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GENERALIZED POWER POMPEIU TYPE INEQUALITIES FOR LOCAL FRACTIONAL INTEGRALS WITH APPLICATIONS TO OSTROWSKI'S INEQUALITY

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Abstract. We establish some generalizations of power Pompeiu's inequality for local fractional integral. Afterwards, these results gave some new generalized Ostrowski type inequalities. Finally, some applications of these inequalities for generalized special means are obtained.

Keywords: Ostrowski's inequality, Pompeiu's mean value theorem, Local fractional integral, Fractal space, Special Means.

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1. Introduction

In 1938, it was obtained the following result by Ostrowski in [8].

Theorem 1.1. *Let $f:[a,b]\rightarrow\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f^{'}:(a,b)\rightarrow\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty=\sup_{t\in(a,b)}|f'(t)|<\infty$. Then, the following inequality holds:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\left\|f'\right\|_\infty \tag{1}$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

Inequality (1) has wide applications in numerical analysis and in the theory of special means; estimating error bounds for mid-point, trapezoid and Simpson rules and other quadrature rules, etc. It has attracted considerable attention and interest from mathematicians and other researchers as shown by hundreds of papers published in the last decade. As a

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result of these studies, one can find by making a simple search in the MathSciNet database of the American Mathematical Society.

In 1946, Pompeiu [9] derived a variant of Lagrange's mean value theorem, now known as *Pompeiu's mean value theorem.* It can be stated as follows:

Theorem 1.2. *For every real valued function f differentiable on an interval $[a,b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a,b]$, there exist a point ξ between x_1 and x_2 such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi).$$

It has been obtained the following Pompeiu type inequality by Dragomir in [4].

Theorem 1.3. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $[a,b]$ not containing 0. Then for any $x \in [a,b]$, we have the inequality*

$$\left| \frac{a+b}{2} \frac{f(x)}{x} + \frac{1}{b-a} \int_a^b f(t)dt \right|$$
$$\leq \frac{b-a}{|x|} \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] \left\| f - lf' \right\|_\infty .$$

where $l(t) = t$ for all $t \in [a,b]$. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Many researcher studied on inequailities by using Pompeiu mean value theorem. For example, it is established OStrowski type inequalities via Pompeiu mean value theorem in [1], [2], [4], [5], [10], [12]. Furthermore, Sarikaya obtained an inequality of Grüss type via variant Pompeiu mean value theorem in [11]. Also, a large number of Pompeiu type inequalities have been studied by mathematicians.

2. Preliminaries

Recall the set R^α of real line numbers and use the Gao-Yang-Kang's idea to describe the definition of the local fractional derivative and local fractional integral, see [14, 16] and so on.

Recently, the theory of Yang's fractional sets [14] was introduced as follows.

For $0 < \alpha \leq 1$, we have the following α-type set of element sets:

Z^α : The α-type set of integer is defined as the set $\{0^\alpha, \pm 1^\alpha, \pm 2^\alpha, ..., \pm n^\alpha, ...\}$.

Q^α : The α-type set of the rational numbers is defined as the set $\{m^\alpha = \left(\frac{p}{q}\right)^\alpha : p, q \in Z, q \neq 0\}$.

J^α : The α-type set of the irrational numbers is defined as the set $\{m^\alpha \neq \left(\frac{p}{q}\right)^\alpha : p, q \in Z, q \neq 0\}$.

R^α : The α-type set of the real line numbers is defined as the set $R^\alpha = Q^\alpha \cup J^\alpha$.

If a^α, b^α and c^α belongs the set R^α of real line numbers, then

(1) $a^\alpha + b^\alpha$ and $a^\alpha b^\alpha$ belongs the set R^α;
(2) $a^\alpha + b^\alpha = b^\alpha + a^\alpha = (a+b)^\alpha = (b+a)^\alpha$;
(3) $a^\alpha + (b^\alpha + c^\alpha) = (a+b)^\alpha + c^\alpha$;
(4) $a^\alpha b^\alpha = b^\alpha a^\alpha = (ab)^\alpha = (ba)^\alpha$;
(5) $a^\alpha (b^\alpha c^\alpha) = (a^\alpha b^\alpha) c^\alpha$;
(6) $a^\alpha (b^\alpha + c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha$;
(7) $a^\alpha + 0^\alpha = 0^\alpha + a^\alpha = a^\alpha$ and $a^\alpha 1^\alpha = 1^\alpha a^\alpha = a^\alpha$.

The definition of the local fractional derivative and local fractional integral can be given as follows.

Definition 2.1. [14] *A non-differentiable function $f : R \to R^{\alpha}$, $x \to f(x)$ is called to be local fractional continuous at x_0, if for any $\varepsilon > 0$, there exists $\delta > 0$, such that*

$$|f(x) - f(x_0)| < \varepsilon^{\alpha}$$

holds for $|x - x_0| < \delta$, where $\varepsilon, \delta \in R$. If $f(x)$ is local continuous on the interval (a, b), we denote $f(x) \in C_{\alpha}(a, b)$.

Definition 2.2. [14] *The local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined by*

$$f^{(\alpha)}(x_0) = \left.\frac{d^{\alpha} f(x)}{dx^{\alpha}}\right|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^{\alpha}(f(x) - f(x_0))}{(x - x_0)^{\alpha}},$$

where $\Delta^{\alpha}(f(x) - f(x_0)) \cong \Gamma(\alpha + 1)(f(x) - f(x_0))$.

If there exists $f^{(k+1)\alpha}(x) = \overbrace{D_x^{\alpha} ... D_x^{\alpha}}^{k+1 \text{ times}} f(x)$ for any $x \in I \subseteq R$, then we denote $f \in D_{(k+1)\alpha}(I)$, where $k = 0, 1, 2, ...$

Lemma 2.1. [15] *Suppose that $f(x) \in C_{\alpha}[a, b]$ and $f(x) \in D_{\alpha}(a, b)$, then for $0 < \alpha \leq 1$ we have a $\alpha-$differential form*

$$d^{\alpha} f(x) = f^{(\alpha)}(x) dx^{\alpha}.$$

Lemma 2.2. [15] *Let I be an interval, $f, g : I \subset R \to R^{\alpha}$ (I° is the interior of I) such that $f, g \in D_{\alpha}(I^{\circ})$. Then, the following differentiation rules are valid.*
(1) $\frac{d^{\alpha}[f(x) \pm g(x)]}{dx^{\alpha}} = f^{(\alpha)}(x) \pm g^{(\alpha)}(x)$;
(2) $\frac{d^{\alpha} f(x) g(x)}{dx^{\alpha}} = f^{(\alpha)}(x) g(x) + f(x) g^{(\alpha)}(x)$;
(3) $\frac{d^{\alpha}\left(\frac{f(x)}{g(x)}\right)}{dx^{\alpha}} = \frac{f^{(\alpha)}(x) g(x) - f(x) g^{(\alpha)}(x)}{[g(x)]^2}$ *where* $g(x) \neq 0$;
(4) $\frac{d^{\alpha}[c f(x)]}{dx^{\alpha}} = c f^{(\alpha)}(x)$ *where c is a constant;*
(5) If $y(x) = (f \circ g)(x)$, then

$$\frac{d^{\alpha} y(x)}{dx^{\alpha}} = f^{(\alpha)}(g(x)) \left(g^{(1)}(x)\right)^{\alpha}.$$

Theorem 2.1 (Generalized mean value theorem)**.** [18]*Suppose that $f(x) \in C_{\alpha}[a, b]$, $f^{(\alpha)}(x) \in C(a, b)$, then we have*

$$\frac{f(x) - f(x_0)}{(x - x_0)^{\alpha}} = \frac{f^{(\alpha)}(\xi)}{\Gamma(\alpha + 1)}$$

where $a < x_0 < \xi < x < b$.

Definition 2.3. [14] *Let $f(x) \in C_{\alpha}[a, b]$. Then the local fractional integral is defined by,*

$${}_aI_b^{\alpha} f(x) = \frac{1}{\Gamma(\alpha + 1)} \int_a^b f(t)(dt)^{\alpha} = \frac{1}{\Gamma(\alpha + 1)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^{\alpha},$$

with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_1, \Delta t_2, ..., \Delta t_{N-1}\}$, where $[t_j, t_{j+1}]$, $j = 0, ..., N - 1$ and $a = t_0 < t_1 < ... < t_{N-1} < t_N = b$ is partition of interval $[a, b]$.

Here, it follows that ${}_aI_b^{\alpha} f(x) = 0$ if $a = b$ and ${}_aI_b^{\alpha} f(x) = -{}_bI_a^{\alpha} f(x)$ if $a < b$. If for any $x \in [a, b]$, there exists ${}_aI_x^{\alpha} f(x)$, then we denoted by $f(x) \in I_x^{\alpha}[a, b]$.

Lemma 2.3. [14]

(1) (Local fractional integration is anti-differentiation) Suppose that $f(x) = g^{(\alpha)}(x) \in C_\alpha[a,b]$*, then we have*

$$_aI_b^\alpha f(x) = g(b) - g(a).$$

(2) (Local fractional integration by parts) Suppose that $f(x), g(x) \in D_\alpha[a,b]$ *and* $f^{(\alpha)}(x)$, $g^{(\alpha)}(x) \in C_\alpha[a,b]$*, then we have*

$$_aI_b^\alpha f(x)g^{(\alpha)}(x) = f(x)g(x)|_a^b - _aI_b^\alpha f^{(\alpha)}(x)g(x).$$

Lemma 2.4. [14] *We have*

i) $\dfrac{d^\alpha x^{k\alpha}}{dx^\alpha} = \dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)} x^{(k-1)\alpha};$

ii) $\dfrac{1}{\Gamma(\alpha+1)} \int_a^b x^{k\alpha}(dx)^\alpha = \dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)} \left(b^{(k+1)\alpha} - a^{(k+1)\alpha}\right),\ k \in R.$

Lemma 2.5 (Generalized Hölder's inequality)**.** [14] *Let* $f, g \in C_\alpha[a,b]$, $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\frac{1}{\Gamma(\alpha+1)} \int_a^b |f(x)g(x)| (dx)^\alpha \leq \left(\frac{1}{\Gamma(\alpha+1)} \int_a^b |f(x)|^p (dx)^\alpha\right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\alpha+1)} \int_a^b |g(x)|^q (dx)^\alpha\right)^{\frac{1}{q}}.$$

Theorem 2.2 (Generalized Ostrowski inequality)**.** [13]*Let* $I \subseteq \mathbb{R}$ *be an interval,* $f : I^0 \subseteq \mathbb{R} \to \mathbb{R}^\alpha$ *(*I^0 *is the interior of* I*) such that* $f \in D_\alpha(I^0)$ *and* $f^{(\alpha)} \in C_\alpha[a,b]$ *for* $a, b \in I^0$ *with* $a < b$ *Then. for all* $x \in [a,b]$*, we have the inequality*

$$\begin{aligned}&\left| f(x) - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\, _aI_b^\alpha f(t) \right| \\ &\leq 2^\alpha \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left[\frac{1}{4^\alpha} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^{2\alpha}\right] (b-a)^\alpha \left\| f^{(\alpha)} \right\|_\infty.\end{aligned}$$

In [6], Erden and Sarikaya proved the following identity and also they established the following inequality by using this identity.

Theorem 2.3 (Generalized Pompeiu's mean value theorem)**.** *Let* $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}^\alpha$ *be a mapping such that* $f \in D_\alpha(a,b)$*, with* $[a,b]$ *not containing* 0 *and for all pairs* $x_1 \neq x_2$ *in* $[a,b]$*, there exist a point* ξ *in* (x_1, x_2) *such that the following equality holds:*

$$\frac{x_1^\alpha f(x_2) - x_2^\alpha f(x_1)}{(x_1 - x_2)^\alpha} = f(\xi) - \frac{\xi^\alpha}{\Gamma(1+\alpha)} f^{(\alpha)}(\xi).$$

Theorem 2.4. *Let* $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}^\alpha$ *be a mapping such that* $f \in C_\alpha[a,b]$ *and* $f \in D_\alpha(a,b)$*, with* $[a,b]$ *not containing* 0*. Then for any* $x \in [a,b]$*, we have the inequality*

$$\begin{aligned}&\left| \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \frac{f(x)}{x^\alpha} (a+b)^\alpha - \frac{1}{(b-a)^\alpha}\, _aI_b^\alpha f(t) \right| \\ &\leq \frac{2^\alpha \Gamma(1+\alpha)(b-a)^\alpha}{\Gamma(1+2\alpha)\, |x|^\alpha} \left[\frac{1}{4^\alpha} + \frac{\left(x - \frac{a+b}{2}\right)^{2\alpha}}{(b-a)^{2\alpha}}\right] \left\| f - lf^{(\alpha)} \right\|_\infty\end{aligned}$$

where $l(t) = \frac{t^\alpha}{\Gamma(1+\alpha)}$, $t \in [a,b]$*, and* $\left\| f - lf^{(\alpha)} \right\|_\infty = \sup_{\xi \in (a,b)} \left| f(\xi) - lf^{(\alpha)}(\xi) \right| < \infty.$

The interested reader is invited to look over the references [3], [7], [14]-[19] for local fractional theory. Also, many researcher studied on generalized Ostrowski type inequalities for local fractions integrals (see, [13]). In addition, Erden and Sarikaya give generalized Pompeiu mean value theorem and some generalized Pompeiu type inequalities for local fractional calculus in [6].

In this study, some generalization of power Pompeiu's type inequalities involving local fractional integrals are obtained and also some new generalized Ostrowski type inequalities are obtained. Finally, applications of these inequalities for special means are also given.

3. Generalized Power Pompeiu's Type Inequalities

Generalized Ostrowski type inequalities can be derived using the following inequality.

Corollary 3.1 (Generalized Pompeiu's Inequality)**.** *With the assumptions of Theorem 2.3 and if* $\left\|f - lf^{(\alpha)}\right\|_\infty = \sup_{t\in(a,b)} \left|f(t) - lf^{(\alpha)}(t)\right| < \infty$ *where* $l(t) = \frac{t^\alpha}{\Gamma(1+\alpha)}$, $t \in [a,b]$, *then*

$$|t^\alpha f(x) - x^\alpha f(t)| \le \left\|f - lf^{(\alpha)}\right\|_\infty |x-t|^\alpha$$

for any $t, x \in [a,b]$.

We can generalize the above inequality for the power function as follows.

Theorem 3.1. *Let* $f : [a,b] \to \mathbb{R}^\alpha$ *be* $f \in D_\alpha(a,b)$ *and* $f \in C_\alpha [a,b]$, $b > a > 0$. *If* $r \in \mathbb{R}$, $r \neq 0$, *then for any* $x \in [a,b]$, *we have the inequality*

$$|t^{r\alpha} f(x) - x^{r\alpha} f(t)| \le \frac{1}{|r|^\alpha} |x^{r\alpha} - t^{r\alpha}| \left\|lf^{(\alpha)} - r^\alpha f\right\|_\infty \tag{2}$$

where $l(t) = \frac{t^\alpha}{\Gamma(1+\alpha)}$, $t \in [a,b]$ *and* $\left\|lf^{(\alpha)} - r^\alpha f\right\|_\infty = \sup_{s\in[a,b]} \left|f^{(\alpha)}(s)l(s) - r^\alpha f(s)\right|$.

Proof. Because of $f \in D_\alpha(a,b)$ and $f \in C_\alpha [a,b]$, $H \in D_\alpha(a,b)$ and $H \in C_\alpha [a,b]$ defined as $H(s) = \frac{f(s)}{s^{r\alpha}}$. Then, for any $t, x \in [a,b]$ with $x \neq t$, we have

$$\frac{1}{\Gamma(\alpha+1)} \int_t^x H^{(\alpha)}(s)(ds)^\alpha = \frac{f(x)}{x^{r\alpha}} - \frac{f(t)}{t^{r\alpha}}. \tag{3}$$

On the other side, using the second and fifth items of Theorem 2.2, we obtain

$$H^{(\alpha)}(s) = \frac{f^{(\alpha)}(s)s^\alpha - r^\alpha \Gamma(1+\alpha) f(s)}{s^{(r+1)\alpha}}. \tag{4}$$

From (3) and (4), we get

$$t^{r\alpha} f(x) - x^{r\alpha} f(t) = x^{r\alpha} t^{r\alpha} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1)} \int_t^x \frac{f^{(\alpha)}(s)\frac{s^\alpha}{\Gamma(1+\alpha)} - r^\alpha f(s)}{s^{(r+1)\alpha}} (ds)^\alpha. \tag{5}$$

Taking the modulus in (5), we have

$$|t^{r\alpha} f(x) - x^{r\alpha} f(t)| \tag{6}$$

$$\le x^{r\alpha} t^{r\alpha} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1)} \left| \int_t^x \frac{\left|f^{(\alpha)}(s)l(s) - r^\alpha f(s)\right|}{s^{(r+1)\alpha}} (ds)^\alpha \right|$$

and therefore we get the inequality

$$
\begin{aligned}
&\left|t^{r\alpha}f(x)-x^{r\alpha}f(t)\right| \\
&\leq x^{r\alpha}t^{r\alpha}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1)}\left|\int\limits_t^x \frac{1}{s^{(r+1)\alpha}}(ds)^{\alpha}\right| \sup_{s\in[x,t]([t,x])}\left|f^{(\alpha)}(s)l(s)-r^{\alpha}f(s)\right|.
\end{aligned}
$$

Applying Lemma 2.4(ii), we can write

$$
\left|t^{r\alpha}f(x)-x^{r\alpha}f(t)\right| \leq \frac{1}{|r|^{\alpha}}x^{r\alpha}t^{r\alpha}\left\|lf^{(\alpha)}-r^{\alpha}f\right\|_{\infty}\left|\frac{1}{t^{r\alpha}}-\frac{1}{x^{r\alpha}}\right|
$$

which competes the proof. □

Theorem 3.2. *Let* $f:[a,b]\to\mathbb{R}^{\alpha}$ *be* $f\in D_{\alpha}(a,b)$ *and* $f\in C_{\alpha}[a,b]$, $b>a>0$. *If* $r\in\mathbb{R}$, $r\neq 0$, *then for any* $x\in[a,b]$, *we have*

$$
\left|t^{r\alpha}f(x)-x^{r\alpha}f(t)\right| \leq \frac{x^{r\alpha}t^{r\alpha}\Gamma(\alpha+1)}{\min\left\{x^{(r+1)\alpha},t^{(r+1)\alpha}\right\}}\left\|lf^{(\alpha)}-r^{\alpha}f\right\|_{1}
$$

where $l(t)=\frac{t^{\alpha}}{\Gamma(1+\alpha)}$, $t\in[a,b]$, *and* $\left\|lf^{(\alpha)}-r^{\alpha}f\right\|_{1}$ *is defined by*

$$
\left\|lf^{(\alpha)}-r^{\alpha}f\right\|_{1}=\frac{1}{\Gamma(\alpha+1)}\int\limits_t^x\left|f^{(\alpha)}(s)l(s)-r^{\alpha}f(s)\right|(ds)^{\alpha}.
$$

Proof. If we utilize the inequality (6), then we obtain the inequality

$$
\begin{aligned}
&\left|t^{r\alpha}f(x)-x^{r\alpha}f(t)\right| \\
&\leq x^{r\alpha}t^{r\alpha}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1)}\left|\int\limits_t^x\frac{\left|f^{(\alpha)}(s)l(s)-r^{\alpha}f(s)\right|}{s^{(r+1)\alpha}}(ds)^{\alpha}\right| \\
&\leq x^{r\alpha}t^{r\alpha}\Gamma(\alpha+1)\left|\frac{1}{\Gamma(\alpha+1)}\int\limits_t^x\left|f^{(\alpha)}(s)l(s)-r^{\alpha}f(s)\right|(ds)^{\alpha}\right| \\
&\times\sup_{s\in[x,t]([t,x])}\left\{\frac{1}{s^{(r+1)\alpha}}\right\} \\
&=\frac{x^{r\alpha}t^{r\alpha}\Gamma(\alpha+1)}{\min\left\{x^{(r+1)\alpha},t^{(r+1)\alpha}\right\}}\left\|lf^{(\alpha)}-r^{\alpha}f\right\|_{1}.
\end{aligned}
$$

The proof is thus completed. □

Now, we prove a generalized power Pompeiu type inequality for $p-$norm.

Theorem 3.3. *Let $f:[a,b]\to\mathbb{R}^{\alpha}$ be $f\in D_{\alpha}(a,b)$ and $f\in C_{\alpha}[a,b]$, $b>a>0$. If $r\in\mathbb{R}$, $r\neq 0$ and $r\neq -\frac{1}{p}$, $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$, then for any $x\in[a,b]$, we have the inequality*

$$|t^{r\alpha}f(x)-x^{r\alpha}f(t)|$$

$$\leq \frac{x^{r\alpha}t^{r\alpha}\Gamma(\alpha+1)^{\frac{1}{p}}}{|1-q(r+1)|^{\frac{\alpha}{q}}}\left|x^{(1-q(r+1))\alpha}-t^{(1-q(r+1))\alpha}\right|^{\frac{1}{q}}\left\|lf^{(\alpha)}-r^{\alpha}f\right\|_{p},$$

where $l(t)=\frac{t^{\alpha}}{\Gamma(1+\alpha)}$, $t\in[a,b]$, and $\left\|lf^{(\alpha)}-r^{\alpha}f\right\|_{p}$ is defined by

$$\left\|lf^{(\alpha)}-r^{\alpha}f\right\|_{p}=\left(\frac{1}{\Gamma(\alpha+1)}\int_{t}^{x}\left|f^{(\alpha)}(s)l(s)-r^{\alpha}f(s)\right|^{p}(ds)^{\alpha}\right)^{\frac{1}{p}}.$$

Proof. Utilizing the inequality (6) and Hölder's integral inequality, we deduce

$$|t^{r\alpha}f(x)-x^{r\alpha}f(t)|\leq x^{r\alpha}t^{r\alpha}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1)}\left|\int_{t}^{x}\frac{\left|f^{(\alpha)}(s)l(s)-r^{\alpha}f(s)\right|}{s^{(r+1)\alpha}}(ds)^{\alpha}\right|$$

$$\leq x^{r\alpha}t^{r\alpha}\Gamma(\alpha+1)\left|\frac{1}{\Gamma(\alpha+1)}\int_{t}^{x}\frac{1}{s^{q(r+1)\alpha}}(ds)^{\alpha}\right|^{\frac{1}{q}}$$

$$\times\left|\frac{1}{\Gamma(\alpha+1)}\int_{t}^{x}\left|f^{(\alpha)}(s)l(s)-r^{\alpha}f(s)\right|^{p}(ds)^{\alpha}\right|^{\frac{1}{p}}.$$

Afterwards, should we apply Lemma 2.4(ii), then we get the inequality

$$|t^{r\alpha}f(x)-x^{r\alpha}f(t)|$$

$$\leq \frac{x^{r\alpha}t^{r\alpha}\Gamma(\alpha+1)^{\frac{1}{p}}}{|1-q(r+1)|^{\frac{\alpha}{q}}}\left\|lf^{(\alpha)}-r^{\alpha}f\right\|_{p}\left|x^{(1-q(r+1))\alpha}-t^{(1-q(r+1))\alpha}\right|^{\frac{1}{q}},$$

which completes the proof. □

4. Generalized Ostrowski Type Results

We give several Ostrowski type inequalities involving local fractional integral.

Theorem 4.1. *Let $f:[a,b]\to\mathbb{R}^{\alpha}$ be $f\in D_{\alpha}(a,b)$ and $f\in C_{\alpha}[a,b]$, $b>a>0$. If $r\in\mathbb{R}$, $r\neq 0$ and $r\neq -1$, then for any $x\in[a,b]$, we have*

$$\left|\frac{\Gamma(1+r\alpha)\left(b^{r+1}-a^{r+1}\right)^{\alpha}}{\Gamma(1+(r+1)\alpha)}f(x)-x^{r\alpha}\ {}_{a}I_{b}^{\alpha}f(t)\right| \tag{7}$$

$$\leq\frac{\left\|lf^{(\alpha)}-r^{\alpha}f\right\|_{\infty}}{|r|^{\alpha}}\times\begin{cases}M_{r}(x), & \text{if } r>0\\ -M_{r}(x), & \text{if } r\in(-\infty,0)\setminus\{-1\}\end{cases}$$

where $l(t)=\frac{t^{\alpha}}{\Gamma(1+\alpha)}$, $t\in[a,b]$ and $\left\|lf^{(\alpha)}-r^{\alpha}f\right\|_{\infty}=\sup_{t\in[a,b]}\left|f^{(\alpha)}(t)l(t)-r^{\alpha}f(t)\right|$, and $M_r(x)$ is defined by

$$M_{r}(x)=\frac{\Gamma(1+r\alpha)\left[2^{\alpha}x^{(r+1)\alpha}-\left(a^{r+1}+b^{r+1}\right)^{\alpha}\right]}{\Gamma(1+(r+1)\alpha)}+\frac{2^{\alpha}x^{(r+1)\alpha}-(a+b)^{\alpha}}{\Gamma(\alpha+1)}.$$

Proof. Integrating both sides of (2) with respect to t from a to b for local fractional integrals, we obtain

$$\left|\frac{\Gamma(1+r\alpha)\left(b^{r+1}-a^{r+1}\right)^{\alpha}}{\Gamma(1+(r+1)\alpha)}f(x)-x^{r\alpha}\ {}_aI_b^{\alpha}f(t)\right|$$

$$\leq \frac{\left\|lf^{(\alpha)}-r^{\alpha}f\right\|_{\infty}}{|r|^{\alpha}\Gamma(\alpha+1)}\int_a^b\left|x^{r\alpha}-t^{r\alpha}\right|(dt)^{\alpha}.$$

Should we take $r>0$, then we have

$$\begin{aligned}&\frac{1}{\Gamma(\alpha+1)}\int_a^b\left|x^{r\alpha}-t^{r\alpha}\right|(dt)^{\alpha}\\&=\frac{1}{\Gamma(\alpha+1)}\int_a^x\left(x^{r\alpha}-t^{r\alpha}\right)(dt)^{\alpha}+\frac{1}{\Gamma(\alpha+1)}\int_x^b\left(t^{r\alpha}-x^{r\alpha}\right)(dt)^{\alpha}\\&=\frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+1)\alpha)}\left[\left(a^{r+1}+b^{r+1}\right)^{\alpha}-2^{\alpha}x^{(r+1)\alpha}\right]+\frac{2^{\alpha}x^{(r+1)\alpha}-(a+b)^{\alpha}}{\Gamma(\alpha+1)}.\end{aligned}$$

On the other side, if we take $r\in(-\infty,0)\setminus\{-1\}$, then we have the equality

$$\begin{aligned}&\frac{1}{\Gamma(\alpha+1)}\int_a^b\left|x^{r\alpha}-t^{r\alpha}\right|(dt)^{\alpha}\\&=\frac{1}{\Gamma(\alpha+1)}\int_a^x\left(t^{r\alpha}-x^{r\alpha}\right)(dt)^{\alpha}+\frac{1}{\Gamma(\alpha+1)}\int_x^b\left(x^{r\alpha}-t^{r\alpha}\right)(dt)^{\alpha}\\&=\frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+1)\alpha)}\left[2^{\alpha}x^{(r+1)\alpha}-\left(a^{r+1}+b^{r+1}\right)^{\alpha}\right]+\frac{(a+b)^{\alpha}-2^{\alpha}x^{(r+1)\alpha}}{\Gamma(\alpha+1)}.\end{aligned}$$

The proof is thus completed. □

Theorem 4.2. *Let $f:[a,b]\to\mathbb{R}^{\alpha}$ be $f\in D_{\alpha}(a,b)$ and $f\in C_{\alpha}[a,b]$, $b>a>0$. If $r\in\mathbb{R}$, $r\neq 0$ and $r\neq 1$, then for any $x\in[a,b]$, we have*

$$\begin{aligned}&\left|\frac{f(x)}{x^{r\alpha}}(b-a)^{\alpha}-\ {}_aI_b^{\alpha}\frac{f(t)}{t^{r\alpha}}\right|\\&\leq\frac{1}{|r|^{\alpha}}\frac{\left\|lf^{(\alpha)}-r^{\alpha}f\right\|_{\infty}}{\Gamma(\alpha+1)}\times\begin{cases}S_r(x), & \text{if } r\in(0,\infty)\setminus\{1\}\\ -S_r(x), & \text{if } r<0\end{cases}\end{aligned}$$

where $l(t)=\frac{t^{\alpha}}{\Gamma(1+\alpha)}$, $t\in[a,b]$ and $\left\|lf^{(\alpha)}-r^{\alpha}f\right\|_{\infty}=\sup_{t\in[a,b]}\left|f^{(\alpha)}(t)l(t)-r^{\alpha}f(t)\right|$ and $S_r(x)$ is defined by

$$S_r(x)=\frac{2^{\alpha}x^{(1-r)\alpha}-\left(a^{1-r}+b^{1-r}\right)^{\alpha}}{\Gamma(\alpha+1)(1-r)^{\alpha}}+\frac{(a+b)^{\alpha}-2^{\alpha}x^{\alpha}}{\Gamma(\alpha+1)x^{r\alpha}}.$$

Proof. Dividing both sides of (2) with $t^{r\alpha}x^{r\alpha}$ and integrating over $t \in [a,b]$ for local fractional integrals, we obtain

$$\left|\frac{f(x)}{x^{r\alpha}}(b-a)^{\alpha} - {}_{a}I_{b}^{\alpha}\frac{f(t)}{t^{r\alpha}}\right| \leq \frac{1}{|r|^{\alpha}}\frac{\left\|lf^{(\alpha)} - r^{\alpha}f\right\|_{\infty}}{\Gamma(\alpha+1)}\int_{a}^{b}\left|\frac{1}{t^{r\alpha}} - \frac{1}{x^{r\alpha}}\right|(dt)^{\alpha}.$$

For $r \in (0,\infty)\setminus\{1\}$, we observe that

$$\frac{1}{\Gamma(\alpha+1)}\int_{a}^{b}\left|\frac{1}{t^{r\alpha}} - \frac{1}{x^{r\alpha}}\right|(dt)^{\alpha}$$

$$= \frac{1}{\Gamma(\alpha+1)}\int_{a}^{x}\left(\frac{1}{t^{r\alpha}} - \frac{1}{x^{r\alpha}}\right)(dt)^{\alpha} + \frac{1}{\Gamma(\alpha+1)}\int_{x}^{b}\left(\frac{1}{x^{r\alpha}} - \frac{1}{t^{r\alpha}}\right)(dt)^{\alpha}.$$

Also, using the Lemma 2.4(ii), we can write

$$\frac{1}{\Gamma(\alpha+1)}\int_{a}^{b}\left|\frac{1}{t^{r\alpha}} - \frac{1}{x^{r\alpha}}\right|(dt)^{\alpha} = \frac{2^{\alpha}x^{(1-r)\alpha} - \left(a^{1-r}+b^{1-r}\right)^{\alpha}}{\Gamma(\alpha+1)\,(1-r)^{\alpha}} + \frac{(a+b)^{\alpha} - 2^{\alpha}x^{\alpha}}{\Gamma(\alpha+1)x^{r\alpha}}$$

for any $x \in [a,b]$.

On the other side, for $r < 0$, we also have

$$\frac{1}{\Gamma(\alpha+1)}\int_{a}^{b}\left|\frac{1}{t^{r\alpha}} - \frac{1}{x^{r\alpha}}\right|(dt)^{\alpha} = \frac{\left(a^{1-r}+b^{1-r}\right)^{\alpha} - 2^{\alpha}x^{(1-r)\alpha}}{\Gamma(\alpha+1)\,(1-r)^{\alpha}} + \frac{2^{\alpha}x^{\alpha} - (a+b)^{\alpha}}{\Gamma(\alpha+1)x^{r\alpha}}$$

for any $x \in [a,b]$.

The proof is thus completed. □

5. Applications For Some Special Means

Let us recall some generalized means:

$$A_{\alpha}(a,b) = \frac{a^{\alpha}+b^{\alpha}}{2^{\alpha}};$$

$$L_{n}(a,b) = \left[\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n+1)\alpha)}\left[\frac{b^{(n+1)\alpha} - a^{(n+1)\alpha}}{(b-a)^{\alpha}}\right]\right]^{\frac{1}{n}},\ n \in Z\setminus\{-1,0\},\ a,b \in R,\ a \neq b.$$

Now, let us reconsider the inequality (7):

$$\left|\frac{\Gamma(1+r\alpha)\left(b^{r+1}-a^{r+1}\right)^{\alpha}}{\Gamma(1+(r+1)\,\alpha)}f(x) - x^{r\alpha}\ {}_{a}I_{b}^{\alpha}f(t)\right| \leq \frac{\left\|lf^{(\alpha)} - r^{\alpha}f\right\|_{\infty}}{|r|^{\alpha}} \times \begin{cases} M_{r}(x), & \text{if } r > 0 \\ -M_{r}(x), & \text{if } r \in (-\infty,0)\setminus\{-1\} \end{cases}$$

where $M_r(x)$ is defined by

$$M_r(x) = \frac{\Gamma(1+r\alpha)\left[2^{\alpha}x^{(r+1)\alpha} - \left(a^{r+1}+b^{r+1}\right)^{\alpha}\right]}{\Gamma(1+(r+1)\alpha)} + \frac{2^{\alpha}x^{(r+1)\alpha} - (a+b)^{\alpha}}{\Gamma(\alpha+1)}.$$

Consider the mapping $f:(0,\infty)\to R^{\alpha}$, $f(t)=t^{n\alpha}$, $n\in Z\backslash\{-1,0\}$. Then, $0<a<b$, we have

$$f\left(\frac{a+b}{2}\right) = [A_{\alpha}(a,b)]^{n}$$

and

$$\frac{1}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(t) = [L_n(a,b)]^{n}.$$

Now, should we use the Lemma 2.4, we obtain

$$\left\|lf^{(\alpha)} - r^{\alpha}f\right\|_{\infty} = \begin{cases} \left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+\alpha)\Gamma(1+(n-1)\alpha)} - r^{\alpha}\right| b^{n\alpha}, & n>1 \\ \left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+\alpha)\Gamma(1+(n-1)\alpha)} - r^{\alpha}\right| a^{n\alpha}, & n\in(-\infty,1]\backslash\{-1,0\} \end{cases}$$

and then we can write the inequality

$$\left|\frac{\Gamma(1+r\alpha)\left(b^{r+1}-a^{r+1}\right)^{\alpha}}{\Gamma(1+(r+1)\alpha)}[A_{\alpha}(a,b)]^{n} - (b-a)^{\alpha}[A_{\alpha}(a,b)]^{r}[L_n(a,b)]^{n}\right|$$

$$\leq \frac{2^{\alpha}\delta_n(a,b)}{|r|^{\alpha}} \times \begin{cases} M_r(x), & \text{if } r>0 \\ -M_r(x), & \text{if } r\in(-\infty,0)\setminus\{-1\} \end{cases}$$

where $\delta_n(a,b)$ is defined by

$$\delta_n(a,b) = \begin{cases} \left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+\alpha)\Gamma(1+(n-1)\alpha)} - r^{\alpha}\right| b^{n\alpha}, & n>1 \\ \left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+\alpha)\Gamma(1+(n-1)\alpha)} - r^{\alpha}\right| a^{n\alpha}, & n\in(-\infty,1]\backslash\{-1,0\} \end{cases}$$

and $M_r(x)$ is defined as

$$M_r(x) = \frac{\Gamma(1+r\alpha)\left[[A_{\alpha}(a,b)]^{(r+1)} - A_{\alpha}(a^{r+1},b^{r+1})\right]}{\Gamma(1+(r+1)\alpha)} + \frac{[A_{\alpha}(a,b)]^{(r+1)} - A_{\alpha}(a,b)}{\Gamma(\alpha+1)}.$$

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