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ON A ROBIN PROBLEM IN ORLICZ-SOBOLEV SPACES

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Abstract. In the present paper, we deal with the existence of solutions to a class of an elliptic equation with Robin boundary condition. The problem is settled in Orlicz-Sobolev spaces and the main tool used is Ekeland's variational principle.

Keywords: Non-homogeneous Robin problem, Variational method, Ekeland variational principle, Orlicz-Sobolev spaces.

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1. Introduction

In this article, we are concerned with a Robin problem settled in Orlicz-Sobolev spaces of the form

$$\begin{cases} -\mathrm{div}(a(|\nabla u(x)|)\nabla u(x)) + a(|u(x)|)u(x) &= \lambda f(u(x)), \ \ x \in \Omega \\ a(|\nabla u(x)|)\frac{\partial u(x)}{\partial \eta} + b(x)|u(x)|^{p-2}u(x) &= 0, \ \ x \in \partial\Omega \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ and has a Lipschitz boundary $\partial\Omega$, $\lambda > 0$ is a real parameter, η is the unit exterior vector on $\partial\Omega$, $b \in L^\infty(\partial\Omega)$ with $\inf_{x\in\partial\Omega} b(x) > 0$, and f is a real valued continuous function. The function $\varphi(t) := a(|t|)t$ is an increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. We want to remark that if we let $a(t) = |t|^{p-2}$, problem (1) turns into the well-known p-Laplace equation. p-Laplace equations have been studied by many authors because of their various applications to different disciplines see, e.g., [2, 20, 33] and references therein.

The study of variational problems in the classical Sobolev and Orlicz-Sobolev spaces is an interesting topic of research due to its significant role in many fields of mathematics, such as approximation theory, partial differential equations, calculus of variations, non-linear potential theory, the theory of quasiconformal mappings, non-Newtonian fluids, image processing, differential geometry, geometric function theory, probability theory, magnetostatics, and capillarity phenomena (see, e.g.,[3, 7, 8, 9, 10, 13, 14, 16, 23, 27, 30, 31]).

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Moreover, problem (1) posses more complicated nonlinearities, for example, it is inhomogeneous, so in the discussions, some special techniques will be needed. However, the inhomogeneous nonlinearities have important physical background. Therefore, equation (1) may represent a variety of mathematical models corresponding to certain phenomenons (see, e.g., [23]), e.g.,

(1) Nonlinear elasticity: $\varphi(t) = \left(1+t^2\right)^{\alpha} - 1$, $\alpha > \frac{1}{2}$,
(2) Plasticity: $\varphi(t) = t^{\alpha}\left(\log\left(1+t\right)\right)^{\beta}$, $\alpha \geq 1, \beta > 0$,
(3) Generalized Newtonian fluids: $\varphi(t) = \int_0^t s^{1-\alpha}\left(\sinh^{-1} s\right)^{\beta} ds$, $0 \leq \alpha \leq 1, \beta > 0$.

Problem (1) is settled in Orlicz-Sobolev spaces and treated by variational approach and the main medium is Ekeland's variational principle. In Section 2, we give the basic knowledge and preliminary results. In Section 3, we show that problem (1) has a nontrivial weak solution. At the end of Section 3 we provide an example to illustrate the main result, i.e. Theorem 3.1. To the authors' best knowledge, the results obtained in the present papers are not covered in the literature, and therefore, it has a potential to contribute it.

2. Preliminaries

To deal with problem (1), we use the theory of Orlicz-Sobolev spaces since problem (1) contains a nonhomogeneous function φ in the differential operator. Therefore, we start with some basic concepts of Orlicz-Sobolev spaces. For more details we refer the readers to the monographs [1], [29], [32], [35], and the papers [13], [23],[26], [31].

The function $a : (0,\infty) \to \mathbb{R}$ is a function such that the mapping, defined by

$$\varphi(t) := \begin{cases} a(|t|)t & \text{for } t \neq 0, \\ 0, & \text{for } t = 0, \end{cases} \tag{2}$$

is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. For the function φ above, let us define

$$\Phi(t) = \int_0^t \varphi(s)ds, \quad \bar{\Phi}(t) = \int_0^t \varphi^{-1}(s)ds \quad t \in \mathbb{R}, \tag{3}$$

then the functions Φ and $\bar{\Phi}$ are complementary N-functions, i.e. Young functions satisfying the following conditions: Φ is a convex, nondecreasing and continuous function; $\Phi(0) = 0$; $\Phi(t) > 0$ for all $t > 0$; $\lim_{t\to 0} \frac{\Phi(t)}{t} = 0$; $\lim_{t\to\infty} \frac{\Phi(t)}{t} = +\infty$ (see e.g., [1],[32],[35]). On the other hand, $\bar{\Phi}$ satisfies the following

$$\bar{\Phi}(t) = \sup\{st - \Phi(s) : \ s \geq 0\}, \quad t \geq 0.$$

Moreover the following Young inequality holds

$$st \leq \Phi(s) + \bar{\Phi}(t) \text{ for } t, s \in \mathbb{R}.$$

These functions allow us to define the Orlicz spaces $L_{\Phi}(\Omega)$ and $L_{\bar{\Phi}}(\Omega)$, respectively.
In the sequel, we use the following assumption:

$$1 < \varphi_0 := \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)} \leq \frac{t\varphi(t)}{\Phi(t)} \leq \varphi^0 := \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)} < \infty. \tag{4}$$

By help of assumption (4), the Orlicz space $L_{\Phi}(\Omega)$ coincides with the equivalence classes of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\int_{\Omega} \Phi(|u(x)|)dx < \infty, \tag{5}$$

and is equipped with the Luxembourg norm

$$|u|_\Phi := \inf\left\{k > 0 : \int_\Omega \Phi(\frac{|u(x)|}{k})\, dx \le 1\right\}. \tag{6}$$

For Orlicz spaces, Hölder inequality reads as follows (see [1],[35])

$$\int_\Omega uv\, dx \le 2\|u\|_{L_\Phi(\Omega)}\|u\|_{L_{\bar{\Phi}}(\Omega)} \quad \text{for all } u \in L_\Phi(\Omega) \text{ and } v \in L_{\bar{\Phi}}(\Omega).$$

The Orlicz-Sobolev space $W^1L_\Phi(\Omega)$ building upon $L_\Phi(\Omega)$ is the space defined by

$$W^1L_\Phi(\Omega) := \left\{u \in L_\Phi(\Omega) : \ \frac{\partial u}{\partial x_i} \in L_\Phi(\Omega),\ i = 1, 2, ..., N\right\}.$$

which becomes a Banach space under the norm

$$\|u\|_{1,\Phi} := |u|_\Phi + \sum_{i=1}^{N} ||\frac{\partial u}{\partial x_i}||_\Phi. \tag{7}$$

The spaces $L_\Phi(\Omega)$ and $W^1L_\Phi(\Omega)$ generalize the usual spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$, respectively, where the role played by the convex mapping $t \longmapsto \frac{|t|^p}{p}$ is assumed by a more general convex function $\Phi(t)$. More clearly, for the case $\Phi(t) := |t|^p$, we replace $L_\Phi(\Omega)$ by $L^p(\Omega)$ and $W^1L_\Phi(\Omega)$ by $W^{1,p}(\Omega)$ and call them *Lebesgue spaces* and *Sobolev spaces*, respectively.

Through this paper, we use the notations $W^1L_\Phi = W^{1,\Phi}$ and $L_\Phi = L^\Phi$, and assume that

$$\text{the function } t \to \Phi(\sqrt{t}) \text{ is convex for all } t \in [0, \infty). \tag{8}$$

Proposition 2.1 (see [24]). *Assume that Ω is a bounded domain with smooth boundary $\partial\Omega$. Then the embedding $W^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$ is compact provided $1 \le r < p^*$, where $p^* := \frac{Np}{N-p}$ if $p < N$ and $p^* := +\infty$ otherwise.*

Proposition 2.2 (see [18, 24]). *Assume that Ω is a bounded domain and has a Lipchitz boundary $\partial\Omega$. Then the embedding $W^{1,p}(\Omega) \hookrightarrow L^r(\partial\Omega)$ is compact provided $1 \le r < p^*$.*

Remark 2.1. *By help of the assumption (4), the Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$ is continuously embedded in the classical Sobolev space $W^{1,\varphi_0}(\Omega)$. On the other hand, $W^{1,\varphi_0}(\Omega)$ is compactly embedded in $L^r(\Omega)$, and hence, $W^{1,\Phi}(\Omega)$ is continuously and compactly embedded in the classical Lebesgue space $L^r(\Omega)$ for all $1 \le r < \varphi_0^*$.*

Proposition 2.3 ([1, 23]). *If (4) and (8) hold then the spaces $L^\Phi(\Omega)$ and $W^{1,\Phi}(\Omega)$ are separable and reflexive Banach spaces.*

Proposition 2.4 (see [25, 31]). *Let define the modular $\rho(u) := \int_\Omega(\Phi(|\nabla u|) + \Phi(|u|))dx : W^{1,\Phi}(\Omega) \to \mathbb{R}$. Then for every $u_n, u \in W^{1,\Phi}(\Omega)$, we have*

(*i*) $\|u\|_{1,\Phi}^{\varphi^0} \le \rho(u) \le \|u\|_{1,\Phi}^{\varphi_0}$ *if* $\|u\|_{1,\Phi} < 1$

(*ii*) $\|u\|_{1,\Phi}^{\varphi_0} \le \rho(u) \le \|u\|_{1,\Phi}^{\varphi^0}$ *if* $\|u\|_{1,\Phi} > 1$

(*iii*) $\|u_n - u\|_{1,\Phi} \to 0 \Leftrightarrow \rho(u_n - u) \to 0$

(*iv*) $\|u_n - u\|_{1,\Phi} \to \infty \Leftrightarrow \rho(u_n - u) \to \infty$

Proposition 2.4 $(iii) - (iv)$ means that norm and modular topology coincide on $L^\Phi(\Omega)$ provided Φ satisfies (4), which enables that Φ satisfies the well-known Δ_2-condition, i.e.,

$$\Phi(2t) \le K\Phi(t) \ \text{ for all } \ t \in [0, \infty).$$

where K is a positive constant (see e.g, [31]).

3. Main results

Definition 3.1. *We say that $u \in W^{1,\Phi}(\Omega)$ is a weak solution of problem (1) iff*

$$\int_\Omega (a(|\nabla u|)\nabla u \cdot \nabla v + a(|u|)uv)dx + \int_{\partial\Omega} b(x)|u|^{p-2}uv d\gamma = \lambda \int_\Omega f(u)v dx, \quad \forall v \in W^{1,\Phi}(\Omega)$$

where $d\gamma$ is the measure on the boundary $\partial\Omega$.

The energy functional corresponding to problem (1) is defined as $I : W^{1,\Phi}(\Omega) \to \mathbb{R}$,

$$I(u) := \int_\Omega (\Phi(|\nabla u|) + \Phi(|u|))dx + \int_{\partial\Omega} \frac{b(x)}{p}|u|^p d\gamma - \lambda \int_\Omega F(u)dx,$$

where $F(u) = \int_0^u f(s)ds$.

Remark 3.1. *The operator*

$$\langle \Lambda'(u), v\rangle = \int_\Omega (a(|\nabla u|)\nabla u \cdot \nabla v + a(|u|)uv)dx,$$

defined from $W^{1,\Phi}(\Omega)$ to its dual space $(W^{1,\Phi}(\Omega))^*$, is of type (S_+), that is, $u_n \rightharpoonup u$ in $W^{1,\Phi}(\Omega)$ and $\limsup\langle \Lambda'(u_n), u_n - u\rangle \leq 0$ imply $u_n \to u$ in $W^{1,\Phi}(\Omega)$, see [23].

We will assume the following assumption.

(F) $f : \Omega \to \mathbb{R}$ is a continuous function and there exist constants $c_1, c_2 > 0$ such that $c_1|t|^{s-1} \leq f(t) \leq c_2|t|^{q-1}$, with $1 \leq s < q < \varphi_0^*$.

The main result of the present paper is the following.

Theorem 3.1. *Suppose that the functions φ and Φ are as defined in Section 2, and condition (F) holds. If in addition, the inequalities*

$$q < \varphi_0, \quad s < p < \varphi_0,$$

hold, then there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$ problem (1) has a nontrivial weak solution in $W^{1,\Phi}(\Omega)$.

To obtain the main result, first we need to prove the following lemmas.

Lemma 3.1. *The functional I is well-defined on $W^{1,\Phi}(\Omega)$ and Fréchet differentiable, i.e., $I \in C^1(W^{1,\Phi}(\Omega), \mathbb{R})$ and its derivative is*

$$\langle I'(u), v\rangle = \int_\Omega (a(|\nabla u|)\nabla u \cdot \nabla v + a(|u|)uv)dx + \int_{\partial\Omega} b(x)|u|^{p-2}uv d\gamma - \lambda \int_\Omega f(u)v dx.$$

Proof. In [30], the authors showed that the operator

$$\Lambda(u) = \int_\Omega (\Phi(|\nabla u|) + \Phi(|u|))dx$$

is well-defined and of class $C^1(W^{1,\Phi}(\Omega), \mathbb{R})$. Moreover, from condition (F), we have $0 \leq F(u) \leq \frac{c_2}{q}|u|^q$. Therefore, considering the continuous embeddings $W^{1,\Phi}(\Omega) \hookrightarrow L^p(\partial\Omega)$ and $W^{1,\Phi}(\Omega) \hookrightarrow L^q(\Omega)$, it follows

$$|I(u)| \leq \Lambda(u) + \int_{\partial\Omega} \frac{b(x)}{p}|u|^p d\gamma + \lambda\frac{c_2}{q}\int_\Omega |u|^q dx < \infty$$

which means that I is well-defined on $W^{1,\Phi}(\Omega)$.
Since $\Lambda \in C^1(W^{1,\Phi}(\Omega), \mathbb{R})$, it is enough to show that the operator J given by

$$J(u) = \int_{\partial\Omega} \frac{b(x)}{p}|u|^p d\gamma - \lambda \int_\Omega F(u)dx$$

is of class $C^1(W^{1,\Phi}(\Omega),\mathbb{R})$. To this end, first, it must be shown that for all $v \in W^{1,\Phi}(\Omega)$

$$\langle J'(u),v\rangle = \lim_{t\to 0}\frac{J(u+tv)-J(u)}{t} = \int_{\partial\Omega} b(x)|u|^{p-2}uv d\gamma - \lambda\int_{\Omega} f(u)v dx,$$

and then it must be obtained that $J' : W^{1,\Phi}(\Omega) \to (W^{1,\Phi}(\Omega))^*$ is continuous.
The continuity properties of $|\cdot|$ and f along with the definition of F, allow us to apply the mean value theorem, that is,

$$\begin{aligned}\langle J'(u),v\rangle &= \lim_{t\to 0}\int_{\partial\Omega}\frac{b(x)}{p}\frac{|u+tv|^p-|u|^p}{t}d\gamma - \lambda\lim_{t\to 0}\int_{\Omega}\frac{F(u+tv)-F(u)}{t}dx\\ &= \lim_{t\to 0}\int_{\partial\Omega} b(x)|u+t\theta v|^{p-2}(u+t\theta v)v d\gamma - \lambda\lim_{t\to 0}\int_{\Omega} f(u+t\theta v)v dx,\end{aligned}$$

where $u, v \in W^{1,\Phi}(\Omega)$ and $0 \le \theta \le 1$. Now, if we apply the Young's inequality along with the inequality $|a + b|^m \le 2^{m-1}(|a|^m + |b|^m)$, for all $a, b \in \mathbb{R}^N$ and $m \ge 1$, consecutively to the both integrands on the right-hand side of the above expression, and use condition (F), it reads

$$|b(x)|u+t\theta v|^{p-2}(u+t\theta v)v| \le b(x)\left(\frac{2^{p-1}(p-1)}{p}|u|^p + (\frac{2^{p-1}(p-1)}{p}+1)|v|^p\right) \tag{9}$$

and

$$|f(u+t\theta v)v| \le c\left(\frac{2^{q-1}(q-1)}{q}|u|^q + (\frac{2^{q-1}(q-1)}{q}+1)|v|^q\right) \tag{10}$$

The right hand sides of the inequalities (9) and (10) belong to $L^1(\Omega)$. Therefore, by the Lebesgue dominated convergence theorem along with the continuity properties of f and $|\cdot|$, we have

$$\begin{aligned}\langle J'(u),v\rangle &= \int_{\partial\Omega} b(x)\lim_{t\to 0}|u+t\theta v|^{p-2}(u+t\theta v)v d\gamma - \lambda\int_{\Omega}\lim_{t\to 0} f(u+t\theta v)v dx\\ &= \int_{\partial\Omega} b(x)|u|^{p-2}uv d\gamma - \lambda\int_{\Omega} f(u)v dx.\end{aligned}$$

Since the right-hand side of the above expression, as a function of v, is a continuous linear functional on $W^{1,\Phi}(\Omega)$, it is the Gateaux differential of J.
Next, we proceed to the continuity of J'. To this end, we assume, for a sequence $u_n \subset W^{1,\Phi}(\Omega)$, that $u_n \to u \in W^{1,\Phi}(\Omega)$. Then, using condition (F), it reads

$$\begin{aligned}|\langle J'(u_n)-J'(u),v\rangle| &\le \left|\int_{\partial\Omega} b(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)v d\gamma\right| + \lambda\left|\int_{\Omega}(f(u)-f(u_n))v dx\right|\\ &\le \int_{\partial\Omega} b(x)(|u_n|^{p-1}+|u|^{p-1})|v| d\gamma + c_2\lambda\int_{\Omega}(|u_n|^{q-1}+|u|^{q-1})|v| dx\end{aligned}$$

Since $u_n \to u \in W^{1,\Phi}(\Omega)$, by the compact embeddings $W^{1,\Phi}(\Omega) \hookrightarrow L^p(\partial\Omega)$ and $W^{1,\Phi}(\Omega) \hookrightarrow L^q(\Omega)$, up to a subsequence still denoted by (u_n), we have
$u_n \to u$ in $L^p(\partial\Omega)$,
$u_n \to u$ in $L^q(\Omega)$,
$u_n(x) \to u(x)$ a.e. $x \in \overline{\Omega}$,
and there exist $w \in L^p(\partial\Omega)$ and $\phi \in L^q(\Omega)$ such that $|u_n(x)| \le w(x)$ and $|u_n(x)| \le \phi(x)$,

a.e. $x \in \partial\Omega$ and a.e. $x \in \Omega$, respectively, for all $n \in \mathbb{N}$. Therefore, using Hölder inequality and considering $b \in L^{\infty}(\partial\Omega)$, it reads

$$\begin{aligned}|\langle J'(u_n) - J'(u), v\rangle| &\leq c_3 \int_{\partial\Omega} (|w|^{p-1} + |u|^{p-1})|v| d\gamma + c_2\lambda \int_{\Omega} (|\phi|^{q-1} + |u|^{q-1})|v| dx \\ &\leq c_4(||w|^{p-1}|_{L^{p/p-1}(\partial\Omega)} + ||u|^{p-1}|_{L^{p/p-1}(\partial\Omega)})|v|_{L^p(\partial\Omega)} \\ &+ c_5(||\phi|^{q-1}|_{L^{q/q-1}(\Omega)} + ||u|^{q-1}|_{L^{q/q-1}(\Omega)})|v|_{L^q(\Omega)} \in L^1(\Omega)\end{aligned}$$

Now, we mention the following inequality given in [15]: for $1 < k < \infty$ there exists a constant $C_k > 0$ such that

$$||\xi|^{k-2}\xi - |\zeta|^{k-2}\zeta| \leq C_k|\xi - \zeta|(|\xi| + |\zeta|)^{k-2}, \ \forall \xi, \zeta \in \mathbb{R}^N.$$

Moreover, considering that $u_n(x) \to u(x)$ a.e. $x \in \overline{\Omega}$ and f is continuous, we obtain that

$$\lim_{n\to\infty} |f(u_n(x)) - f(u(x))| = 0, \ \lim_{n\to\infty} |b(x)(|u_n(x)|^{p-2}u_n(x) - |u(x)|^{p-2}u(x))| = 0.$$

If we take into account the above inequalities and apply the Lebesgue dominated convergence theorem once more, it reads

$$\lim_{n\to\infty} \int_{\Omega} |f(u_n) - f(u)| = 0, \ \lim_{n\to\infty} \int_{\partial\Omega} |b(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)| = 0,$$

which means, as a conclusion, that

$$\lim_{n\to\infty} \sup \|J'(u_n) - J'(u)\|_{(W^{1,\Phi}(\Omega))^*} = 0.$$

Therefore, $J' : W^{1,\Phi}(\Omega) \to (W^{1,\Phi}(\Omega))^*$ is continuous. □

Lemma 3.2. *There exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$ there exist $\tau, \delta > 0$ such that for all $u \in W^{1,\Phi}(\Omega)$ with $\|u\|_{1,\Phi} = \delta < 1$ we have $I(u) \geq \tau > 0$.*

Proof. By (F), we have $F(u) \leq \frac{c_2}{q}|u|^q$. From Proposition 2.4 and the continuous embedding $W^{1,\Phi}(\Omega) \hookrightarrow L^q(\Omega)$, it follows

$$\begin{aligned}I(u) &= \int_{\Omega} (\Phi(|\nabla u|) + \Phi(|u|))dx + \int_{\partial\Omega} \frac{b(x)}{p}|u|^p d\gamma - \lambda \int_{\Omega} F(u)dx \\ &\geq \int_{\Omega} (\Phi(|\nabla u|) + \Phi(|u|))dx - c_2\lambda \int_{\Omega} \frac{|u|^q}{q} dx \\ &\geq \|u\|_{1,\Phi}^{\varphi^0} - \frac{c_2\lambda}{q}\|u\|_{1,\Phi}^q \\ &\geq (\|u\|_{1,\Phi}^{\varphi^0 - q} - \frac{c_2\lambda}{q})\|u\|_{1,\Phi}^q\end{aligned}$$

If we define the function

$$\Psi(\delta) := \delta^{\varphi^0 - q} - \frac{c_2\lambda}{q}$$

then, Ψ is continuous at $\delta = 0$ and $\Psi(0) \neq 0$. If we put

$$\lambda^* = \frac{q}{2c_2}\delta^{\varphi^0 - q} \tag{11}$$

then for any $\lambda \in (0, \lambda^*)$, there is a real number $\delta_0 := \left(\frac{1}{2}\delta^{\varphi^0} + \frac{c_2\lambda}{q}\right)^{1/(\varphi^0 - q)} > 0$ such that in a neighborhood of the origin, where Ψ has the same sign with $\Psi(\delta_0)$, it holds

$$\tau := \frac{1}{2}\delta^{\varphi^0} = \Psi(\delta_0) > 0.$$

Therefore, for all $\lambda \in (0, \lambda^*)$ and any $u \in W^{1,\Phi}(\Omega)$ with $\|u\|_{1,\Phi} = \delta$, we have $I(u) \geq \tau > 0$. □

Lemma 3.3. *There exists $\theta \in W^{1,\Phi}(\Omega)$ such that $\theta \geq 0$, $\theta \neq 0$ it holds $I(t\theta) < 0$ provided $t > 0$ is small enough.*

Proof. First, we note that for $0 < t < 1$ and $s > 0$ it holds $\Phi(ts) \leq t^{\varphi_0}\Phi(s)$. Indeed, from the assumption (4), we have

$$\varphi_0 \leq \frac{z\varphi(z)}{\Phi(z)}, \ \ \forall z \geq 0,$$

from which we can proceed as follows

$$\int_{ts}^{s} \frac{\varphi_0}{z} dz \leq \int_{ts}^{s} \frac{\varphi(z)}{\Phi(z)} dz$$
$$\log s^{\varphi_0} - \log(ts)^{\varphi_0} \leq \log \Phi(s) - \log \Phi(ts)$$
$$\log \Phi(ts) - \log t^{\varphi_0} \leq \log \Phi(s)$$

and hence we obtain that

$$\Phi(ts) \leq t^{\varphi_0}\Phi(s).$$

Moreover, from (F), we have $F(u) \geq \frac{c_1}{s}|u|^s$. Thus,

$$\begin{aligned} I(t\theta) &\leq \int_\Omega \Phi(|\nabla(t\theta)|) + \Phi(|(t\theta)|)dx + \int_{\partial\Omega} \frac{b(x)}{p}|t\theta|^p d\gamma - c_1\lambda \int_\Omega \frac{|t\theta|^s}{s} dx \\ &\leq t^{\varphi_0}\rho(\theta) + \frac{t^p}{p}\int_{\partial\Omega} b(x)|\theta|^p d\gamma - \frac{c_1\lambda t^s}{s}\int_\Omega |\theta|^s dx \\ &\leq t^p\left(\rho(\theta) + \int_{\partial\Omega} b(x)|\theta|^p d\gamma\right) - \frac{c_1\lambda t^s}{s}\int_\Omega |\theta|^s dx, \end{aligned}$$

for $0 < t < 1$ and $1 \leq s < p < \varphi_0$. Thus,

$$I(t\theta) < 0$$

for $t < \eta^{1/(p-s)}$ with

$$0 < \eta < \min\left\{1, \frac{\frac{c_1\lambda}{s}\int_\Omega |\theta|^s dx}{\rho(\theta) + \int_{\partial\Omega} b(x)|\theta|^p d\gamma}\right\}.$$

□

Proof. (**Proof of Theorem 3.1**) By Lemma 3.2, I is bounded from below on the ball $\overline{B(0;\delta)} = \{u \in W^{1,\Phi}(\Omega) : \|u\|_{1,\Phi} \leq \delta\}$, and therefore, there is a constant $\underline{c}$ such that $\underline{c} := \inf_{\overline{B(0;\delta)}} I$. Then, by Lemma 3.3, it follows that

$$-\infty < \underline{c} := \inf_{\overline{B(0;\delta)}} I < 0. \tag{12}$$

Moreover from Lemma 3.2, we have

$$\inf_{\partial B(0;\delta)} I > 0. \tag{13}$$

If we combine (12) and (13), it reads

$$0 < \varepsilon < \inf_{\partial B(0;\delta)} I - \inf_{\overline{B(0;\delta)}} I. \tag{14}$$

Since I is bounded from below and weakly lower semicontinuous, we can apply Ekeland's variational principle, given in [21], to the functional $I : \overline{B(0;\delta)} \to \mathbb{R}$. Therefore, we can find $u_\varepsilon \in \overline{B(0;\delta)}$ such that

$$I(u_\varepsilon) < \inf_{\overline{B(0;\delta)}} I + \varepsilon \tag{15}$$

$$I(u_\varepsilon) < I(u) + \varepsilon \|u - u_\varepsilon\|_{1,\Phi}, \ u \neq u_\varepsilon \tag{16}$$

From (14) and (15), we obtain that

$$I(u_\varepsilon) \leq \inf_{\overline{B(0;\delta)}} I + \varepsilon \leq \inf_{B(0;\delta)} I + \varepsilon < \inf_{\partial B(0;\delta)} I$$

which means that $u_\varepsilon \in B(0;\delta)$. Let's define the functional $\overline{I} : \overline{B(0;\delta)} \to \mathbb{R}$ such that

$$\overline{I}(u) := I(u) + \varepsilon \|u - u_\varepsilon\|_{1,\Phi}$$

which is a perturbation of I. Then, from the above expressions, u_ε is a minimum point of $\overline{I}$. Hence, if we put $u = u_\varepsilon + tv$ and take $v \in B(0;1)$, it reads

$$\frac{\overline{I}(u_\varepsilon + tv) - \overline{I}(u_\varepsilon)}{t} \geq 0,$$

provided $t > 0$ is small enough. The last inequality above leads us to

$$\frac{I(u_\varepsilon + tv) - I(u_\varepsilon)}{t} + \varepsilon \|v\|_{1,\Phi} > 0.$$

If we let $t \to 0$, we obtain that

$$\langle I'(u_\varepsilon), v\rangle + \varepsilon \|v\|_{1,\Phi} \geq 0. \tag{17}$$

If we replace v by $-v$ in the lines above, we obtain

$$-\langle I'(u_\varepsilon), v\rangle + \varepsilon \|v\|_{1,\Phi} \geq 0, \tag{18}$$

which means, along with (17),

$$|\langle I'(u_\varepsilon), v\rangle| \leq \varepsilon \|v\|_{1,\Phi}. \tag{19}$$

By the definition of norm on $(W^{1,\Phi}(\Omega))^*$ for I', it follows that

$$\|I'(u_\varepsilon)\|_{(W^{1,\Phi}(\Omega))^*} \leq \varepsilon. \tag{20}$$

Therefore, as a corollary of Ekeland's variational principle, (15),(16) and (20) guarantee that there is a minimizing sequence $(\omega_n) \in B(0;\delta)$ of I such that

$$I(\omega_n) \to \underline{c} = \inf_{\overline{B(0;\delta)}} I \text{ and } I'(\omega_n) \to 0 \text{ in } (W^{1,\Phi}(\Omega))^* \text{ (or equivalently } \|I'(\omega_n)\|_{(W^{1,\Phi}(\Omega))^*} \to 0). \tag{21}$$

To see how (21) holds, continue picking out the functions u_ε consecutively which provides a sequence consisting of functions u_ε. Then, choose a sequence of $\omega_n := u_{\varepsilon=1/n}$, i.e., put $\varepsilon = 1/n$ in (15) and (20), and repeat the steps infinitely times (i.e., $n \to \infty$), which leads to $\varepsilon \to 0$, and consequently to (21).

On the other hand, from (21), (ω_n) is bounded in $W^{1,\Phi}(\Omega)$. By reflexivity of $W^{1,\Phi}(\Omega)$, for a convenient subsequence (still denoted by ω_n), we have $\omega_n \rightharpoonup \omega$ in $W^{1,\Phi}(\Omega)$. Moreover, since

$$\|I'(\omega_n)\|_{(W^{1,\Phi}(\Omega))^*} = \sup_{\|\omega_n - \omega\|_{1,\Phi} \leq 1} |\langle I'(\omega_n), \omega_n - \omega\rangle|,$$

(21) yields to

$$\langle I'(\omega_n), \omega_n - \omega\rangle \to 0,$$

which means that

$$\langle I'(\omega_n), \omega_n - \omega\rangle = \int_\Omega (a(|\nabla \omega_n|)\nabla\omega_n \nabla(\omega_n - \omega) + a(|\omega_n|)\omega_n(\omega_n - \omega))dx +$$
$$\int_{\partial\Omega} b(x)|\omega_n|^{p-2}\omega_n(\omega_n - \omega)d\gamma - \lambda \int_\Omega f(\omega_n)(\omega_n - \omega)dx \to 0.$$

If we use the compact embeddings $W^{1,\Phi}(\Omega) \hookrightarrow L^p(\partial\Omega)$ and $W^{1,\Phi}(\Omega) \hookrightarrow L^q(\Omega)$ along with the Hölder inequality, (F) and consider the fact that $b \in L^\infty(\partial\Omega)$, it reads

$$\left|\int_{\partial\Omega} b(x)|\omega_n|^{p-2}\omega_n(\omega_n - \omega)d\gamma\right| \leq |b(x)|\omega_n|^{p-1}|_{L^{p/(p-1)}(\partial\Omega)}|\omega_n - \omega|_{L^p(\partial\Omega)} \to 0,$$

and

$$\left|\int_\Omega f(\omega_n)(\omega_n - \omega)dx\right| \leq c_2 \int_\Omega ||\omega_n|^{q-1}(\omega_n - \omega)dx| \leq ||\omega_n|^{q-1}|_{L^{q/(q-1)}(\Omega)}|\omega_n - \omega|_{L^q(\Omega)} \to 0.$$

Therefore, we must have

$$\langle \Lambda'(\omega_n), \omega_n - \omega\rangle = \int_\Omega (a(|\nabla \omega_n|)\nabla\omega_n \nabla(\omega_n - \omega) + a(|\omega_n|)\omega_n(\omega_n - \omega))dx \to 0.$$

Since the operator Λ' is of type (S_+), we obtain that $\omega_n \to \omega$ in $W^{1,\Phi}(\Omega)$. Therefore, by (21), we have

$$I(\omega) = \underline{c} < 0 \text{ and } I'(\omega) = 0.$$

As a result, we infer that ω is a nontrivial weak solution to problem (1) for any $\lambda \in (0, \lambda^*)$. The proof is completed. $\square$

Example 3.1. *As an example, we can choose $a(t) = |t|^{p-2}$ and $f(t) = |t|^{\beta-1}$. Then problem (1) turns into*

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u &= \lambda|u|^{\beta-1} \text{ in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial\eta} + b(x)|u|^{p-2}u &= 0 \text{ on } \partial\Omega \end{cases} \tag{22}$$

which is the well-known p-Laplacian equation with Robin boundary condition. The energy functional corresponding to problem (22) will be $\Upsilon : W^{1,p}(\Omega) \to \mathbb{R}$,

$$\Upsilon(u) := \frac{1}{p}\int_\Omega (|\nabla u|^p + |u|^p)dx + \frac{1}{p}\int_{\partial\Omega} b(x)|u|^p d\gamma - \frac{\lambda}{\beta}\int_\Omega |u|^\beta dx.$$

It is obvious that the all conditions of Theorem 3.1 hold, and hence, problem (22) has a nontrivial weak solution.

Remark 3.2. *We want to note that the results of the present paper obtained for the constant exponents p, q, s can be extended to variable exponent case $p = p(x), q = q(x), s = s(x)$ under some appropriate assumptions. To be more clear, for variable exponent case, problem (1) will turn to*

$$\begin{cases} -\mathrm{div}(a(|\nabla u|)\nabla u) + a(|u|)u &= \lambda f(x, u) \text{ in } \Omega \\ a(|\nabla u|)\frac{\partial u}{\partial\eta} + b(x)|u|^{p(x)-2}u &= 0 \text{ on } \partial\Omega \end{cases} \tag{23}$$

If we replace the assumption (F) with the following

(F1) *$f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and there exist $c_1, c_2 > 0$ such that*

$$c_1|t|^{s(x)-1} \leq f(x,t) \leq c_2|t|^{q(x)-1}$$

where $s, q \in C(\overline{\Omega})$ such that $1 < \inf_{x\in\overline{\Omega}} s(x) := s^- \leq s(x) \leq q(x) < \varphi_0^*$,

then Theorem 3.1. will be as follows:

Theorem 3.2. *Suppose that condition (F1) holds. If in addition, the inequalities*

$$\sup_{x\in\overline{\Omega}} q(x) := q^+ < \varphi_0, \quad p^+ < \varphi_0, \quad s^+ < p^-$$

hold, then there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$ problem (23) has a nontrivial solution in $W^{1,\Phi}(\Omega)$.

Remark 3.3. *If we let $a(t) = |t|^{p(x)-2}$ and $f(x,t) = |t|^{\beta(x)-1}$ in Example 3.1, problem (22) turns into the well-known $p(x)$-Laplace equation which is formulated in the variable exponent Sobolev spaces $W^{k,p(x)}(\Omega)$, see, e.g., the papers* [4, 5, 6, 11, 12, 18, 22, 28, 36, 37] *and the monographs* [17, 19, 34] *for the detailed background.*

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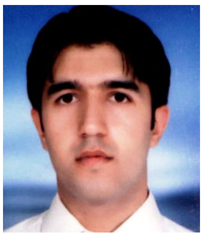

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