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AN APPROXIMATE ANALYTICAL SOLUTION OF ONE-DIMENSIONAL GROUNDWATER RECHARGE BY SPREADING

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Abstract. The present paper discusses the problem of one dimensional groundwater recharge in the vertical direction. The groundwater is recharged by spreading of water in vertical direction and the moisture content of soil increases. On the basis of linear and nonlinear conductivity and diffusivity functions, three cases are considered for Brooks-Corey model. The governing nonlinear partial differential equations has been solved by homotopy analysis method. The proper value of convergence control parameter for convergent solution has been chosen from c_0-curve. The numerical and graphical solutions are presented.

Keywords: Groundwater; Moisture content; HAM.

AMS Subject Classification: 76S05, 65Nxx, 74H10

1. Introduction

One dimensional groundwater recharge problem is related to hydrology, environment engineering, soil mechanics, water resource engineering etc. The flow of water in unsaturated soil has been considered with some specific assumptions. The saturated zone, in which relatively all pores and fractures are saturated with water. In the unsaturated zone, the pore space is partly filled by air and partly by water. The unsaturated zone is the part of the subsurface between the ground surface and the groundwater table (see figure 1). Moisture content is the quantity of water contained in a soil. Moisture content is used in a wide range of scientific and technical areas. In the dry soil there is no moisture, so its value 0 and is 1 when the medium is fully saturated by water. So the range of moisture content is 0 (completely dry) to 1 (fully saturated by water). The water flow in the unsaturated zone is complicated by the fact that the soil's permeability to water depends on its water saturation [1]. The water flow through soil is unsteady and slightly saturated because the moisture content is time dependent function and all pores are not completely filled with water.

The Richard's equation is one of the most well-known equations to describe the behavior of unsaturated zones in soil. One dimensional groundwater recharge has great importance

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in many branches of science and engineering. The problem of groundwater flow has been discussed by many researchers with different aspects, like as Klute [10] reduced diffusion equation to an ordinary differential equation and applied a forward integration and iteration method, Verma [27] obtained solution of a one dimensional groundwater recharge for constant diffusivity and linear conductivity by laplace transform, Mehta [15] obtained singular perturbation solution of one dimensional flow in unsaturated porous media with small diffusivity coefficient, Prasad et al. [24] developed numerical model to simulate moisture flow through unsaturated zones using the finite element method, Desai [6] obtained composite expansion solution for groundwater recharge in vertical direction, Mehta and Patel [16] obtained solution of Burger's equation for one dimensional groundwater recharge by spreading in porous media, Joshi et al. [8] obtained solution of one dimensional vertical groundwater recharge by group theoretical approach, Nasseri et al. [17] studied solution of advection-diffusion equation on the basis of the simplified Brooks-Corey model for soil conductivity and diffusivity.

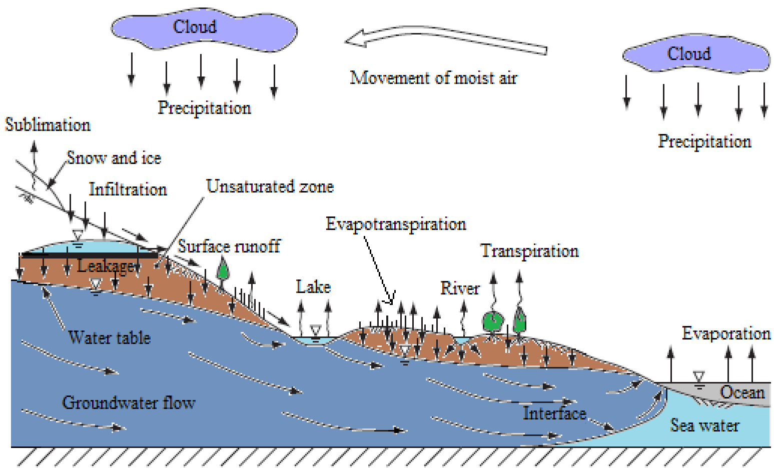

FIGURE 1. The hydrological cycle [3].

The main goal of the present work is to obtained solution of one dimensional groundwater recharge by spreading. It is assumed that the groundwater recharge takes place over the large basin of such geological location that the sides are bounded by rigid boundaries while the bottom by a thick layer of water table for investigated flow problem. Here the flow takes place downward direction through unsaturated porous medium up to depth L (L is length of basin). On the basis of linear and nonlinear conductivity and diffusivity functions, three cases are considered for Brooks-Corey model. The mathematical formulation leads to one dimensional nonlinear partial differential equation which is solved by homotopy analysis method. The solution expresses in series form and it gives moisture content of soil.

2. Mathematical Formulation

The equation of continuity for water flow through unsaturated porous medium is governed by

$$\frac{\partial}{\partial t}(\rho\Theta) = -\nabla \cdot M \tag{1}$$

where ρ is the fluid density, Θ is the moisture content and M is a mass flux of moisture. Darcy's law for the motion of water in unsaturated porous medium is expressed as [2]

$$v = -\kappa \nabla \Phi \tag{2}$$

where v is the volume flux of moisture, κ is the hydraulic conductivity, $\nabla\Phi$ is the gradient of the whole moisture potential. The mass flux of moisture M is the product of fluid density ρ and volume flux of moisture v, i.e. $M = \rho v$. Thus (1) and (2) gives us

$$\frac{\partial}{\partial t}(\rho\Theta) = \nabla(\rho\kappa\nabla\Phi). \tag{3}$$

Consider the relation $\Phi = \psi - z$ for the system in which flow takes place in the vertical direction only where ψ is the pressure potential. The vertical downward direction is considered as the positive direction of z-axis. Considering only the one dimensional vertical flow for incompressible fluid, (3) becomes

$$\frac{\partial\Theta}{\partial t} = \frac{\partial}{\partial z}\left(\kappa\frac{\partial\psi}{\partial z}\right) - \frac{\partial\kappa}{\partial z}. \tag{4}$$

Considering Θ and ψ to be related by single valued function and assume that $D = \kappa\frac{\partial\psi}{\partial\Theta}$ is the soil water diffusivity. Thus (4) reduces to

$$\frac{\partial\Theta}{\partial t} = \frac{\partial}{\partial z}\left(D\frac{\partial\Theta}{\partial z}\right) - \frac{\partial\kappa}{\partial z}. \tag{5}$$

This equation is known as Richard's equation [1, 10, 17, 25] which is one of the most important equations expressing water content in unsaturated porous medium with broad applications in hydrology, engineering, and soil sciences.

3. Discussions

We consider three nonlinear forms of Richard's equation (5) for linear and nonlinear conductivity and diffusivity coefficients. We will focus on Brooks-Corey model, with conductivity and diffusivity assumed to be of the form $\kappa = \kappa_0\Theta^k$ and $D = D_0\Theta^n$ with $k \geq 1,\ n \geq 0$ [4, 17, 28]. We used Brooks-Corey model [4, 17, 28] for the three cases as (i) Linear diffusivity and nonlinear conductivity (ii) Linear diffusivity and linear conductivity (iii) Nonlinear diffusivity and linear conductivity.

3.1. **Linear diffusivity and nonlinear conductivity.** For the case, consider D is the constant as $D = D_0$ and the nonlinear conductivity represented as $\kappa = \kappa_0\Theta^2$, $\kappa_0 = \frac{D_0}{2L}$ [15]. Then (5) results

$$\frac{\partial\Theta}{\partial t} = D_0\frac{\partial^2\Theta}{\partial z^2} - \frac{D_0}{L}\Theta\frac{\partial\Theta}{\partial z}. \tag{6}$$

Using dimensionless variables,

$$Z = \frac{z}{L} \text{ and } T = \frac{tD_0}{L^2}$$

(6) reduces to the governing equation

$$\frac{\partial\Theta}{\partial T} = \frac{\partial^2\Theta}{\partial Z^2} - \Theta\frac{\partial\Theta}{\partial Z}. \tag{7}$$

3.2. **Linear diffusivity and linear conductivity.** Assume that the linear diffusivity D as $D = D_0\Theta$ and the linear conductivity represented as $\kappa = \kappa_0\Theta$, $\kappa_0 = \frac{D_0}{2L}$ [15]. Then (5) becomes

$$\frac{\partial\Theta}{\partial t} = D_0\Theta\frac{\partial^2\Theta}{\partial z^2} + D_0\left\{\frac{\partial\Theta}{\partial z}\right\}^2 - \frac{D_0}{2L}\frac{\partial\Theta}{\partial z}. \tag{8}$$

Using dimensionless variables,

$$Z = \frac{z}{L} \text{ and } T = \frac{tD_0}{L^2}$$

(8) gives us

$$\frac{\partial\Theta}{\partial T} = \Theta\frac{\partial^2\Theta}{\partial Z^2} + \left\{\frac{\partial\Theta}{\partial Z}\right\}^2 - \frac{1}{2}\frac{\partial\Theta}{\partial Z}. \tag{9}$$

3.3. **Nonlinear diffusivity and linear conductivity.** Consider D is nonlinear function of Θ as $D = D_0\Theta^2$ and assuming the hydraulic conductivity as $\kappa = \kappa_0\Theta$, $\kappa_0 = \frac{D_0}{2L}$ [15]. Then (5) reduces to

$$\frac{\partial\Theta}{\partial t} = D_0\Theta^2\frac{\partial^2\Theta}{\partial z^2} + 2D_0\Theta\left\{\frac{\partial\Theta}{\partial z}\right\}^2 - \frac{D_0}{2L}\frac{\partial\Theta}{\partial z}. \tag{10}$$

Using dimensionless variables,

$$Z = \frac{z}{L} \text{ and } T = \frac{tD_0}{L^2}$$

(10) becomes

$$\frac{\partial\Theta}{\partial T} = \Theta^2\frac{\partial^2\Theta}{\partial Z^2} + 2\Theta\left\{\frac{\partial\Theta}{\partial Z}\right\}^2 - \frac{1}{2}\frac{\partial\Theta}{\partial Z}. \tag{11}$$

The equations (7), (9) and (11) are solved with boundary conditions by homotopy analysis method. The solutions of these equations represent moisture content of soil at a depth Z and time T. For definiteness of the physical problem, the water will flow in vertical downward direction and hence set of boundary conditions are given by:

$$\Theta(0, T) = 0.01 \text{ and } \Theta(1, T) = 1. \tag{12}$$

4. Homotopy Analysis Method

Homotopy analysis method was first proposed by Liao [11] in his Ph.D. thesis. This method is used to solve the nonlinear differential equations. Many researchers have successfully employed this technique to solve different types of nonlinear ODEs as well as nonlinear PDEs [5, 7, 9, 11, 12, 13, 14, 18, 19, 20, 21, 22, 23, 26]. Liao [11, 12, 13, 14] has discussed homotopy analysis solutions of nonlinear ordinary and partial differential equations, Darvishi and Khani [5] have discussed homotopy series solution of the foam drainage equation, Ghotbi et al. [7] have discussed an analytical approach of infiltration in unsaturated soils by HAM, Kheiri et al. [9] have obtained approximation of modified Burgers-Korteweg-de Vries equation and the Newell -Whitehead equation by using HAM, Vajravelu and Van Gorder [26] have discussed solutions of nonlinear ordinary and partial differential equations by HAM, Patel and Desai [19, 20, 21, 22, 23] have discussed convergent solution of one dimensional nonlinear partial differential equation arising in fluid flow through porous media by HAM with the help of c_0-curve.

Let $\mathcal{N}[\phi(Z, T; q)] = 0$ denote a nonlinear partial differential equation, $\phi(Z, T; q)$ be an unknown function which represents Θ at depth Z for a given time T for $0 \leq q \leq 1$.

We use the auxiliary linear operator $\mathcal{L}[\phi(Z,T;q)] = \frac{\partial^2\phi(Z,T;q)}{\partial Z^2}$ and the initial approximation of $\Theta(Z,T)$ is $\Theta_0(Z,T) = (1+T)Z + 0.01(1-Z^2) - TZ^3$ which satisfy both boundary conditions.

The zeroth-order deformation equation [11] is constructed as

$$(1-q)\mathcal{L}[\phi(Z,T;q) - \Theta_0(Z,T)] = c_0 q H(Z,T)\mathcal{N}[\phi(Z,T;q)] \tag{13}$$

where $q \in [0,1]$ the embedding-parameter, c_0 nonzero convergence control parameter, $H(Z,T)$ nonzero auxiliary function.

When $q=0$ and $q=1$, (13) gives us

$$\phi(Z,T;0) = \Theta_0(Z,T) \text{ and } \phi(Z,T;1) = \Theta(Z,T). \tag{14}$$

According to (14) as q increases from 0 to 1, $\phi(Z,T;q)$ continuously varies from $\Theta_0(Z,T)$ to $\Theta(Z,T)$. The solution is considered as

$$\phi(Z,T;q) = \Theta_0(Z,T) + \sum_{m=1}^{\infty}\Theta_m(Z,T)q^m \tag{15}$$

where

$$\Theta_m(Z,T) = \frac{1}{m!}\frac{\partial^m\phi(Z,T;q)}{\partial q^m}\bigg|_{q=0}. \tag{16}$$

The auxiliary linear operator, the initial approximation, the convergence control parameter and the auxiliary function are assume in such a way that the series of $\phi(Z,T;q)$ with respect to q at $q=1$

$$\Theta(Z,T) = \Theta_0(Z,T) + \sum_{m=1}^{\infty}\Theta_m(Z,T) \tag{17}$$

converges.

Define $\overrightarrow{\Theta_n} = \{\Theta_0(Z,T), \Theta_1(Z,T), \ldots, \Theta_n(Z,T)\}$. Differentiating the zeroth order deformation equation (13) m times with respect to q and putting $q=0$ and then finally dividing them by $m!$, we have the mth-order deformation equation

$$\mathcal{L}[\Theta_m(Z,T) - \chi_m\Theta_{m-1}(Z,T)] = c_0 H(Z,T)\mathcal{R}_m(\overrightarrow{\Theta_{m-1}}) \tag{18}$$

subject to the boundary conditions

$$\Theta_m(0,T) = 0 \text{ and } \Theta_m(1,T) = 0,\ m \geq 1 \tag{19}$$

where

$$\mathcal{R}_m(\overrightarrow{\Theta_{m-1}}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\phi(Z,T;q)]}{\partial q^{m-1}}\bigg|_{q=0}, m \geq 1 \tag{20}$$

and

$$\chi_m = \begin{cases} 0 & \text{if } m \leq 1, \\ 1 & \text{if } m > 1. \end{cases} \tag{21}$$

For simplicity, we assume that $H(Z,T) = 1$. Thus the solution of the mth-order deformation equation (18) is

$$\Theta_m(Z,T) = \chi_m\Theta_{m-1}(Z,T) + c_0\mathcal{L}^{-1}[\mathcal{R}_m(\overrightarrow{\Theta_{m-1}})] + C_1 Z + C_2 \tag{22}$$

where C_1 and C_2 are constants or functions of T. Hence the homotopy analysis solution is as

$$\Theta(Z,T) = \Theta_0(Z,T) + \Theta_1(Z,T) + \Theta_2(Z,T) + \cdots . \tag{23}$$

As discussed by many researchers [5, 7, 9, 12, 14, 18, 19, 20, 21, 22, 23, 26] the convergence of homotopy series solution is strongly depends on convergence control parameter c_0 which occurs in solution (23). The proper value of c_0 is chosen with the help of c_0-curve which is suggested by Liao [12].

4.1. **Solution of equation** (7)**.** According to (7), we define a nonlinear operator $\mathcal{N}$ as

$$\mathcal{N}[\phi(Z,T;q)] = \frac{\partial^2\phi(Z,T;q)}{\partial Z^2} - \phi(Z,T;q)\frac{\partial\phi(Z,T;q)}{\partial Z} - \frac{\partial\phi(Z,T;q)}{\partial T}. \tag{24}$$

Apply homotopy analysis method to (20) with above mentioned linear operator and initial approximation, we get

$$\mathcal{R}_m(\overrightarrow{\Theta_{m-1}}) = \frac{\partial^2\Theta_{m-1}}{\partial Z^2} - \sum_{i=0}^{m-1}\Theta_i\frac{\partial\Theta_{m-1-i}}{\partial Z} - \frac{\partial\Theta_{m-1}}{\partial T}, m \geq 1 \tag{25}$$

and the solution (23) is of the form

$$\begin{aligned}\Theta(Z,T) = (1+T)Z + 0.01(1-Z^2) - TZ^3 + c_0\Big\{&0.29581Z - 0.015Z^2 - 0.3333Z^3\\ &+ 0.0025Z^4 + 0.04999Z^5 + 1.135TZ - 0.005TZ^2 - \frac{4TZ^3}{3} + 0.005TZ^4\\ &+ \frac{TZ^5}{5} - \frac{0.005TZ^6}{3} + \frac{4T^2Z}{105} - \frac{T^2Z^3}{6} + \frac{T^2Z^5}{5} - \frac{T^2Z^7}{14}\Big\} + \cdots . \end{aligned}\tag{26}$$

Here the c_0-curves of $\Theta_Z(0.5, 0.5)$ and $\Theta_Z(1, 0)$ are used to choose proper value of c_0. Figure 2 represents the c_0-curves of $\Theta_Z(0.5, 0.5)$ (DotDashed) and $\Theta_Z(1, 0)$ (Thick) for 10th order approximation which is plotted using Mathematica BVPh package [13]. The line segment almost parallel to horizontal axis gives valid interval of c_0 [12]. We chosen $c_0 = -0.5$ from valid range of c_0 (see figure 2).

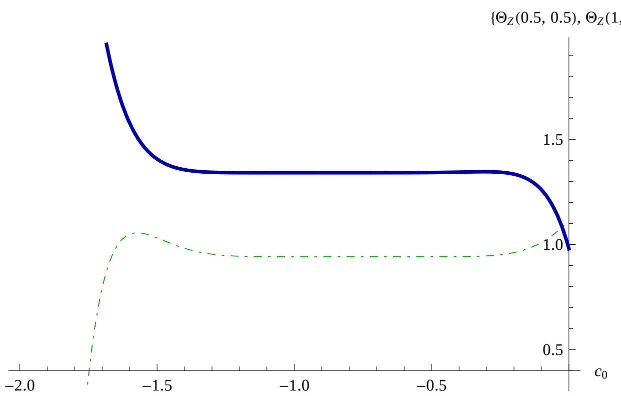

FIGURE 2. The c_0-curves of $\Theta_Z(0.5, 0.5)$ (DotDashed) and $\Theta_Z(1, 0)$ (Thick).

The approximate analytical solution of (7) is expressed in the series form which represents the moisture content $\Theta(Z,T)$ of soil at depth Z for a given time T. The numerical

TABLE 1. Numerical values of the moisture content Θ.

T	$Z=0.0$	$Z=0.1$	$Z=0.2$	$Z=0.3$	$Z=0.4$	$Z=0.5$	$Z=0.6$	$Z=0.7$	$Z=0.8$	$Z=0.9$	$Z=1.0$
0.1	0.01000	0.09433	0.17946	0.26614	0.35517	0.44743	0.54394	0.64587	0.75463	0.87196	1.00000
0.2	0.01000	0.09433	0.17946	0.26614	0.35517	0.44744	0.54395	0.64589	0.75465	0.87197	1.00000
0.3	0.01000	0.09433	0.17946	0.26614	0.35518	0.44745	0.54397	0.64590	0.75467	0.87199	1.00000
0.4	0.01000	0.09433	0.17947	0.26615	0.35519	0.44747	0.54399	0.64592	0.75469	0.87200	1.00000
0.5	0.01000	0.09434	0.17948	0.26617	0.35521	0.44749	0.54400	0.64594	0.75470	0.87201	1.00000
0.6	0.01000	0.09434	0.17949	0.26619	0.35523	0.44751	0.54403	0.64596	0.75472	0.87202	1.00000
0.7	0.01000	0.09435	0.17951	0.26621	0.35526	0.44753	0.54405	0.64598	0.75473	0.87203	1.00000
0.8	0.01000	0.09436	0.17953	0.26623	0.35529	0.44756	0.54407	0.64600	0.75474	0.87204	1.00000
0.9	0.01000	0.09438	0.17955	0.26626	0.35532	0.44759	0.54410	0.64601	0.75475	0.87205	1.00000
1.0	0.01000	0.09439	0.17958	0.26630	0.35536	0.44763	0.54413	0.64603	0.75476	0.87205	1.00000

representation of the solution is obtained using Mathematica coding [13]. Table 1 indicates the numerical values of $\Theta(Z,T)$.

The graphical representation of solution (26) is also obtained using Mathematica coding [13]. Figure 2 represents the graph of $\Theta(Z,T)$ versus depth Z for fixed time $T = 0.2, 0.4, 0.6, 0.8, 1$ and figure 3 represents the graph of $\Theta(Z,T)$ versus depth Z and time T.

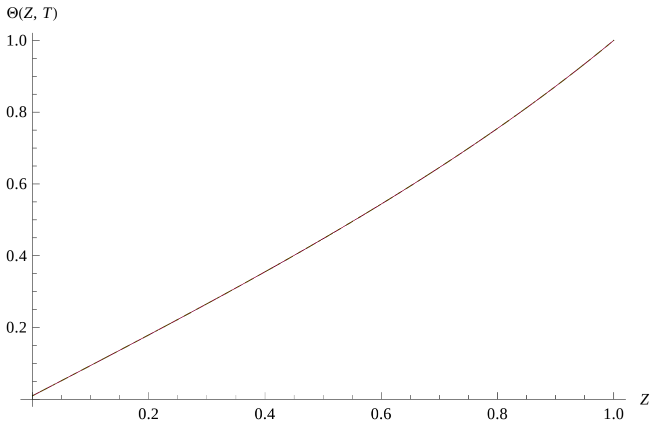

FIGURE 3. The graph of $\Theta(Z,T)$ v/s Z for fixed $T = 0.2, 0.4, 0.6, 0.8, 1$.

4.2. **Solution of equation** (9)**.** Apply homotopy analysis method to nonlinear partial differential equation (9) with nonlinear operator $\mathcal{N}[\phi(Z,T;q)]$ as

$$\mathcal{N}[\phi(Z,T;q)] = \phi(Z,T;q)\frac{\partial^2\phi(Z,T;q)}{\partial Z^2} + \left\{\frac{\partial\phi(Z,T;q)}{\partial Z}\right\}^2 - \frac{1}{2}\frac{\partial\phi(Z,T;q)}{\partial Z} - \frac{\partial\phi(Z,T;q)}{\partial T}. \tag{27}$$

Now (20) becomes

$$\mathcal{R}_m(\overrightarrow{\Theta_{m-1}}) = \sum_{i=0}^{m-1}\Theta_i\frac{\partial^2\Theta_{m-1-i}}{\partial Z^2} - \sum_{i=0}^{m-1}\frac{\partial\Theta_i}{\partial Z}\frac{\partial\Theta_{m-1-i}}{\partial Z} - \frac{1}{2}\frac{\partial\Theta_{m-1}}{\partial Z} - \frac{\partial\Theta_{m-1}}{\partial T}, m \geq 1 \tag{28}$$

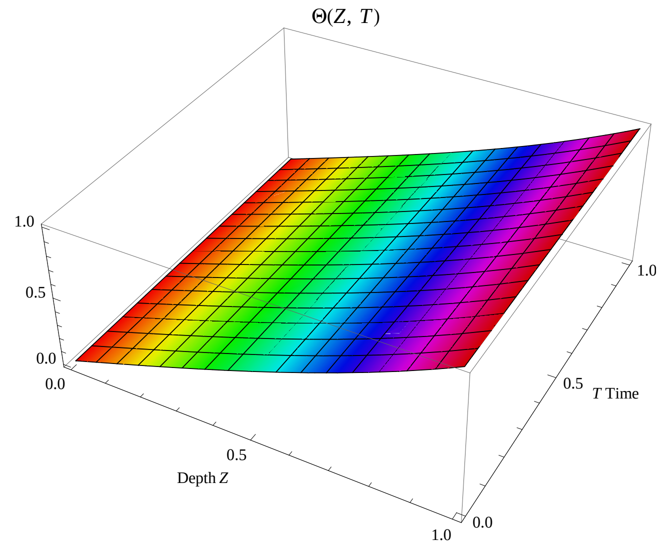

FIGURE 4. The graph of $\Theta(Z,T)$ v/s Z and T.

and the homotopy analysis solution of (9) is

$$\Theta(Z,T) = (1+T)Z + 0.01(1-Z^2) - TZ^3 + c_0\Big\{ -0.12495Z + 0.2499Z^2 - 0.175Z^3 + 0.00005Z^4 + 0.05Z^5 + 0.135TZ + 0.75TZ^2 - 0.02TZ^3 - 0.875TZ^4 + 0.01TZ^5 + 0.5T^2Z^2 - T^2Z^4 + 0.5T^2Z^6\Big\} + \cdots \tag{29}$$

which gives the moisture content at depth Z for time T.

Using Mathematica, we plotted the c_0-curves of $\Theta_Z(0.5,0.5)$ (DotDashed) and $\Theta_Z(1,0)$ (Thick) (see figure 5) and the proper value of $c_0 = -0.5$ is chosen from this c_0-curve.

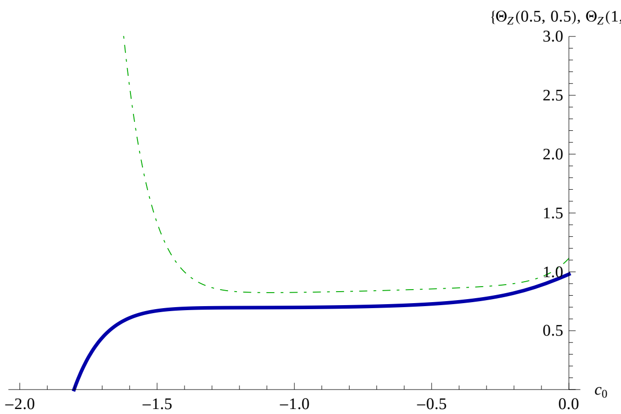

FIGURE 5. The c_0-curves of $\Theta_Z(0.5,0.5)$ (DotDashed) and $\Theta_Z(1,0)$ (Thick).

The numerical values of solution are given in table 2 and graphical interpretations are obtained (see figures 6-7). Figure 6 represents the graph of $\Theta(Z,T)$ v/s Z for a fixed time $T = 0.2, 0.4, ..., 1$ and figure 7 presents the graph of $\Theta(Z,T)$ v/s depth Z and time T.

TABLE 2. Numerical values of the moisture content Θ.

T	$Z=0.0$	$Z=0.1$	$Z=0.2$	$Z=0.3$	$Z=0.4$	$Z=0.5$	$Z=0.6$	$Z=0.7$	$Z=0.8$	$Z=0.9$	$Z=1.0$
0.1	0.01000	0.17640	0.30690	0.41732	0.51574	0.60640	0.69164	0.77285	0.85093	0.92650	1.00000
0.2	0.01000	0.18033	0.31083	0.42024	0.51767	0.60760	0.69238	0.77331	0.85122	0.92665	1.00000
0.3	0.01000	0.18412	0.31459	0.42315	0.51974	0.60903	0.69333	0.77392	0.85158	0.92681	1.00000
0.4	0.01000	0.18778	0.31822	0.42605	0.52194	0.61065	0.69448	0.77468	0.85201	0.92700	1.00000
0.5	0.01000	0.19133	0.32174	0.42896	0.52426	0.61244	0.69578	0.77555	0.85252	0.92720	1.00000
0.6	0.01000	0.19477	0.32516	0.43188	0.52668	0.61437	0.69724	0.77654	0.85308	0.92742	1.00000
0.7	0.01000	0.19811	0.32850	0.43481	0.52919	0.61643	0.69882	0.77763	0.85371	0.92767	1.00000
0.8	0.01000	0.20135	0.33176	0.43775	0.53177	0.61861	0.70052	0.77882	0.85439	0.92792	1.00000
0.9	0.01000	0.20451	0.33496	0.44071	0.53443	0.62088	0.70232	0.78009	0.85512	0.92820	1.00000
1.0	0.01000	0.20758	0.33811	0.44368	0.53715	0.62324	0.70421	0.78143	0.85590	0.92849	1.00000

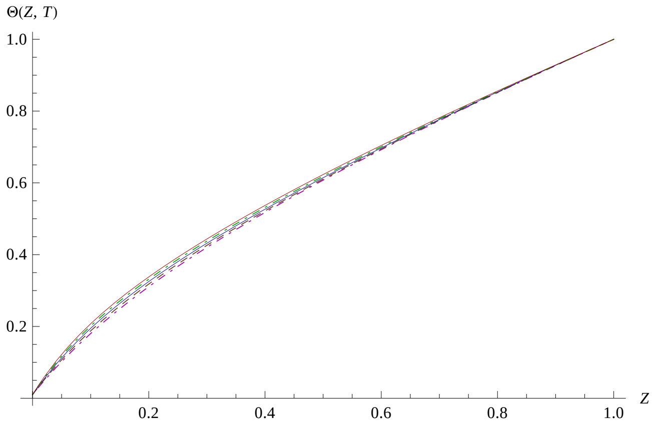

FIGURE 6. The graph of $\Theta(Z,T)$ v/s Z for fixed $T = 0.2$(*lowermost graph*), 0.4, 0.6, 0.8, 1(*uppermost graph*).

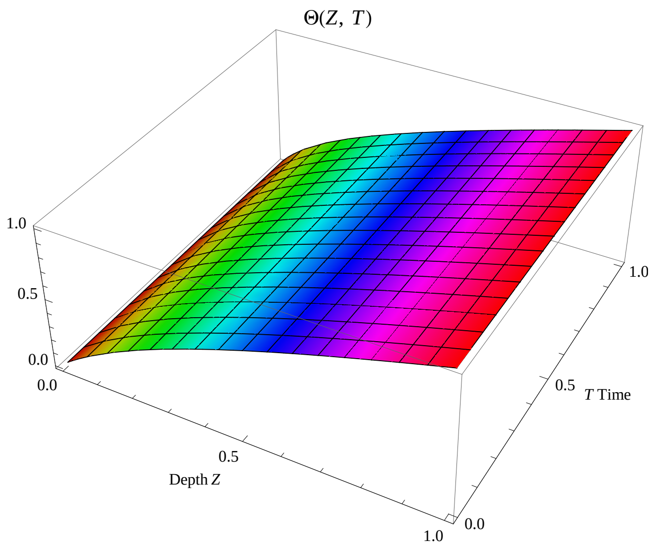

FIGURE 7. The graph of $\Theta(Z,T)$ v/s Z and T.

4.3. **Solution of equation** (11)**.** Let us consider the nonlinear operator $\mathcal{N}[\phi(Z,T;q)]$ from (11) as

$$\mathcal{N}[\phi(Z,T;q)] = \phi(Z,T;q)^2 \frac{\partial^2 \phi(Z,T;q)}{\partial Z^2} + 2\phi(Z,T;q)\left\{\frac{\partial \phi(Z,T;q)}{\partial Z}\right\}^2 - \frac{1}{2}\frac{\partial \phi(Z,T;q)}{\partial Z} - \frac{\partial \phi(Z,T;q)}{\partial T}. \quad (30)$$

Using (30) in (20), we get

$$\mathcal{R}_m(\overrightarrow{\Theta_{m-1}}) = \sum_{j=0}^{m-1} \Theta_j \sum_{i=0}^{m-1-j} \Theta_i \frac{\partial^2 \Theta_{m-1-j-i}}{\partial Z^2} - 2\sum_{j=0}^{m-1} \Theta_j \sum_{i=0}^{m-1-j} \frac{\partial \Theta_i}{\partial Z}\frac{\partial \Theta_{m-1-j-i}}{\partial Z} - \frac{1}{2}\frac{\partial \Theta_{m-1}}{\partial Z} - \frac{\partial \Theta_{m-1}}{\partial T}, m \geq 1 \quad (31)$$

and (23) gives us the homotopy analysis solution of (11) as

$$\begin{aligned}\Theta(Z,T) = {} & (1+T)Z + 0.01(1-Z^2) - TZ^3 + c_0\Big\{0.031767Z - 0.240001Z^2 \\ & + \frac{0.5044Z^3}{3} - 0.009999Z^4 + 0.0501Z^5 - \frac{0.000001Z^6}{3} + 0.1251TZ \\ & - 0.23TZ^2 + 0.9997TZ^3 + 0.085TZ^4 - 0.9997TZ^5 + 0.02TZ^6 \\ & - 0.0001TZ^7 + 0.01T^2Z^2 + T^2Z^3 - 0.03T^2Z^4 - 2T^2Z^5 + 0.03T^2Z^6 \\ & + T^2Z^7 - 0.01T^2Z^8 + \frac{T^3Z^3}{3} - T^3Z^5 + T^3Z^7 - \frac{T^3Z^9}{3}\Big\} + \cdots . \quad (32)\end{aligned}$$

The convergent homotopy analysis solution is obtained using the proper value of c_0. We plotted the c_0-curves of $\Theta_Z(0.5, 0.5)$ (DotDashed) and $\Theta_Z(1, 0)$ (Thick) in figure 8. The proper value of $c_0 = -0.5$ is chosen for numerical and graphical representations of solution.

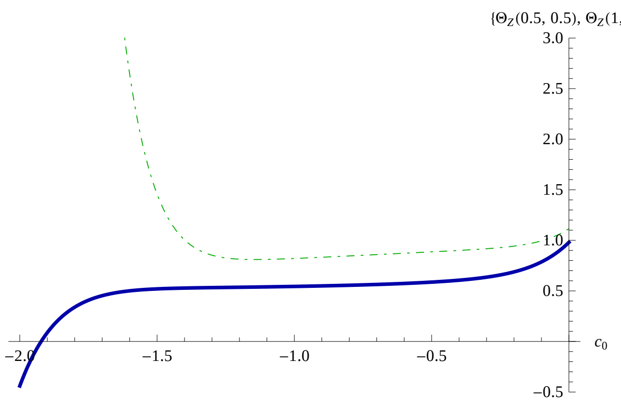

FIGURE 8. The c_0-curves of $\Theta_Z(0.5, 0.5)$ (DotDashed) and $\Theta_Z(1, 0)$ (Thick).

The numerical values of the moisture content are given in table 3. The graph of $\Theta(Z,T)$ v/s depth Z for a fixed time T is given in figure 9. In the figure 10, the graph of $\Theta(Z,T)$ v/s depth Z and time T is presented.

5. Conclusions

We discussed the one dimensional groundwater recharge by spreading through unsaturated porous medium. The homotopy analysis method is adopted to solve the governing equations. The series solutions are obtained for the equations which presented on the basis of linear and nonlinear conductivity and diffusivity functions. The solutions are satisfy both boundary conditions. The numerical and graphical representations of solutions are given. Moisture content of soil is increase when depth increases for a given time.

TABLE 3. Numerical values of the moisture content Θ.

T	$Z=0.0$	$Z=0.1$	$Z=0.2$	$Z=0.3$	$Z=0.4$	$Z=0.5$	$Z=0.6$	$Z=0.7$	$Z=0.8$	$Z=0.9$	$Z=1.0$
0.1	0.01000	0.12580	0.25712	0.39116	0.51643	0.62666	0.72116	0.80265	0.87461	0.93981	1.00000
0.2	0.01000	0.13388	0.27120	0.40647	0.52880	0.63465	0.72559	0.80498	0.87586	0.94040	1.00000
0.3	0.01000	0.14194	0.28480	0.42071	0.54007	0.64204	0.72996	0.80745	0.87720	0.94098	1.00000
0.4	0.01000	0.14994	0.29789	0.43396	0.55038	0.64899	0.73431	0.81004	0.87860	0.94157	1.00000
0.5	0.01000	0.15788	0.31047	0.44628	0.55990	0.65558	0.73864	0.81272	0.88007	0.94216	1.00000
0.6	0.01000	0.16575	0.32253	0.45775	0.56877	0.66191	0.74297	0.81547	0.88159	0.94277	1.00000
0.7	0.01000	0.17355	0.33408	0.46845	0.57709	0.66803	0.74728	0.81829	0.88316	0.94338	1.00000
0.8	0.01000	0.18125	0.34513	0.47848	0.58496	0.67398	0.75159	0.82115	0.88478	0.94400	1.00000
0.9	0.01000	0.18886	0.35568	0.48790	0.59245	0.67978	0.75588	0.82406	0.88643	0.94463	1.00000
1.0	0.01000	0.19637	0.36576	0.49678	0.59963	0.68546	0.76016	0.82699	0.88812	0.94527	1.00000

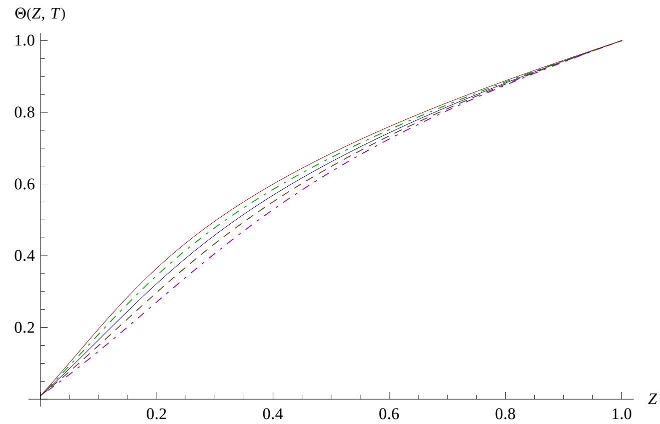

FIGURE 9. The graph of $\Theta(Z,T)$ v/s Z for fixed T = 0.2(*lowermost graph*), 0.4, 0.6, 0.8, 1(*uppermost graph*).

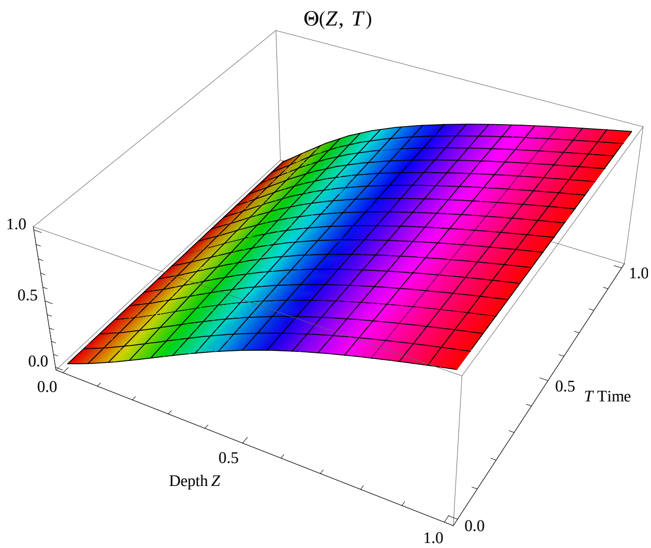

FIGURE 10. The graph of $\Theta(Z,T)$ v/s Z and T.

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