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ON SOME PROPERTIES OF NORMAL Γ-IDEALS IN NORMAL Γ-SEMIGROUPS

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ABSTRACT. We introduce the concept of normal Γ-ideal and bi-Γ-ideal in normal Γ-semigroups. We characterize the (normal) Γ-semigroup and normal regular Γ-semigroup in terms of elementary properties of bi-Γ-ideal proving the various equivalent conditions. In particular, we establish, among the other things, that if I_1, I_2 are any two normal Γ-ideals of a Γ-semigroup S, then their product $I_1\Gamma I_2$ and $I_2\Gamma I_1$ are also normal Γ-ideals of S and $I_1\Gamma I_2 = I_2\Gamma I_1$. Finally, we show that the minimal normal Γ-ideal of a Γ-semigroup S is a Γ-group.

Keywords: Γ-semigroup, normal ideal

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1. Introduction and Preliminaries

A semigroup is an algebraic structure consisting of a nonempty set S together with an associative binary operation [2], [8]. The formal investigation of semigroups was initiated in the beginning of 20th century. Semigroups are significant in different fields of mathematics, for instance, language and coding theory, combinatorics, automata theory and mathematical analysis. The concept of the Γ-semigroup was given by M. K. Sen [12] in 1981 as a generalization of semigroups and ternary semigroups. Many classical properties of semigroups are also true for Γ-semigroups. Many authors extended and generalized the results of semigroups to Γ-semigroups. Our paper is inspired by the rapid development of the theory of Γ-semigroup investigated by lot of researchers, for instance, by Chattopadhyay, Chinram, Sen and Saha [5], [6], [10], [14] in different directions.

The purpose of this paper is to study some semigroup-theoretic results in the framework of the broader perspective of Γ-semigroups motivated by generalizations, and with the possible applications to, the ideal theory of other algebraic structures. Our results will provide an analogue for normal ideals and bi-ideals of semigroups in [1] giving the description of

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the characterization of Γ-semigroup via normal Γ-ideals and particularly bi-Γ-ideals. In fact, the class of normal Γ-ideals in normal Γ-semigroups is a generalization of the class of the normal ideals in normal semigroups.

For the information of our readers, we now recall and present some necessary definitions, a few notions and auxiliary results that will be used throughout this paper.

Suppose A and B are two nonempty sets. Let S be the set of all mappings from A to B and Γ be the set of all mappings from B to A. Now, the usual mapping product of two elements of S cannot be defined. However, if we consider f, g from S and α, β from Γ, then, the usual mapping products $f \cdot \alpha \cdot g$ and $\alpha \cdot f \cdot \beta$ are defined. Moreover, $f \cdot \alpha \cdot g \in S$ and $\alpha \cdot f \cdot \beta \in \Gamma$ and $f \cdot \alpha \cdot (g \cdot \beta \cdot h) = f \cdot (\alpha \cdot g \cdot \beta) \cdot h = (f \cdot \alpha \cdot g) \cdot \beta \cdot h$ for all $f, g, h \in S$ and $\alpha, \beta \in \Gamma$. This is the main motivation for Sen [11] to define the Γ-semigroup. We follow the definition of the Γ-semigroup by M. K. Sen and N. K. Saha [9] given in 1986 as follows:

Definition 1.1. *Let S and Γ be two nonempty sets. Then, a triple of the form $(S, \Gamma, \cdot)$ is called a Γ-semigroup, where $\cdot$ is a ternary operation $S \times \Gamma \times S \to S$ such that $(x \cdot \alpha \cdot y) \cdot \beta \cdot z = x \cdot \alpha \cdot (y \cdot \beta \cdot z)$ for all $x, y, z \in S$ and all $\alpha, \beta \in \Gamma$. Let T be a nonempty subset of $(S, \Gamma, \cdot)$. Then, T is called a sub-Γ-semigroup of $(S, \Gamma, \cdot)$ if $a \cdot \gamma \cdot b \in T$ for all $a, b \in T$ and $\gamma \in \Gamma$. Furthermore, a Γ-semigroup S is said to be commutative if $a \cdot \gamma \cdot b = b \cdot \gamma \cdot a$ for all $a, b \in S$ and $\gamma \in \Gamma$.*

Example 1.1. [6] *Let $S = [0, 1]$ and $\Gamma = \{\frac{1}{n} : n$ is a positive integer $\}$. Then, S is a Γ-semigroup under the usual multiplication. Next, let $K = [0, 1/2]$. We have that K is a nonempty subset of S and $a \cdot \gamma \cdot b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$. Then, K is a sub Γ-semigroup of S.*

The above example depicts that every semigroup is a Γ-semigroup and thus, Γ-semigroup is a generalization of semigroups. For other examples of Γ-semigroups, we can see [7], [8], [11].

Notation 1. For subsets A, B of a Γ-semigroup S, the product set $A \cdot B$ of the pair (A, B) relative to S is defined as $A \cdot \Gamma \cdot B = \{a \cdot \gamma \cdot b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}$ and for $A \subseteq S$, the product set $A \cdot A$ relative to S is defined as $A^2 = A \cdot A = A \cdot \Gamma \cdot A$.

Notation 2. We denote by $\mathcal{B}(S)$, the set of all bi-Γ-ideals of S and $\mathfrak{B}(S)$, the set of nonempty subsets of the Γ-semigroup S.

Definition 1.2. *A Γ-semigroup S is called normal if $s\Gamma S = S\Gamma s$ for all $s \in S$.*

Definition 1.3. *An ideal I of a Γ-semigroup S is called normal if $s\Gamma I = I\Gamma s$ for all $s \in S$.*

If there is no way of any confusion, we identify the Γ-semigroup $(S, \Gamma, \cdot)$ by S. Throughout this paper, for the sake of clarity, we denote $a \cdot \gamma \cdot b$ by $a\gamma b$. A sub Γ-semigroup T of a Γ-semigroup S is called normal if $s\Gamma T = T\Gamma s$ for all $s \in S$. A Γ-semigroup S is called left (right) regular if for every element $s \in S$, there exists an element $a \in S$ such that $s = a\gamma_1 s\gamma_2 s (s = s\gamma_1 s\gamma_2 a)$ for all $\gamma_1, \gamma_2 \in \Gamma$. A Γ-semigroup S is called intra-regular if for any $s \in S$, there exist elements $a, b \in S$ such that $s = a\gamma_1 s\gamma_2 s\gamma_3 b$ for all $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$. A Γ-semigroup S is called completely regular if for any element $s \in S$ there exists an element $a \in S$ such that $s = s\gamma_1 a$ and $s\gamma_1 a = a\gamma_2 s$ for all $\gamma_1, \gamma_2 \in \Gamma$. We denote and define the principal left Γ-ideal, right Γ-ideal and bi-Γ-ideal of S generated by $s \in S$ as follows: $(s)_L = s \cup S\Gamma s$, $(s)_R = s \cup s\Gamma S$, $(s)_B = s \cup s^2 \cup s\Gamma S\Gamma s$. An element a of S is called a zero element of S if $s\gamma a = a\gamma s = a$ for $s \in S$ and $\gamma \in \Gamma$.

2. Main Results

We begin by sketching some elementary properties of (normal) Γ-semigroups and thereafter giving a characterization of the (normal) Γ-semigroups in terms of (normal) Γ-ideals and bi-Γ-ideals. Most of the results of this paper is purely ideal-theoretic. We start with the following Proposition:

Proposition 2.1. *Suppose I is any Γ-ideal of a Γ-semigroup S. Then, we have*

(i): $I\Gamma(s)_B = I\Gamma(s)_L = I\Gamma s$ *for all* $s \in S$,

(ii): $(s)_B\Gamma I = (s)_R\Gamma I = s\Gamma I$ *for all* $s \in S$.

Proof. Suppose $s \in S$. Then we have

$$\begin{aligned} I\Gamma(s)_L &= I\ \Gamma(s \cup S\Gamma s) \\ &= I\Gamma s \cup I\ \Gamma(S\Gamma s) \\ &= I\Gamma s \cup (I\Gamma S)\Gamma s \\ &\subseteq I\Gamma s \subseteq I\Gamma(s)_L, \end{aligned}$$

and

$$\begin{aligned} I\Gamma(s)_B &= I\Gamma(s \cup s^2 \cup s\Gamma S\Gamma s) \\ &= I\Gamma s \cup I\Gamma s^2 \cup I\Gamma(s\Gamma S\Gamma s) \\ &= I\Gamma s \cup (I\Gamma s)\Gamma s \cup (I\Gamma s\Gamma S)\Gamma s \\ &\subseteq I\Gamma s \subseteq I\Gamma(s)_B. \end{aligned}$$

So, $I\Gamma(s)_B = I\Gamma(s)_L = I\Gamma s$ *for all* $s \in S$.
In a similar fashion, we can prove that

$$(s)_B\Gamma I = (s)_R\Gamma I = s\Gamma I$$

for all $s \in S$.

Theorem 2.1. *The following assertions are equivalent for a Γ-ideal I of S:*

(i): *I is normal;*
(i): $Y\Gamma I = I\Gamma Y$ *for all* $Y \in \mathcal{B}(S)$;
(iii): $(s)_B\Gamma I = I\Gamma(s)_B$ *for all* $s \in S$;
(iv): $(s)_B\Gamma I = I\Gamma(s)_L$ *for all* $s \in S$;
(v): $(s)_B\Gamma I = I\Gamma s$ *for all* $s \in S$;
(vi): $(s)_R\Gamma I = I\Gamma(s)_B$ *for all* $s \in S$;
(vii): $(s)_R\Gamma I = I\Gamma(s)_L$ *for all* $s \in S$;
(viii): $(s)_R\Gamma I = I\Gamma s$ *for all* $s \in S$;
(ix): $s\Gamma I = I\Gamma(s)_B$ *for all* $s \in S$;
(x): $s\Gamma I = I\Gamma(s)_L$ *for all* $s \in S$.

Proof. $(i) \Rightarrow (ii)$. *Let I be normal. Suppose Y is any nonempty subset of S and for $y \in Y, x \in I$, $y\gamma x \in Y\Gamma I$ for $\gamma \in \Gamma$. Then, we obtain that $y\gamma x \in y\Gamma I = I\Gamma y \subseteq I\Gamma Y$ for $\gamma \in \Gamma$ and therefore $Y\Gamma I \subseteq I\Gamma Y$. In a similar fashion, we can observe that $I\Gamma Y \subseteq Y\Gamma I$ for all $Y \in \mathcal{B}(S)$.*
$(ii) \Rightarrow (iii), (iii) \Rightarrow (iv)$. *They are straightforward.*
$(iv) \Rightarrow (v), (v) \Rightarrow (vi), (vi) \Rightarrow (vii), (vii) \Rightarrow (viii), (viii) \Rightarrow (ix), (ix) \Rightarrow (x)$ *are consequences of Proposition 2.1.*

Theorem 2.2. *Suppose I_1, I_2 are any two normal Γ-ideals of S. Then, their product $I_1\Gamma I_2$ and $I_2\Gamma I_1$ are also normal Γ-ideals of S and $I_1\Gamma I_2 = I_2\Gamma I_1$.*
Proof. We have $I_1\Gamma I_2 = I_2\Gamma I_1$ by Theorem 2.1. For $s \in S$, we have $s\Gamma(I_1\Gamma I_2) = (s\Gamma I_1)\Gamma I_2 = (I_1\Gamma s)\Gamma I_2 = I_1\Gamma(s\Gamma I_2) = I_1\Gamma(I_2\Gamma s) = (I_1\Gamma I_2)\Gamma s$. Hence, $I_1\Gamma I_2$ is normal.

Theorem 2.3. *Let I be a Γ-ideal of a regular Γ-semigroup S. Then, the following assertions are equivalent:*

- **(i):** *I is normal; and for all idempotents $e \in S$;*
- **(ii):** $e\Gamma I = I\Gamma e$;
- **(iii):** $(e)_B\Gamma I = I\Gamma(e)_B$;
- **(iv):** $(e)_B\Gamma I = I\Gamma(e)_L$;
- **(v):** $(e)_B\Gamma I = I\Gamma e$;
- **(vi):** $(e)_R\Gamma I = I\Gamma(e)_B$;
- **(vii):** $(e)_R\Gamma I = I\Gamma(e)_L$;
- **(viii):** $(e)_R\Gamma I = I\Gamma e$;
- **(ix):** $e\Gamma I = I\Gamma(e)_B$;
- **(x):** $e\Gamma I = I\Gamma(e)_L$.

Proof. $(i) \Rightarrow (ii)$. It is obvious. Furthermore, the equivalence of (ii) to (x) can be shown similar to the equivalence of (i) and (v) to (x) in the proof of Theorem 2.2.
$(ii) \Rightarrow (i)$. Suppose $s \in S$. As S is regular, there exists $x \in S$ such that $s = s\gamma_1 x\gamma_2 s$ and $x\gamma_1 s$ is idempotent for $\gamma_1, \gamma_2 \in \Gamma$. It follows that $s\Gamma I = (s\gamma_1 x\gamma_2 s)\Gamma I = s\Gamma((x\gamma_3 s)\Gamma I) = s\Gamma(I\Gamma(x\gamma_4 s)) = (s\Gamma I\Gamma x)\gamma_5 s \subseteq I\Gamma s$. In a similar fashion, we can show the reverse inclusion relation $I\Gamma s \subseteq s\Gamma I$ for all $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 \in \Gamma$. Consequently, we obtain $s\Gamma I = I\Gamma s$ for all $s \in S$.

Proposition 2.2. *Suppose I is any normal Γ-ideal of S. Then, we have that $s\Gamma I$ is a Γ-ideal of S for any $s \in S$.*
Proof. Suppose I is a normal Γ-ideal of S and $s \in S$. Then, it follows that $(s\Gamma I)\Gamma S = s\Gamma(I\Gamma S) \subseteq s\Gamma I$ and $S\Gamma(s\Gamma I) = S\Gamma(I\Gamma s) = (S\Gamma I)\Gamma s \subseteq I\Gamma s$. Hence, $s\Gamma I$ is a Γ-ideal of S.

We will need the following analogue in the context of Γ-semigroup of a famous result of Lajos [7] that the product of a bi-ideal and a nonempty subset of a semigroup is also a bi-ideal of the semigroup.

Theorem 2.4. [5] *The product of a bi-ideal and a nonempty subset of a semigroup S is also a bi-ideal of S.*

Theorem 2.5. *Any minimal Γ-ideal of a Γ-semigroup S is a zero element of $\mathcal{B}(S)$.*
Proof. Suppose I is a minimal Γ-ideal of S. Then it is obvious that $I \in \mathcal{B}(S)$. Suppose B is any bi-Γ-ideal of S. Then, we obtain $B\Gamma I \subseteq S\Gamma I \subseteq I$. Therefore, by Γ-semigroup analogue of Theorem 2.4 and the minimality of I, we have $B\Gamma I = I$. In a similar fashion, we can show that $I\Gamma B = I$ for all $B \in \mathcal{B}(S)$.

Theorem 2.6. *Any minimal normal Γ-ideal of a Γ-semigroup S is a Γ-group.*
Proof. Suppose I is a minimal normal Γ-ideal of a Γ-semigroup S and $s \in S$. Then, we obtain $I\Gamma s = s\Gamma I \subseteq S\Gamma I \subseteq I$. So, by Proposition 2.2 and the minimality of I, we obtain $I\Gamma s = s\Gamma I = I$. This shows that $I\Gamma s = s\Gamma I = I$ for all $s \in S$. Hence, I is a Γ-group.

Theorem 2.7. *The following conditions based on a Γ-semigroup S are equivalent:*

- **(i):** *S is normal*
- **(ii):** $B\Gamma S = S\Gamma B$ for all $B \in \mathcal{B}(S)$;
- **(iii):** $(s)_B\Gamma S = S\Gamma(s)_B$ for all $s \in S$;

(iv): $(s)_B\Gamma S = S\Gamma(s)_L$ *for all* $s \in S$*;*
(v): $(s)_B\Gamma S = S\Gamma s$ *for all* $s \in S$*;*
(vi): $(s)_R\Gamma S = S\Gamma(s)_B$ *for all* $s \in S$*;*
(vii): $(x)_R\Gamma S = S\Gamma(s)_L$ *for all* $s \in S$*;*
(viii): $(s)_R\Gamma S = S\Gamma s$ *for all* $s \in S$*;*
(ix): $s\Gamma S = S\Gamma(s)_B$ *for all* $s \in S$*;*
(x): $s\Gamma S = S\Gamma(s)_L$ *for all* $s \in S$*;*
(xi): $\mathcal{B}(S)$ *is normal;*
(xii): $(s)_B\Gamma\mathcal{B}(S) = \mathcal{B}(S)\Gamma(s)_B$ *for all* $s \in S$*;*
(xiii): $(s)_B\Gamma\mathcal{B}(S) = \mathcal{B}(S)\Gamma(s)_L$ *for all* $s \in S$*;*
(xiv): $(s)_B\Gamma\mathcal{B}(S) = \mathcal{B}(S)\Gamma s$ *for all* $s \in S$*;*
(xv): $(s)_R\Gamma\mathcal{B}(S) = \mathcal{B}(S)\Gamma(s)_B$ *for all* $s \in S$*;*
(xvi): $(s)_R\Gamma\mathcal{B}(S) = \mathcal{B}(S)\Gamma(s)_L$ *for all* $s \in S$*;*
(xvii): $(s)_R\Gamma\mathcal{B}(S) = \mathcal{B}(S)\Gamma s$ *for all* $s \in S$*;*
(xviii): $s\Gamma\mathcal{B}(S) = \mathcal{B}(S)\Gamma(s)_B$ *for all* $s \in S$*;*
(xix): $s\Gamma\mathcal{B}(S) = \mathcal{B}(S)\Gamma(s)_L$ *for all* $s \in S$*;*
(xxx): $s\Gamma\mathcal{B}(S) = \mathcal{B}(S)\Gamma s$ *for all* $s \in S$*.*

Proof. *As the* Γ*-semigroup* S *is a* Γ*-ideal of itself, we obtain that from (i) to (x) are equivalent by Theorem 2.1.*
$(i) \Rightarrow (xi)$ *Suppose (i) is true, then, we show (xi). Suppose* I *and* B *are any two bi-*Γ*-ideals of* S *and* $x \in I$*. Therefore, we have* $x\Gamma B \subseteq x\Gamma S = S\Gamma x \subseteq S\Gamma I \subseteq \mathcal{B}(S)\Gamma I$ *and hence,* $I\Gamma\mathcal{B}(S) \subseteq \mathcal{B}(S)\Gamma I$*. In a similar fashion, we can show that the reverse inclusion is true. Therefore, we have that* $I\Gamma\mathcal{B}(S) = \mathcal{B}(S)\Gamma I$ *for all* $I \in \mathcal{B}(S)$ *and also that* $\mathcal{B}(S)$ *is normal.*
$(xi) \Rightarrow (xii)$*. It is straightforward.*
$(xii) \Rightarrow (i)$*. Suppose* $s \in S$*. Then, for some* $I \in \mathcal{B}(S)$*, we obtain* $s\Gamma S \subseteq (s)_B\Gamma S = I\Gamma(s)_B \subseteq S\Gamma(s)_B \subseteq S\Gamma s$*. In a similar fashion, we can see that the reverse inclusion relation is true. Therefore,* S *is normal. The remaining part of the proof is straightforward.*

Corollary 2.1. *Every one-sided* Γ*-ideal of a normal* Γ*-semigroup* S *is a* Γ*-ideal of* S*.*

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