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ON THE SPECTRA OF CYCLES AND PATHS

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Abstract. Energy of a graph was defined by E. Hückel as the sum of absolute values of the eigenvalues of the adjacency matrix during the search for a method to obtain approximate solutions of Schrödinger equation which include the energy of the corresponding system for a class of molecules. The set of eigenvalues is called the spectrum of the graph and the spectral graph theory dealing with spectrums is one of the most interesting subareas of graph theory. There are a lot of results on the energy of many graph types. Two classes, cycles and paths, show serious differences from others as the eigenvalues are trigonometric algebraic numbers. Here, we obtain the polynomials and recurrence relations for the spectral polynomials of these two graph classes. In particular, we prove that one can obtain the spectra of C_{2n} and P_{2n+1} without detailed calculations just in terms of the spectra of C_n and P_n, respectively.

Keywords: Spectrum of a graph, graph energy, recurrence relation, path, cycle

AMS Subject Classification: 05C30, 05C38

1. Introduction

Throughout this paper, let $G = (V, E)$ be a simple connected graph, that is a graph with no loops nor multiple edges. Two vertices u and v of G are called adjacent if there is an edge e of G connecting u to v. Let $v_1, v_2, \cdots, v_n$ be the vertices of G. The $n \times n$ matrix $A = (a_{ij})$ defined by

$$a_{ij} = \begin{cases} 1, & if\ v_i\ and\ v_j\ are\ adjacent \\ 0, & otherwise. \end{cases}$$

is called the adjacency matrix of the graph G. With slight abuse of language, we shall call the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of a square $n \times n$ matrix A which are the roots of the equation $|A - \lambda I_n| = 0$ as the eigenvalues of the graph G. The polynomial on the left hand side of this equation is called the characteristic (or spectral) polynomial of A (and of the graph G). The set of all eigenvalues of the adjacency matrix A is called the spectrum of the graph G, denoted by $S(G)$. For more detailed information about the fundamental topics on graphs and spectrums of some well-known graphs, see [2], [8], [4], [5], [6], [11],

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[13], [14], [16], [17] and [21].

The energy of G defined as the sum of the absolute values of the eigenvalues of G is the basis for the subfield of graph theory called spectral graph theory, see [1], [12], [8], [15], [18], [19], [20].

We shall denote a path graph by P_n and a cycle graph by C_n. The spectrum of path and cycle graphs are known in literature, [18], [8]. These two spectra show differences with the other graph types as these two are the only ones the elements of which can be stated in terms of the roots of unity. The authors, in [10], found the characteristic polynomials of some graph types including path and cycle graphs, and also gave the recurrence formulae for these graphs. In this paper, we shall consider the spectrum of path and cycle graphs and find these spectra in terms of spectra of some smaller graphs. In particular, we give the spectrum of P_{2n+1} in terms of the spectrum of P_n, and the spectrum of C_{2n} in terms of the spectrum of C_n.

2. Spectrum of Cycle Graphs and Their Recurrences

It is known that the spectrum of a cycle graph C_n is given by

$$S(C_n) = \left\{ \lambda_i : \lambda_i = 2\cos\left(\frac{2\pi i}{n}\right), \quad i = 0, 1, 2, \cdots, n-1 \right\}$$

see [8], [10], [18]. If we note that the elements of $S(C_n)$ are all algebraic numbers defined by means of trigonometrical functions, then we can obtain the eigenvalues of some cycle graph in terms of the eigenvalues of a smaller cycle graph. We first need the following result:

Lemma 2.1. *Let C_n be a cycle graph. Then*

$$\lambda_k = \lambda_{n-k}$$

for every $k = 1, 2, \cdots, n-1$.

Proof. Using the properties of cosine function, we have

$$\begin{aligned}
\lambda_{n-k} &= 2\cos(\frac{2\pi(n-k)}{n}) \\
&= 2\cos(2\pi - \frac{2\pi k}{n}) \\
&= 2\cos(\frac{2\pi k}{n}) \\
&= \lambda_k.
\end{aligned}$$

□

Lemma 2.1 enables one to calculate only $\lambda_0, \lambda_1, \cdots, \lambda_{\lfloor n/2 \rfloor}$ instead of calculating all $\lambda_0, \lambda_1, \cdots, \lambda_{n-1}$ as follows:

Let the spectrum of C_n be

$$S(C_n) = \{\lambda_0, \lambda_1, \cdots, \lambda_{n-2}, \lambda_{n-1}\}$$

and the spectrum of C_{2n} be

$$S(C_{2n}) = \{\mu_0, \mu_1, \cdots, \mu_{2n-2}, \mu_{2n-1}\}$$

where $n \geq 3$. Then the relations between λ_i's and μ_j's are given below:

Theorem 2.1. *For $j = 0, 1, \cdots, n-1$, the spectrum of C_{2n} can be given as below using the spectrum of C_n:*

- *if $n \equiv 0 \pmod 4$, then*

$\mu_{2j} = \lambda_j, \mu_n = -2, \mu_{\frac{n}{2}} = \mu_{\frac{3n}{2}} = 0;$

$$\mu_{2j+1} = \begin{cases} \sqrt{\lambda_{2j+1}+2}, & for\ j = 0, 1, \cdots, \frac{n-4}{4}\ \ or\ \ \frac{3n}{4}, \frac{3n+4}{4}, \cdots, n-1 \\ -\sqrt{\lambda_{2j+1}+2}, & for\ j = \frac{n}{4}, \frac{n+4}{4}, \cdots, \frac{3n-4}{4}, \end{cases}$$

- *if $n \equiv 1 \pmod 4$, then*

$\mu_{2j} = \lambda_j, \mu_n = -2,$

$$\mu_{2j+1} = \begin{cases} \sqrt{\lambda_{2j+1}+2}, & for\ j = 0, 1, \cdots, \frac{n-5}{4}\ \ or\ \ \frac{3n+1}{4}, \frac{3n+5}{4}, \ldots, n-1 \\ -\sqrt{\lambda_{2j+1}+2}, & for\ j = \frac{n-1}{4}, \frac{n+3}{4}, \cdots, \frac{3n-3}{4}, \end{cases}$$

- *if $n \equiv 2 \pmod 4$, then*

$\mu_{2j} = \lambda_j, \mu_n = -2, \mu_{\frac{n}{2}} = \mu^{\frac{3n}{2}} = 0,,$

$$\mu_{2j+1} = \begin{cases} \sqrt{\lambda_{2j+1}+2}, & for\ j = 0, 1, \cdots, \frac{n-6}{4}\ \ or\ \ \frac{3n+2}{4}, \frac{3n+6}{4}, \cdots, n-1 \\ -\sqrt{\lambda_{2j+1}+2}, & for\ j = \frac{n+2}{4}, \frac{n+6}{4}, \cdots, \frac{3n-6}{4}, \end{cases}$$

and

- *if $n \equiv 3 \pmod 4$, then*

$\mu_{2j} = \lambda_j, \mu_n = -2,$

$$\mu_{2j+1} = \begin{cases} \sqrt{\lambda_{2j+1}+2}, & for\ j = 0, 1, \cdots, \frac{n-3}{4}\ \ or\ \ \frac{3n-1}{4}, \frac{3n+5}{4}, \cdots, n-1 \\ -\sqrt{\lambda_{2j+1}+2}, & for\ j = \frac{n+1}{4}, \frac{n+5}{4}, \cdots, \frac{3n-5}{4}. \end{cases}$$

Proof. Recall that $S(C_n) = \{\lambda_0, \lambda_1, \cdots, \lambda_{n-2}, \lambda_{n-1}\}$. Further we know that

$$\lambda_i = 2\cos\left(\frac{2\pi i}{n}\right) for\ i = 0, 1, \cdots, n-1.$$

As $S(C_{2n}) = \{\mu_0, \mu_1, \cdots, \mu_{2n-2}, \mu_{2n-1}\}$, we similarly know that

$$\mu_{2j} = 2\cos\left(\frac{2\pi 2j}{2n}\right) = 2\cos\left(\frac{2\pi j}{n}\right) for\ j = 0, 1, \cdots, 2n-1.$$

Now it is clear that

$$\mu_{2j} = 2\cos\left(\frac{2\pi j}{n}\right) = \lambda_j$$

for $j = 0, 1, \cdots, n-1$. Also, using double angle formulae, we have

$$\mu_k = \mp\sqrt{\lambda_k + 2}$$

as

$$\cos\left(\frac{2\pi k}{n}\right) = 2\cos^2\left(\frac{2\pi k}{2n}\right) - 1$$

$$\cos\left(\frac{2\pi k}{n}\right) + 1 = 2\cos^2\left(\frac{2\pi k}{2n}\right)$$

$$2\cos\left(\frac{2\pi k}{n}\right)+2=4\cos^2\left(\frac{2\pi k}{2n}\right)$$

$$\mp\sqrt{2\cos\left(\frac{2\pi k}{n}\right)+2}=2\cos\left(\frac{2\pi k}{2n}\right)$$

where $k=2j+1$ with

$$j=\begin{cases}0,1,2,\cdots,\frac{n-2}{2} & if\ n\ is\ even\\ 0,1,2,\cdots,\frac{n-3}{2} & if\ n\ is\ odd.\end{cases}$$

There are four possible cases:

If $n\equiv 0\ (mod\ 4)$, then we have the distribution of μ_j's as follows:

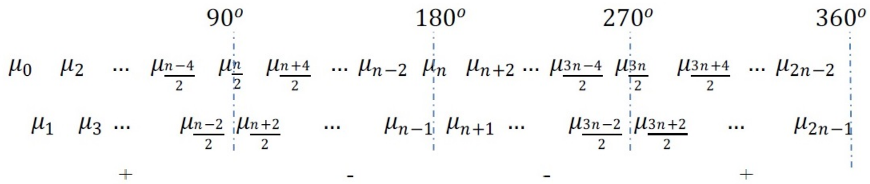

Figure 1.1 The case $n\equiv 0\ (mod\ 4)$ for cycle graph C_{2n}

Then we have μ_{2j+1} as asserted.

If $n\equiv 1\ (mod\ 4)$, then we have the distribution of μ_j's as follows:

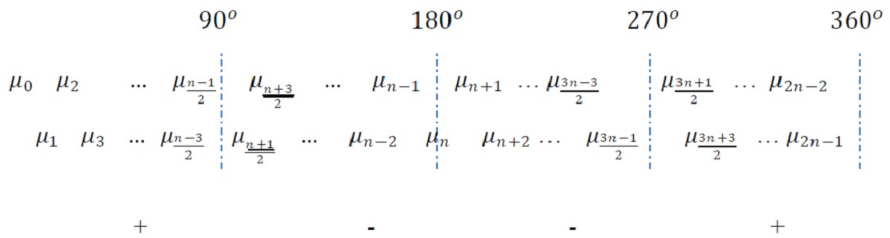

Figure 1.2 The case $n\equiv 1\ (mod\ 4)$ for cycle graph C_{2n}

Then we have the required values.

If $n\equiv 2\ (mod\ 4)$, then we have the distribution of μ_j's as follows:

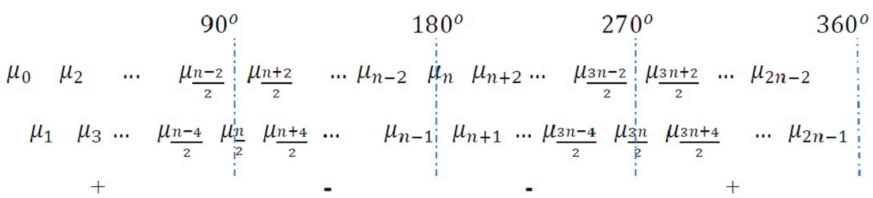

Figure 1.3 The case $n\equiv 2\ (mod\ 4)$ for cycle graph C_{2n}

Then we have the asserted values for μ_{2j+1}.

If $n\equiv 3\ (mod\ 4)$, then we have the distribution of μ_j's as follows:

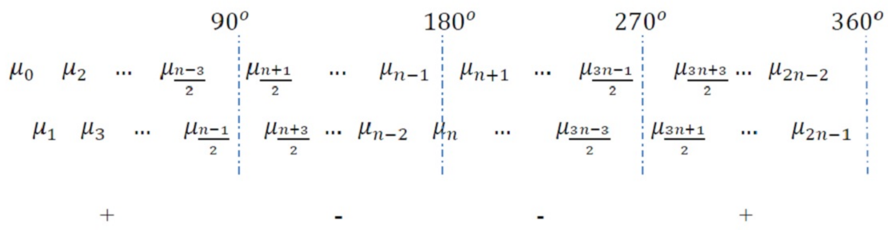

Figure 1.4 The case $n \equiv 3\ (mod\ 4)$ for cycle graph C_{2n}

similarly giving the result. □

Example 2.1. *For $n = 4$, the spectrum of C_4 is shown by $\{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$ and the spectrum of C_8 is shown by $\{\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7\}$. By using the spectrum of C_4, we can find the spectrum of C_8:*

By Lemma 2.1, we can write

$$\mu_1 = \mu_7,\ \ \mu_2 = \mu_6,\ \ \mu_3 = \mu_5$$

and we can also say that $\lambda_1 = \lambda_3$. By Theorem 2.1, we have $\mu_4 = \lambda_2 = -2$.

$$\begin{aligned} S\left(C_8\right) &= \{\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7\} \\ &= \{2, \sqrt{\lambda_1+2}, \lambda_1, -\sqrt{\lambda_3+2}, \lambda_2, -\sqrt{\lambda_3+2}, \lambda_3, \sqrt{\lambda_1+2}\} \\ &= \{2, \sqrt{\lambda_1+2}, \lambda_1, -\sqrt{\lambda_1+2}, \lambda_2, -\sqrt{\lambda_1+2}, \lambda_1, \sqrt{\lambda_1+2}\} \\ &= \left\{2, \sqrt{\lambda_1+2}^{(2)}, \lambda_1^{(2)}, -\sqrt{\lambda_1+2}^{(2)}, -2\right\} \\ &= \left\{2, \sqrt{2}^{(2)}, 0^{(2)}, -\sqrt{2}^{(2)}, -2\right\}. \end{aligned}$$

Example 2.2. *For $n = 5$, the spectrum of C_5 is shown by $\{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ and the spectrum of C_{10} is shown by $\{\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8, \mu_9\}$. By using the spectrum of C_5, we find the spectrum of C_{10} as follows. By Lemma 2.1, $\mu_1 = \mu_9, \mu_2 = \mu_8, \mu_3 = \mu_7, \mu_4 = \mu_6$ and also we can say that $\lambda_1 = \lambda_4, \lambda_2 = \lambda_3$. By Theorem 2.1, we get $\mu_5 = -2$. Therefore*

$$\begin{aligned} S\left(C_{10}\right) &= \{\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8, \mu_9\} \\ &= \left\{2, \sqrt{\lambda_1+2}, \lambda_1, -\sqrt{\lambda_3+2}, \lambda_2, -2, \lambda_3, -\sqrt{\lambda_3+2}, \lambda_4, \sqrt{\lambda_1+2}\right\} \\ &= \left\{2, \sqrt{\lambda_1+2}, \lambda_1, -\sqrt{\lambda_2+2}, \lambda_2, -2, \lambda_2, -\sqrt{\lambda_2+2}, \lambda_1, \sqrt{\lambda_1+2}\right\} \\ &= \left\{2, \sqrt{\lambda_1+2}^{(2)}, \lambda_1^{(2)}, -\sqrt{\lambda_2+2}^{(2)}, \lambda_2^{(2)}, -2\right\}. \end{aligned}$$

Example 2.3. *For $n = 6$, let the spectrum of C_6 be $\{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ and the spectrum of C_{12} be $\{\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8, \mu_9, \mu_{10}, \mu_{11}\}$. By using the spectrum of C_6, find* *the spectrum of C_{12}. By Lemma 2.1 and Theorem 2.1,*

$$\begin{aligned} S\left(C_{12}\right) &= \{\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8, \mu_9, \mu_{10}, \mu_{11}\} \\ &= \Big\{2, \sqrt{\lambda_1+2}, \lambda_1, 0, \lambda_2, -\sqrt{\lambda_1+2}, -2, -\sqrt{\lambda_1+2}\,, \\ &\quad \lambda_2, 0, \lambda_1, \sqrt{\lambda_1+2}\Big\} \\ &= \left\{2, \sqrt{\lambda_1+2}^{(2)}, \lambda_1^{(2)}, 0^{(2)}, \lambda_2^{(2)}, -\sqrt{\lambda_1+2}^{(2)}, -2\right\}. \end{aligned}$$

Example 2.4. *For $n = 7$, the spectrum of C_7 is shown by*

$$\{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}$$

and the spectrum of C_{14} is shown by

$$\{\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8, \mu_9, \mu_{10}, \mu_{11}, \mu_{12}, \mu_{13}\}.$$

By using the spectrum of C_7, we find the spectrum of C_{14}.

By Lemma 2.1 and Theorem 2.1

$$\begin{aligned} S(C_{14}) &= \{\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8, \mu_9, \mu_{10}, \mu_{11}, \mu_{12}, \mu_{13}\} \\ &= \Big\{2, \sqrt{\lambda_1+2}, \lambda_1, \sqrt{\lambda_3+2}, \lambda_2, -\sqrt{\lambda_2+2}, \lambda_3, -2, \lambda_3, \\ &\quad -\sqrt{\lambda_2+2}, \lambda_2, \sqrt{\lambda_3+2}, \lambda_1, \sqrt{\lambda_1+2}\Big\} \\ &= \{2, \sqrt{\lambda_1+2}^{(2)}, \lambda_1^{(2)}, \sqrt{\lambda_3+2}^{(2)}, \lambda_2^{(2)}, -\sqrt{\lambda_2+2}^{(2)}, \lambda_3^{(2)}, -2\}. \end{aligned}$$

3. Spectrum of Path Graphs and Their Recurrences

It is known that the spectrum of a path graph P_n is given by

$$S(P_n) = \left\{\lambda_i : \lambda_i = 2\cos\left(\frac{\pi i}{n+1}\right), \quad i = 1, 2, \ldots, n\right\}$$

see [8], [10], [18]. Like $S(C_n)$, the elements of $S(P_n)$ are all algebraic numbers defined by means of cosine function. We shall now obtain the eigenvalues of the path graph P_{2n+1} in terms of the eigenvalues of the smaller path graph P_n. We first have

Lemma 3.1. *Let P_n be a path graph. Then*

$$\lambda_k = -\lambda_{n+1-k}$$

for every $k = 1, 2, \ldots, n$.

Proof. Using the properties of cosine function, we have

$$\begin{aligned} \lambda_{n+1-k} &= 2\cos(\frac{\pi(n+1-k)}{n+1}) \\ &= 2\cos(\pi - \frac{\pi k}{n+1}) \\ &= -2\cos(\frac{\pi k}{n+1}) \\ &= -\lambda_k. \end{aligned}$$

□

Lemma 3.1 enables one to calculate only $\lambda_0, \lambda_1, \cdots, \lambda_{\lfloor (n+1)/2 \rfloor}$ instead of calculating all $\lambda_0, \lambda_1, \cdots, \lambda_{2n+1}$.

Let the spectrum of P_n be

$$S(P_n) = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$$

and the spectrum of P_{2n+1} be

$$S(P_{2n+1}) = \{\mu_1, \mu_2, \cdots, \mu_{2n}, \mu_{2n+1}\}$$

where $n \geq 3$. Then the relation between λ_i's and μ_j's is given below:

Theorem 3.1. *For $j = 1, 2, \ldots, n$, the spectrum of P_{2n+1} can be given as below using the spectrum of P_n:*

- *if n is odd, then*

$$\mu_{2j} = \lambda_j;$$

$$\mu_{2j-1} = \begin{cases} \sqrt{\lambda_{2j-1} + 2}, & for\ j = 1, 2, \cdots, \frac{n+1}{2}, \\ -\sqrt{\lambda_{2n-2j+3} + 2}, & for\ j = \frac{n+3}{2}, \frac{n+5}{2}, \cdots, n+1. \end{cases}$$

- *if n is even, then*

$$\mu_{2j} = \lambda_j, \quad \mu_{n+1} = 0;$$

$$\mu_{2j-1} = \begin{cases} \sqrt{\lambda_{2j-1} + 2}, & for\ j = 1, 2, \cdots, \frac{n}{2}, \\ -\sqrt{\lambda_{2n-2j+3} + 2}, & for\ j = \frac{n+4}{2}, \frac{n+6}{2}, \cdots, n+1. \end{cases}$$

Proof. Let $S(P_n) = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. Further we know that

$$\lambda_i = 2\cos\left(\frac{\pi i}{n+1}\right) for\ i = 1, 2, \cdots, n.$$

If $S(P_{2n+1}) = \{\mu_1, \mu_2, \cdots, \mu_{2n}, \mu_{2n+1}\}$, we similarly know that

$$\mu_{2j} = 2\cos\left(\frac{\pi 2j}{2n+1+1}\right) = 2\cos\left(\frac{\pi j}{n+1}\right)$$

for $j = 1, 2, \cdots, 2n+1$. Now it is clear that

$$\mu_{2j} = 2\cos\left(\frac{\pi j}{n+1}\right) = \lambda_j$$

for $j = 1, 2, \cdots, n$. Also using double angle formulae, we have

$$\mu_k = \mp\sqrt{\lambda_k + 2}$$

as

$$\cos\left(\frac{\pi k}{n+1}\right) = 2\cos^2\left(\frac{\pi k}{2n+2}\right) - 1$$

$$\cos\left(\frac{\pi k}{n+1}\right) + 1 = 2\cos^2\left(\frac{\pi k}{2n+2}\right)$$

$$2\cos\left(\frac{\pi k}{n+1}\right) + 2 = 4\cos^2\left(\frac{\pi k}{2n+2}\right)$$

$$\mp\sqrt{2\cos\left(\frac{\pi k}{n+1}\right) + 2} = 2\cos\left(\frac{\pi k}{2n+2}\right)$$

where $k = 2j - 1$ with

$$j = \begin{cases} 1, 2, \cdots, \frac{n}{2} & if\ n\ is\ even \\ 1, 2, \cdots, \frac{n+1}{2} & if\ n\ is\ odd. \end{cases}$$

There are two possible cases:

If n is odd, then we have the distribution of μ_j's as follows:

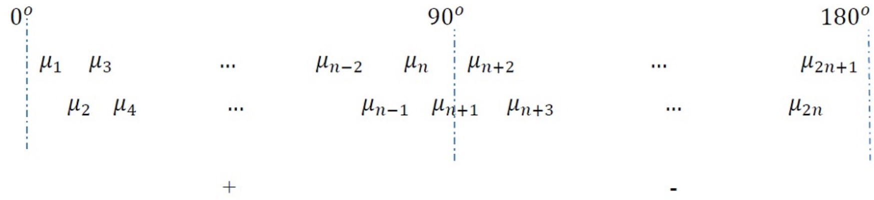

Figure 3.1 The case n is odd for path graph P_{2n+1}

Then we have

$$\mu_{2j-1} = \begin{cases} \sqrt{\lambda_{2j-1}+2}, & for\ j = 1, 2, \cdots, \frac{n+1}{2} \\ -\sqrt{\lambda_{2n-2j+3}+2}, & for\ j = \frac{n+3}{2}, \frac{n+5}{2}, \cdots, n+1 \end{cases}$$

and

$$\mu_{2j} = \lambda_j.$$

If n is even, then we have the distribution of μ_j's as follows:

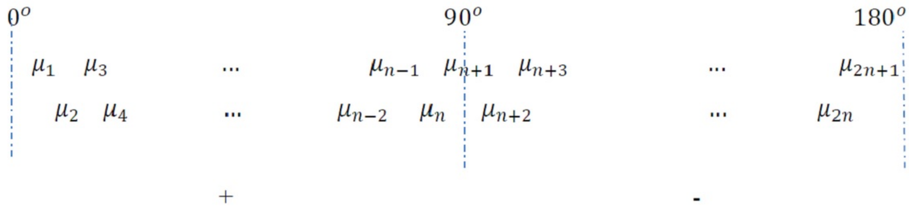

Figure 3.2 The case n even for path graph P_{2n+1}

Then we have

$$\mu_{2j-1} = \begin{cases} \sqrt{\lambda_{2j-1}+2}, & for\ j = 1, 2, \cdots, \frac{n}{2} \\ -\sqrt{\lambda_{2n-2j+3}+2}, & for\ j = \frac{n+4}{2}, \frac{n+6}{2}, \cdots, n+1 \end{cases}$$

and

$$\mu_{2j} = \lambda_j, \quad \mu_{n+1} = 0.$$

□

Example 3.1. *For $n = 7$, the spectrum of P_7 is shown by $\{\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_7\}$ and the spectrum of P_{15} is shown by $\{\mu_1, \mu_2, \mu_3, \cdots, \mu_{15}\}$. By using the spectrum of P_7, we can find the spectrum of P_{15} as follows:*

By Theorem 3.1, we get

$$\begin{aligned} S(P_{15}) &= \{\sqrt{\lambda_1+2}, \lambda_1, \sqrt{\lambda_3+2}, \lambda_2, \sqrt{\lambda_5+2}, \lambda_3, \sqrt{\lambda_7+2}, \lambda_4, -\sqrt{\lambda_7+2}, \lambda_5, \\ &\quad -\sqrt{\lambda_5+2}, \lambda_6, -\sqrt{\lambda_3+2}, \lambda_7, -\sqrt{\lambda_1+2}\} \end{aligned}$$

Also by Lemma 3.1,

$$\lambda_1 = -\lambda_7, \quad \lambda_2 = -\lambda_6, \quad \lambda_3 = -\lambda_5, \quad \lambda_4 = 0,$$

and then

$$\begin{aligned} S(P_{15}) &= \{\sqrt{\lambda_1+2}, \lambda_1, \sqrt{\lambda_3+2}, \lambda_2, \sqrt{-\lambda_3+2}, \lambda_3, \sqrt{-\lambda_1+2}, 0, -\sqrt{-\lambda_1+2}, \\ &\quad -\lambda_3, -\sqrt{-\lambda_3+2}, -\lambda_2, -\sqrt{\lambda_3+2}, -\lambda_1, -\sqrt{\lambda_1+2}\}. \end{aligned}$$

Example 3.2. *For $n = 8$, the spectrum of P_8 is shown by $\{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_8\}$ and the spectrum of P_{17} is shown by $\{\mu_1, \mu_2, \mu_3, \cdots, \mu_{17}\}$. By using the spectrum of P_8, we can obtain the spectrum of P_{17}.*

By Theorem 3.1, one has

$$\begin{aligned} S(P_{17}) &= \Big\{\sqrt{\lambda_1+2}, \lambda_1, \sqrt{\lambda_3+2}, \lambda_2, \sqrt{\lambda_5+2}, \lambda_3, \sqrt{\lambda_7+2}, \lambda_4, 0, \lambda_5, -\sqrt{\lambda_7+2}, \\ &\quad \lambda_6, -\sqrt{\lambda_5+2}, \lambda_7, -\sqrt{\lambda_3+2}, \lambda_8, -\sqrt{\lambda_1+2}\Big\} \end{aligned}$$

Also by Lemma 3.1,

$$\lambda_1 = -\lambda_8, \quad \lambda_2 = -\lambda_7, \quad \lambda_3 = -\lambda_6, \quad \lambda_4 = -\lambda_5$$

and then

$$\begin{aligned} S(P_{17}) &= \Big\{\sqrt{\lambda_1+2}, \lambda_1, \sqrt{\lambda_3+2}, \lambda_2, \sqrt{-\lambda_4+2}, \lambda_3, \sqrt{-\lambda_2+2}, \lambda_4, 0, -\lambda_4, \\ &\quad -\sqrt{-\lambda_2+2}, -\lambda_3, -\sqrt{-\lambda_4+2}, -\lambda_2, -\sqrt{\lambda_3+2}, -\lambda_1, -\sqrt{\lambda_1+2}\Big\}. \end{aligned}$$

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