 u_k : 1 \leq j \leq m_i, 1 \leq i \leq n, 0 \leq k \leq m\}$. We note that $|V(G)| = 2 + m + m_1 + m_2 + \ldots + m_n$ and $|E(G)| = 2(m + m_1 + m_2 + \ldots + m_n) + 1$.

Define vertex labeling $f : V(G) \to \{1, 2, \ldots, 2 + m + m_1 + m_2 + \ldots + m_n\}$ as follows:
Let p be the largest prime number such that $p \leq m + 2$.
$f(v_0) = 1$, $f(u_0) = p$.
Now, label the vertices $u_1, u_2, \ldots, u_m$ from the set $\{2, 3, 4, \ldots, m + 2\} - \{p\}$.

$$f(v_{1j}) = \begin{cases} 2 + m + 2j; & 1 \leq j \leq \left\lfloor \frac{m_1}{2} \right\rfloor \\ 3 + m + 2(m_1 - j); & \left\lfloor \frac{m_1}{2} \right\rfloor < j \leq m_1 \end{cases}$$

$$f(v_{ij}) = \begin{cases} 2 + m + m_1 + m_2 + \ldots + m_{i-1} + 2j; & 1 \leq j \leq \left\lfloor \frac{m_i}{2} \right\rfloor \text{ and } 2 \leq i \leq n \\ 3 + m + m_1 + m_2 + \ldots + m_{i-1} + 2(m_i - j); & \left\lfloor \frac{m_i}{2} \right\rfloor < j \leq m_i \text{ and } 2 \leq i \leq n \end{cases}$$

In view of the above defined labeling pattern we have $e_f(0) = m + m_1 + m_2 + \ldots + m_n$ and $e_f(1) = 1 + m + m_1 + m_2 + \ldots + m_n$.
Thus, $|e_f(0) - e_f(1)| \leq 1$.
Hence, $\left(K_{1,m} \cup \bigcup_{i=1}^{n} C_{m_i}\right) + K_1$ is a divisor cordial graph. □

Example 2.4. Divisor cordial labeling of the graph $(K_{1,6} \cup C_4 \cup C_5 \cup C_3) + K_1$ is shown in the FIGURE 4.

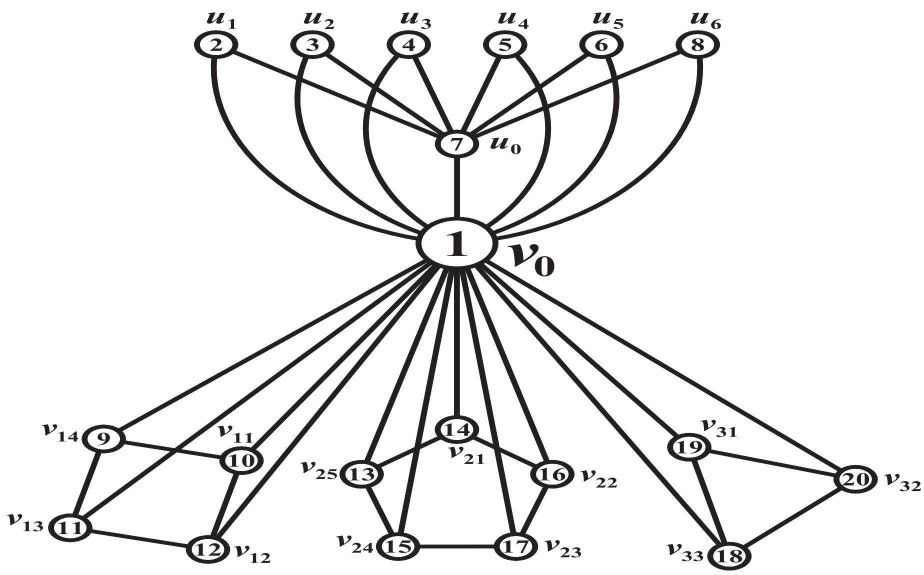

FIGURE 4. Divisor cordial labeling of $(K_{1,6} \cup C_4 \cup C_5 \cup C_3) + K_1$.

Theorem 2.5. *The barycentric subdivision $S(K_{2,n})$ of $K_{2,n}$ is a divisor cordial graph.*

Proof. Let $\{u, v\}$ and $\{v_1, v_2, \ldots, v_n\}$ be the bipartition of the complete bipartite graph $K_{2,n}$. Let $e_i = uv_i$, $e'_i = vv_i$ for $i = 1, 2, \ldots, n$. To obtain barycentric subdivision $G = S(K_{2,n})$ of $K_{2,n}$ subdivide each edge of $K_{2,n}$ by the vertices $u_1, u_2, \ldots, u_n, w_1, w_2, \ldots, w_n$ where each u_i is added between u and v_i for $i = 1, 2, \ldots, n$ and each w_i is added between v and v_i for $i = 1, 2, \ldots, n$. We note that $|V(G)| = 3n + 2$ and $|E(G)| = 4n$.

Define vertex labeling $f : V(G) \to \{1, 2, \ldots, 3n + 2\}$ as follows:
$f(u) = 1$, $f(v) = 2$.

$$f(v_i) = \begin{cases} 3i + 1; & i \equiv 0(mod\ 2) \text{ and } 1 \le i \le n \\ 3i; & i \equiv 1(mod\ 2) \text{ and } 1 \le i \le n \end{cases}$$

$$f(w_i) = \begin{cases} 3i; & i \equiv 0(mod\ 2) \text{ and } 1 \le i \le n \\ 3i + 1; & i \equiv 1(mod\ 2) \text{ and } 1 \le i \le n \end{cases}$$

$f(u_i) = 3i + 2; 1 \le i \le n$
In view of the above defined labeling pattern we have $e_f(0) = e_f(1) = 2n$.
Thus, $|e_f(0) - e_f(1)| \le 1$.
Hence, the barycentric subdivision $S(K_{2,n})$ of $K_{2,n}$ is a divisor cordial graph. □

Example 2.5. Divisor cordial labeling of the graph $S(K_{2,7})$ is shown in the FIGURE 5.

FIGURE 5. Divisor cordial labeling of $S(K_{2,7})$.

Theorem 2.6. *The barycentric subdivision $S(K_{3,n})$ of $K_{3,n}$ is a divisor cordial graph.*

Proof. Let $\{u, v, w\}$ and $\{x_1, x_2, \ldots, x_n\}$ be the bipartition of the complete bipartite graph $K_{3,n}$. Let $e_i = ux_i$, $e'_i = vx_i$ and $e"_i = wx_i$for $i = 1, 2, \ldots, n$. To obtain barycentric subdivision $G = S(K_{3,n})$ of $K_{3,n}$ subdivide each edge of $K_{3,n}$ by the vertices $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n$ where each u_i is added between u and x_i for $i = 1, 2, \ldots, n$, each v_i is added between v and x_i for $i = 1, 2, \ldots, n$ and each w_i is added between w and x_i for $i = 1, 2, \ldots, n$. We note that $|V(G)| = 4n+3$ and $|E(G)| = 6n$.

Define vertex labeling $f : V(G) \to \{1, 2, \ldots, 4n+3\}$ as follows:
$f(u) = 1$, $f(v) = 2$, $f(w) = 3$.

$$f(x_i) = \begin{cases} 4i+1; & i \equiv 0(mod\ 3) \text{ and } 1 \le i \le n \\ 4i+3; & i \equiv 1 \text{ or } 2(mod\ 3) \text{ and } 1 \le i \le n \end{cases}$$

$$f(u_i) = \begin{cases} 4i+2; & i \equiv 0 \text{ or } 2(mod\ 3) \text{ and } 1 \le i \le n \\ 4i+1; & i \equiv 1(mod\ 3) \text{ and } 1 \le i \le n \end{cases}$$

$$f(w_i) = \begin{cases} 4i+3; & i \equiv 0(mod\ 3) \text{ and } 1 \le i \le n \\ 4i+2; & i \equiv 1(mod\ 3) \text{ and } 1 \le i \le n \\ 4i+1; & i \equiv 2(mod\ 3) \text{ and } 1 \le i \le n \end{cases}$$

$f(v_i) = 4i; 1 \le i \le n$
In view of the above defined labeling pattern we have $e_f(0) = e_f(1) = 3n$.
Thus, $|e_f(0) - e_f(1)| \le 1$.
Hence, the barycentric subdivision $S(K_{3,n})$ of $K_{3,n}$ is a divisor cordial graph. □

Example 2.6. Divisor cordial labeling of the graph $S(K_{3,5})$ is shown in the Figure 6.

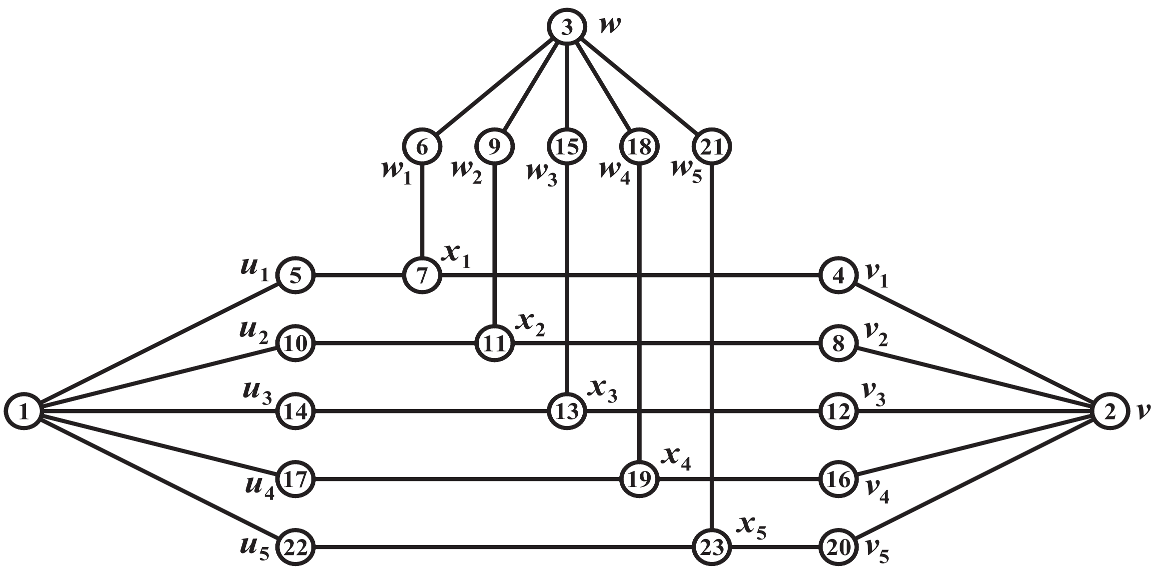

FIGURE 6. Divisor cordial labeling of $S(K_{3,5})$.

3. Concluding Remark

The divisor cordial labeling is a variant of cordial labeling. As all the graphs do not admit divisor cordial labeling it is very interesting to investigate graphs or graph families which are divisor cordial. Here we proved that the graphs $AC_n + K_1$, $\left(\bigcup_{i=1}^{n} C_{m_i}\right) + K_1$, $\left(P_m \cup \bigcup_{i=1}^{n} C_{m_i}\right) + K_1$ and $\left(K_{1,m} \cup \bigcup_{i=1}^{n} C_{m_i}\right) + K_1$ are divisor cordial graphs. We also proved that the barycentric subdivision of complete bipartite graphs $K_{2,n}$ and $K_{3,n}$ admit divisor cordial labeling.

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