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BOUNDEDLY SOLVABILITY OF FIRST ORDER DELAY DIFFERENTIAL OPERATORS WITH PIECEWISE CONSTANT ARGUMENTS

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Abstract. Using the methods of operator theory, we investigate all boundedly solvable extensions of a minimal operator generated by first order delay differential-operator expression with piecewise constant argument in the Hilbert space of vector-functions at finite interval. Also spectrum of these extensions is studied.

Keywords: Boundedly solvable operator, differential operator with piecewise constant argument, spectrum

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1. Introduction

Recently, some properties of differential equations with piecewise constant arguments of various types as retarded, advanced and mixed types have been investigated, intensively in [1],[2],[4],[5],[11] (see also references therein). Many of investigations (existence, uniqueness, stability, oscillation, periodicity of solutions and ets.) are devoted to differential equations (linear and nonlinear) for different order (for example, see [3],[10],[12]). Differential equations with piecewise constant arguments are investigation subjects of many problems in life sciences such as physics, chemistry, biomedicine, mechanical engineering etc.

Using the methods of operator theory many investigations and technical difficulties in these processes may be facilitated.

In this sense in Section 2 of this work by using methods operator theory all boundedly solvable extensions of minimal operator generated by delay type differential-operator expression for first order with piecewise argument in the Hilbert space of vector-functions at finite interval have been described in terms of boundary values. Later on, in Section 3, structure of spectrum of these extensions has been investigated. Finally, the obtained results have been supported by application.

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2. Description of Solvable Extensions

First, we will give some necessary definitions.

Let H be any Hilbert space and $0 < T < \infty$.

Definition 2.1. [7] $L^2(H,(0,T))$, *is the set of Lebesgue measurable $H-$valued vector functions $f(\,\cdot\,)$ for which*

$$\int_0^T \|f(t)\|^2 dt < +\infty.$$

$L^2(H,(0,T))$ *is a Hilbert space with inner product in form*

$$(f,g)_{L^2(H,(0,T))} = \int_0^T (f(t),g(t))_H dt,\ \ f,g \in L^2(H,(0,T)).$$

Definition 2.2. [7] *The completion of the set $C^1(H,[0,T])$ of $H-$valued continuously differentiable vector-functions on $[0,T]$ with respect to the norm*

$$\|f\|_{W_2^1(H,(0,T))} = \left(\|f\|^2_{L^2(H,(0,T))} + \|f'\|^2_{L^2(H,(0,T))}\right)^{1/2}$$

is called Sobolev space of vector-functions on interval $[0,T]$ and will be denoted by $W_2^1(H,(0,T))$.

$W_2^1(H,(0,T))$ *is a Hilbert space with inner product by following form*

$$(f,g)_{W_2^1(H,(0,T))} = (f,g)_{L^2(H,(0,T))} + (f',g')_{L^2(H,(0,T))},\ \ f,g \in W_2^1(H,(0,T)).$$

Definition 2.3. [7] *Let $A : D(A) \subset H \to H$ be a linear densely defined closed operator. The operator A in H is called boundedly solvable, if A is one-to-one, $D(A) = H$ and inverse operator A^{-1} is a bounded in H.*

In the Hilbert space $L^2(H,(0,T))$ of vector-functions consider piecewise constant argument differential-operator expression for first order in the form

$$l(u) = u'(t) + A(t)u(t) + B(t)u([t]), 0 \leq t \leq T < \infty, \tag{1}$$

where H is a separable Hilbert space with inner product $(,\,\cdot\,,)_H$ and norm $\|\cdot\|_H$; operator-function $A(\,\cdot\,) : [0,T] \to L(H)$ and $B(\,\cdot\,) : [0,T] \to L(H)$ are continuous on the uniformly operator topology.

It is known that any vector function $u \in W_2^1(H,(0,T))$ can be represented in the form

$$u(t) = u([t]) + \int_{[t]}^t u'(x)dx.$$

In this case the expression $l(\,\cdot\,)$ can be rewritten as form

$$l(u) = u'(t) + A(t)u(t) + B(t)u(t) - B(t)\int_{[t]}^t u'(x)dx,$$

that is,

$$l(u) = (1 - B(t)V)\,u'(t) + (A(t) + B(t))\,u(t),$$

where $Vu(t) = \int\limits_{[t]}^{t} u(x)dx$. Note that the linear operator

$$V : L^2(H, (0, T)) \to L^2(H, (0, T)), \quad Vu(t) = \int\limits_{[t]}^{t} u(x)dx$$

is bounded. Indeed for any $u \in L^2(H, (0, T))$, we have

$$\begin{aligned} \|Vu\|^2_{L^2(H,(0,T))} &= \int\limits_0^T \|Vu(t)\|^2_H dt = \int\limits_0^T \| \int\limits_{[t]}^{t} u(x)dx\|^2_H dt \le \int\limits_0^T \left(\int\limits_{[t]}^{t} \|u(x)\|dx \right)^2 dt \\ &\le \int\limits_0^T \left(\int\limits_{[t]}^{t} 1^2 dx \right) \left(\int\limits_{[t]}^{t} \|u(x)\|^2_H dx \right) dt \le \int\limits_0^T \left(\int\limits_{[t]}^{t} \|u(x)\|^2_H dx \right) dt \\ &\le \int\limits_0^T \left(\int\limits_0^T \|u(x)\|^2_H dx \right) dt \le T\|u\|^2_{L^2(H,(0,T))}. \end{aligned}$$

Hence $V \in L\left(L^2(H, (0, T))\right)$ and $\|V\| \le \sqrt{T}$.

Now denote

$$S(\,\cdot\,) : H \to H,$$
$$S(t) = 1 - B(t)V, \; 0 \le t \le T.$$

Furthermore it will be assumed that

$$0 \notin \sigma(S(t)) \text{ for each } 0 \le t \le T$$

(where $\sigma(\,\cdot\,)$ is a spectrum set of an operator).

Then the differential-operator expression $l(\,\cdot\,)$ can be written in following form

$$l(u) = S(t)m(u),$$

where

$$m(u) = u'(t) + C(t)u(t) \tag{2}$$

and

$$C(t) = S^{-1}(t)(A(t) + B(t)), \; 0 \le t \le T$$

in the Hilbert space $L^2(H, (0, T))$.

The minimal M_0 and maximal M operators corresponding to (2) can be constructed by standart way (see [8]).

Throughout this work the following operators

$$L_0 = S(\,\cdot\,)M_0,$$
$$L_0 : \overset{o}{W}{}_2^1 (H, (0, T)) \subset L^2(H, (0, T)) \to L^2(H, (0, T))$$

and

$$L = S(\,\cdot\,)M,$$
$$L : W_2^1(H, (0, T)) \subset L^2(H, (0, T)) \to L^2(H, (0, T))$$

are called the minimal and maximal operators corresponding differential expression (1) in $L^2(H, (0, T))$, respectively.

It is said that if $\widetilde{L}$ is solvable, then $\widetilde{M}^{-1} = \widetilde{L}^{-1}S^{-1}$, and if $\widetilde{M}$ is solvable then $\widetilde{L}^{-1} =$

$\widetilde{M}^{-1}S^{-1}$. Hence to describe all boundedly solvable extensions of the minimal operator L_0 in $L^2(H,(0,T))$ it is sufficient to describe all boundedly solvable extensions of the minimal operator M_0.

Now let $U(t,s),\ t,s\in[0,T]$ be the family of evolution operators corresponding to the homogeneous differential equation

$$\begin{cases} U_t(t,s)f + C(t)U(t,s)f = 0,\ t,s\in[0,T], \\ U(s,s)f = f,\ f\in H. \end{cases}$$

The operator $U(t,s)$ for $t,s\in[0,T]$ is a linear continuous, boundedly invertible in H and

$$U^{-1}(t,s) = U(s,t),\ s,t\in[0,T]$$

(for more detailed analysis of this concept see [6]).

Based on similar result in [9], the following theorem can be easily proved.

Theorem 2.1. *Each boundedly solvable extension $\widetilde{L}$ of the minimal operator L_0 in $L^2(H,(0,T))$ is generated by the differential-operator expression (1) and boundary condition*

$$(K+E)u(0) = KU(0,T)u(T), \tag{3}$$

where $K\in L(H)$ and E is an identity operator in H. The operator K is determined uniquely by the extension $\widetilde{L}$, i.e $\widetilde{L}=L_K$.

On the contrary, the restriction of the maximal operator L to the linear manifold of vector-functions satisfies the condition (3) for some bounded operator $K\in L(H)$ is a boundedly solvable extension of the minimal operator L_0 in the $L^2(H,(0,T))$.

3. Spectrum of Solvable Extension

In this section, the spectrum structure of solvable extensions of a minimal operator L_0 in $L^2(H,(0,T))$ is investigated.

Theorem 3.1. *In order to get $\lambda\in\sigma(L_K)$ the necessary and sufficient condition is*

$$0\in\sigma\left(E+K-KU(0,T)exp\left\{\lambda\int_0^T S^{-1}(x)dx\right\}\right).$$

Proof. Consider the following problem of the spectrum for a boundedly solvable extension L_K of the minimal operator L_0, that is, $L_K u = \lambda u + f,\ \lambda\in\mathbb{C},\ f\in L^2(H,(0,T))$. Then

$$u'(t) + C(t)u(t) = \lambda S^{-1}u(t) + S^{-1}f,$$
$$(K+E)u(0) = KU(0,T)u(T).$$

From this, we obtained that

$$u'(t) = S^{-1}(t)(\lambda - C(t))u(t) + S^{-1}(t)f(t)$$

with boundary condition

$$(K+E)u(0) = KU(0,T)u(T).$$

The general solution of above differential equation is in the form

$$\begin{aligned} u(t) &= exp\left\{\lambda\int_0^t S^{-1}(x)dx\right\} exp\left\{-\int_0^t S^{-1}(x)C(x)dx\right\} f_0 \\ &+ \int_0^t exp\left\{\lambda\int_s^t S^{-1}(x)dx\right\} exp\left\{-\int_s^t S^{-1}(x)C(x)dx\right\} S^{-1}(s)f(s)ds \\ &= exp\left\{\lambda\int_0^t S^{-1}(x)dx\right\} U(t,0)f_0 \\ &+ \int_0^t exp\left\{\lambda\int_s^t S^{-1}(x)dx\right\} exp\left\{-\int_s^t S^{-1}(x)C(x)dx\right\} S^{-1}(s)f(s)ds,\ f_0\in H. \end{aligned}$$

On the other hand, from boundary condition it implies that

$$\left(E+K-KU(0,T)exp\left\{\lambda\int_0^T S^{-1}(x)dx\right\}\right) f_0$$

$$= KU(0,T)\int_0^T exp\left\{\lambda\int_s^T S^{-1}(x)dx\right\} exp\left\{-\int_s^T S^{-1}(x)C(x)dx\right\} S^{-1}(s)f(s)ds.$$

From this it is obtained that for the solvability of last equation on f_0 the necessary and sufficient condition is

$$0\in\rho\left(E+K-KU(0,T)exp\left\{\lambda\int_0^T S^{-1}(x)dx\right\}\right).$$

Once of such $\lambda\in\mathbb{C}$ the operator $(L_K-\lambda)^{-1}$ will be linear bounded operator $L^2(H,(0,T))$. This means that in order to get $\lambda\in\sigma(L_K)$ the necessary and sufficient condition is the condition

$$0\in\sigma\left(E+K-KU(0,T)exp\left\{\lambda\int_0^T S^{-1}(x)dx\right\}\right).$$

Consequently, the claim of theorem is clear. □

We note that the obtained results can be extended to the differential-operator expression in form

$$\begin{aligned} l(u) &= u'(t)+A(t)u(t)+B(t)u([t])+C(t)u([t+1]), \\ & \quad 0<t<T,\ 1<T<\infty \text{ (advanced type) [1]} \end{aligned}$$

and

$$l(u)=u'(t)+A(t)u(t)+\sum_{k=0}^{n} B_k u([t-k]),\ t>0 \text{ (retarted type) [5] }.$$

Example 3.1. *Consider the differential equation with piecewise constant argument in scalar case in the form*

$$x'(t)+mx(t)+Mx([t])=\sigma(t),\ t\in[0,T],\ T<\infty$$

with initial condition

$$x(0) = x_0,$$

where $\sigma \in C[0,T],\ M \neq 0$ *and* $m,\ M \in \mathbb{R}$.

In this case, if we replace $x(t)$ *with* $u(t)$ *defined as*

$$u(t) = x(t) - x_0,\ 0 < t < T,$$

then we have

$$\begin{aligned} &u'(t) + mu(t) + Mu([t]) = \sigma(t) - mx_0 - Mx_0, \\ &u(0) = 0. \end{aligned}$$

In this case $A(t) = m,\ B(t) = M,\ S(t) = 1 - MV$. *Since* $\|V\| \leq \sqrt{T}$, *then for any* $M < \frac{1}{\sqrt{T}}$ *the operator* $S(\ \cdot\ )$ *boundedly solvable and by Theorem 2.1 therefore the above considered boundary value problem have uniquely solution in* $L^2(H,(0,T))$.

Moreover,

$$u(t) \ = \ \int_0^t exp\left\{-(m+M)\int_s^t (1-MV(x))^{-2}dx\right\}(1-MV(s))^{-1}(\sigma(s) - mx_0 - Mx_0)ds.$$

Note that similar problem has been considered in [4] .

Remark 3.1. *In some situation the solvability problems for the delay type differential equation with a piecewise constant arguments can be transformed to the boundary value problems for ordinary differential equation in higher order.*

Now consider the following delay differential equation with piecewise constant argument in model case for $0 < t < T,\ ab \neq 0$

$$u'(t) + au(t) + b(t)u([t]) = 0,\ a, b \in \mathbb{R},$$

with boundary condition

$$u(0) = u_0.$$

Then from this it is obtained that

$$au(t) = -u'(t) - bu([t]),\ 0 < t < T.$$

In this case

$$\begin{aligned} &u''(t) = -au'(t),\ 0 < t < T,\ t \neq n,\ n \leq T \\ &u(0) = 0, \\ &u'(0) = -au(0) - bu(0) = -(a+b)u_0. \end{aligned}$$

Later on

$$\begin{aligned} &u'(t) = -au(t) + c, \\ &c = au(t) + u'(t). \end{aligned}$$

Then

$$c = au_0 + u_0' = au_0 - au_0 - bu_0 = -bu_0.$$

Consequently,

$$au(t) = -u'(t) - bu_0.$$

On the other words

$$u'(t) + au(t) = -bu_0.$$

From this

$$\begin{aligned} u(t) &= e^{-at}c + \int\limits_0^t e^{-a(t-s)}(-bu_0)ds \\ &= ce^{-at} - bu_0e^{-at}\int\limits_0^t e^{as}ds \\ &= ce^{-at} - bu_0e^{-at}\frac{1}{a}\left(e^{at}-1\right) \\ &= \left(c+bu_0)e^{-at}\right) - \frac{bu_0}{a}. \end{aligned}$$

Therefore

$$u_0 = c + bu_0 - \frac{bu_0}{a},$$

i.e.

$$c = \frac{a-ba+b}{a}u_0.$$

In this case, we have

$$u(t) = \left(\frac{a+b}{a}u_0e^{-at}\right) - \frac{bu_0}{a},\ 0 < t < T.$$

This idea can be applicated to the many problems of solvability of mentioned type differential equations. For example, in case when $a(t) \neq 0,\ t \geq 0$ *and* $a(\ \cdot\ ) \in C^1[0,\infty)$ *the differential equation*

$$u'(t) + a(t)u([t]) = 0,\ t \neq n,\ n = 1,2,...$$

can be rewritten in the form

$$u([t]) = \frac{-1}{a(t)}u'(t),\ t \geq 0.$$

From this it is clear that

$$\left(\frac{-1}{a(t)}u'(t)\right)' = 0,\ t \neq 1,2,...$$

That is,

$$a^2(t)u''(t) - a'(t)u'(t) = 0,\ t \geq 0,\ t = 1,2,...$$

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