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DEGREE OF APPROXIMATION BY PRODUCT $(\overline{N}, p_n, q_n)(E, q)$ SUMMABILITY OF FOURIER SERIES OF A SIGNAL BELONGING TO $Lip(\alpha, r)$-CLASS

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Abstract. Approximation of periodic functions by different linear summation methods have been studied by many researchers. Further, for sharpening the estimate of errors out of the approximations several product summability methods were introduced by different investigators. In this paper a new theorem has been established on $(\overline{N}, p_n, q_n)(E, q)$-summability of Fourier series of a function belonging to $f \in Lip(\alpha, r)$ class that generalizes several known results.

Keywords: Degree of approximation, Fourier series, $Lip(\alpha, r)$-class, $(\overline{N}, p_n, q_n)(E, s)$-mean, Lebesgue integral.

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1. Introduction

The Theory of Summability is a wide field of Mathematics as regards to the study of Analysis and Functional Analysis. It has many applications in Numerical Analysis (to study the rate of convergence), Operator Theory (approximation of functions of positive linear operators), the theory of orthogonal series and approximation theory etc. Approximation Theory has been originated from a well known theorem of Weierstrass, while approximating a continuous function in an given interval by a polynomial. Later the study was extended to approximate the piecewise continuous periodic function by trigonometric polynomials. Next for estimating the errors out of the approximations, it was noticed that the error is minimum if the coefficients of n^{th} trigonometric polynomial are the Fourier coefficients. Thus n^{th} partial sum of Fourier series is a better estimate for the approximation of a periodic function. Next for accuracy of estimations to a certain degree, different

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linear summation methods of Fourier series of 2π periodic functions on real line $\mathbb{R}$ (that is, Cesàro mean, Nörlund mean, Matrix mean etc.) were introduced. Much of advanced in the theory of trigonometric approximations has been studied by different investigators for periodic functions of $Lip(\alpha, r)$-class. The degree of approximation of functions belonging to $Lip(\alpha)$, $Lip(\alpha, r)$, $Lip(\xi(t), r)$, $(r \geq 1)$-class has been studied by various investigators like Pradhan *et al.* [13], Lal [3], Mishra *et al.* ([4], [5], [6], [7], [8] and [9]), Paikray *et al.* [12], Deepmala et al. [1] and Misra *et al.* [10]. Recently, Nigam [11] has established a result on product summability of a function of Lipschitz class for sharpening the estimate of errors. In an attempt to make an advance study in this direction a new theorem on $(\overline{N}, p_n, q_n)(E, q)$ product summability for a function f belongs to $Lip(\alpha, r)$-class has been established.

2. Definition and Notations

Let $\sum u_n$ be a given infinite series with the sequence of partial sum $\{s_n\}$. Let $\{p_n\}$ and $\{q_n\}$ be sequences of positive real numbers such that,

$$P_n = \sum_{k=0}^{n} p_k \text{ and } Q_n = \sum_{k=0}^{n} q_k.$$

The sequence to sequence transformation,

$$t_n^{\overline{N}} = \frac{1}{R_n} \sum_{k=0}^{n} p_k q_k s_k, \tag{1}$$

where $R_n = p_0 q_n + p_1 q_{n-1} + \dots + p_n q_0 \neq 0$, $(p_{-1} = q_{-1} = R_{-1} = 0)$, defines the $(\overline{N}, p_n, q_n)$ mean of the sequence $\{s_n\}$ generated by the sequence of coefficients $\{p_n\}$ and $\{q_n\}$, and it is denoted by $\{t_n^{\overline{N}}\}$.

If $\lim\limits_{n\to\infty} t_n^{\overline{N}} \to s$, then the series $\sum u_n$ is $(\overline{N}, p_n, q_n)$ summable to s.

The necessary and sufficient conditions for regularity of $(\overline{N}, p_n, q_n)$ summability are

(i) $\frac{p_k q_k}{R_n} \to 0$, for each integer $k \geq 0$ as $n \to \infty$ and

(ii) $|\sum_{k=0}^{n} p_k q_k| < C|\mathbb{R}_n|$,

where, C is any positive integer independent of n.

The sequence to sequence transformation,

$$E_n^s = \frac{1}{(1+s)^n} \sum_{v=0}^{n} \binom{n}{v} s^{n-v} s_v \tag{2}$$

defines the (E, s) mean of the sequence $\{s_n\}$ and let it denoted by $\{E_n^s\}$.

If $E_n^s \to s$ as $n \to \infty$, then $\sum u_n$ is summable to s with respect to (E, s) summability and (E, s) method is regular (see [2]).

Now we define, a new composite transformation $(\overline{N}, p_n, q_n)$ over (E, s) of $\{s_n\}$ as,

$$T_n^{\overline{N}E} = \frac{1}{R_n} \sum_{k=0}^{n} p_k q_k \left(E_k^s\right) = \frac{1}{R_n} \sum_{k=0}^{n} p_k q_k \left\{ \frac{1}{(1+s)^k} \sum_{v=0}^{k} \binom{k}{v} s^{k-v} s_v \right\}. \tag{3}$$

If $T_n^{\overline{N}E} \to s$ as $n \to \infty$, then $\sum u_n$ is summable to s by $(\overline{N}, p_n, q_n)(E, s)$ summability method.

Let $s_n \to s$ implies $E_n^s(s_n) \to s$ as $n \to \infty$, thus (E, s) method is regular. Now we may write, $T_n^{\overline{N}E} = t_n^{\overline{N}}(E_n^s(s_n)) \to s$ as $n \to \infty$. Therefore, $(\overline{N}, p_n, q_n)(E, s)$ method is also regular.

Remark 2.1. *If we put $q_n = 1$, in equation (1) then $(\overline{N}, p_n, q_n)$-summability reduces to $(\overline{N}, p_n)$-summability and for $p_n = 1$, it reduces to $(\overline{N}, q_n)$-summability. If we put $p_n = 1$ and $q_n = 1$, then $(\overline{N}, p_n, q_n)$-summability reduces to $(C, 1)$-summability.*

Let f be a 2π periodic function (signal) belonging to $L^r[0, 2\pi]$ $(r \geq 1)$, then the Fourier series of f is given by

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx). \tag{4}$$

Let $s_n(f)$ be the n^{th} partial sum of the Fourier series (4), then

$$s_n(f) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx). \tag{5}$$

As regards to the Lipschitz classes, we have:

A signal (function) $f \in Lip(\alpha)$, if

$$|f(x+t) - f(x)| = O(|t|^{\alpha}) \ (0 < \alpha \leq 1) \ (t > 0)$$

and $f \in Lip(\alpha, r)$, if

$$\left(\int_{[0,2\pi]} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O(|t|^{\alpha}) \ (0 < \alpha \leq 1) \ (t > 0) \ (r \geq 1).$$

Remark 2.2. *If we take $r \to \infty$, then the $Lip(\alpha, r)$-class reduces to the $Lip(\alpha)$-class.*

Furthermore as regards to norm, we have the L_∞-norm of a function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$\|f\|_\infty = \sup\{|f(x)| : x \in \mathbb{R}\}$$

and L_r-norm of a function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$\|f\|_r = \left(\int_{[0,2\pi]} |f(x)|^r dx\right)^{\frac{1}{r}} \ (r \geq 1).$$

The degree of approximation of a function $f : \mathbb{R} \to \mathbb{R}$ by a trigonometric polynomial (t_n) of order n under $\|.\|_\infty$ is defined by

$$\|t_n - f(x)\|_\infty = \sup\left\{|t_n(x) - f(x)| : x \in \mathbb{R}\right\},$$

and the degree of approximation of $E_n(f)$ of a function $f \in L_r$ is given by

$$E_n(f) = \min_{t_n} \|t_n - f\|_r.$$

We use the following notations throughout the paper:

$$\begin{aligned} \psi(t) &= f(x+t) + f(x-t) \text{ and} \\ \widetilde{K_n}(t) &= \frac{1}{2\pi R_n} \sum_{k=0}^{n} p_k q_k \left\{ \frac{1}{(1+s)^k} \sum_{v=0}^{k} \binom{k}{v} s^{k-v} \frac{\cos\frac{t}{2} - \cos(v+\frac{1}{2})t}{\sin\frac{t}{2}} \right\}. \end{aligned}$$

3. Known Theorem

Dealing with the $(C,2)(E,1)$ product mean, in 2013 Nigam [11] proved the following theorem.

Theorem 3.1. *If is a 2π periodic function, Lebesgue integrable on $[0,2\pi]$ belonging to the class $Lip(\alpha,r)$ $(0<r<\infty)$ then its degree of approximation by $(C,2)(E,1)$ mean of Fourier series (4) is given by*

$$\|C_n^2E_n^1 - f\|_\infty = O\left\{(n+1)^{1/r-\alpha}\right\} \ (0<\alpha<1), \tag{6}$$

where δ is an arbitrary number such that $0<\delta<\frac{1}{r}-\alpha$, where $\frac{1}{r}+\frac{1}{s}=1$, $(1\le r<\infty)$, and $C_n^2E_n^1$ is $(C,2)(E,1)$ mean of the series (4).

4. Main Theorem

Theorem 4.1. *If f is a 2π periodic function, Lebesgue integrable on $[0,2\pi]$ belonging to the class $Lip(\alpha,r)$ $(0<r<\infty)$ then its degree of approximation by $(\overline{N},p_n,q_n)(E,q)$ mean of Fourier series (4) is given by*

$$\|T_n^{\overline{N}E} - f\|_\infty = O\left\{(n+1)^{1/r-\alpha}\right\} \ (0<\alpha<1) \ (r\ge 1), \tag{7}$$

where δ is an arbitrary number such that $0<\delta<\frac{1}{r}-\alpha$, with $\frac{1}{r}+\frac{1}{s}=1$, $(1\le r<\infty)$, and $T_n^{\overline{N}E}$ is $(\overline{N},p_n,q_n)(E,q)$ mean of the series (4).

To prove the above theorem, first we need to prove the following lemmas.

Lemma 4.1. *Let $|K_n(t)| = O(n)$, $\left(0\le t\le \frac{1}{n+1}\right)$.*

Proof. For $0\le t\le\frac{1}{n+1}$, we have $\sin nt\le n\sin t$, thus

$$\begin{aligned}
|K_n(t)| &= \frac{1}{2\pi r_n}\left|\sum_{k=0}^{n} p_kq_k\left\{\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\frac{sin(v+\frac{1}{2})t}{sin\frac{t}{2}}\right\}\right| \\
&\le \frac{1}{2\pi r_n}\left|\sum_{k=0}^{n} p_kq_k\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}(2v+1)\frac{sin\frac{t}{2}}{sin\frac{t}{2}}\right| \\
&= \frac{1}{2\pi r_n}\left|\sum_{k=0}^{n} p_kq_k\frac{1}{(1+q)^k}(2k+1)\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\right| \\
&\le \frac{(2n+1)}{2\pi r_n}\sum_{k=0}^{n} p_kq_k\frac{1}{(1+q)^k}\left|\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\right| \\
&= \frac{(2n+1)}{2\pi r_n}\sum_{k=0}^{n} p_kq_k\frac{1}{(1+q)^k}(1+q)^k \\
&= O(n).
\end{aligned}$$

□

Lemma 4.2. *Let $|K_n(t)| = O(\frac{1}{t})$, $\left(\frac{1}{n+1}<t\le\pi\right)$.*

Proof. For $\frac{1}{n+1} < t \leq \pi$, using Jordans lemma, $sin\ \frac{t}{2} \geq \frac{t}{\pi}$ and sin $nt \leq 1$, we have

$$\begin{aligned}
|K_n(t)| &= \frac{1}{2\pi r_n}\left|\sum_{k=0}^{n}\left\{\frac{1}{(1+q)^n}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\frac{sin(v+\frac{1}{2})t}{sin\frac{t}{2}}\right\}\right| \\
&\leq \frac{1}{2\pi r_n}\left|\sum_{k=0}^{n}p_kq_k\left\{\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\frac{1}{sin\frac{t}{2}}\right\}\right| \\
&\leq \frac{1}{2\pi r_n}\left|\sum_{k=0}^{n}p_kq_k\left\{\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\left(\frac{\pi}{t}\right)\right\}\right| \\
&\leq \frac{1}{2\pi r_n}\left(\frac{\pi}{t}\right)\sum_{k=0}^{n}p_kq_k\frac{1}{(1+q)^k}\left|\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\right| \\
&= \frac{1}{2tr_n}\sum_{k=0}^{n}p_kq_k \\
&= O\left(\frac{1}{t}\right).
\end{aligned}$$

□

5. Proof of the Theorem 4.1

Proof. The integral representation of $s_n(f;x)$ is given by

$$s_n(f;x) = -\frac{1}{\pi}\int_0^{\pi}\psi(t)\frac{\sin(t/2)-\sin(n+1/2)t}{2\sin(t/2)}.$$

Therefore, we have

$$s_n(f;x) - f(x) = \frac{1}{2\pi}\int_{[0,\pi]}\psi(t)\frac{\sin(n+\frac{1}{2})t}{\sin(\frac{t}{2})}dt.$$

By equation (3) we have,

$$\begin{aligned}
\|T_n^{\overline{N}E} - f\| &= \frac{1}{2\pi r_n}\sum_{k=0}^{n}p_kq_k\int_0^{\pi}\frac{\phi(t)}{\sin\frac{t}{2}}\frac{1}{(1+q)^k}\left\{\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\sin\left(v+\frac{1}{2}\right)tdt\right\} \\
&= \int_0^{\pi}\phi(t)K_n(t)dt \\
&= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right]\phi(t)K_n(t)dt \\
&= I_1 + I_2.
\end{aligned}$$

If $f \in (L_r, t^\alpha) \Rightarrow \phi(t) \in (L_r, t^\alpha)$ and by using Hölder inequality and Lemma 4.1, we have

$$\begin{aligned}
|I_1| &\leq \int_0^{\frac{1}{n+1}} \left| \frac{\phi(t)}{t^\alpha} . t^\alpha K_n(t) \right| dt \\
&= \left(\int_0^{\frac{1}{n+1}} \left| \frac{\phi(t)}{t^\alpha} \right|^r dt \right)^{\frac{1}{r}} \left(\int_0^{\frac{1}{n+1}} \left| t^\alpha K_n(t) \right|^r dt \right)^{\frac{1}{r}} \\
&= O(1) \left(\int_0^{\frac{1}{n+1}} t^\alpha n^r dt \right)^{\frac{1}{r}} \\
&= O\left(\frac{1}{n+1} \right)^\alpha \left(\frac{n^r}{n+1} \right)^{\frac{1}{r}} \\
&= O\left(\left(\frac{1}{n+1} \right)^\alpha \frac{1}{(n+1)^{\frac{1}{r}-1}} \right) \\
&= O\left(\left(\frac{1}{n+1} \right)^\alpha \frac{1}{(n+1)^{-\frac{1}{r}}} \right) \\
&= O\left(\left(\frac{1}{n+1} \right)^\alpha (n+1)^{\frac{1}{r}} \right) \\
&= O\left(\frac{1}{(n+1)^{\alpha-1/r}} \right). \qquad (8)
\end{aligned}$$

Next, by using Hölder inequality and Lemma 4.2, we have

$$\begin{aligned}
&|I_2| \leq \int_{\frac{1}{n+1}}^{\pi} \left| \frac{\phi(t)}{t^\alpha} . t^\alpha K_n(t) \right| dt \\
&= \left(\int_{\frac{1}{n+1}}^{\pi} \left| \frac{\phi(t)}{t^\alpha} \right|^r dt \right)^{\frac{1}{r}} \left(\int_{\frac{1}{n+1}}^{\pi} \left| t^\alpha K_n(t) \right|^r dt \right)^{\frac{1}{r}} \\
&= O(1) \left(\int_{\frac{1}{n+1}}^{\pi} \left| \left(\frac{t^\alpha}{t} \right) \right|^r dt \right)^{\frac{1}{r}} \\
&= O(1) \left(\int_{\frac{1}{\pi}}^{n+1} \left| \frac{(\frac{1}{y})^\alpha}{\frac{1}{y}} \right|^r \frac{dy}{y^2} \right)^{\frac{1}{r}} \\
&= O\left((n+1) \left(\frac{1}{n+1} \right)^\alpha \right) \left(\int_\varepsilon^\pi \frac{1}{y^2} dy \right)^{\frac{1}{r}}, \text{ for some } \frac{1}{\pi} \leq \ \varepsilon \leq \ n+1 \\
&= O\left(\frac{1}{(n+1)^{\alpha-1/r}} \right). \qquad (9)
\end{aligned}$$

Clearly, from (8) and (9),

$$\left| T_n^{\overline{N}E} - f(x) \right| = O\left(\frac{1}{(n+1)^{\alpha-1/r}} \right), \ (r \geq 1).$$

This implies,

$$\| T_n^{\overline{N}E} - f(x) \|_\infty = \sup_{-\pi < x < \pi} | T_n^{\overline{N}E} - f(x) \| = O\left(\frac{1}{(n+1)^{\alpha-1/r}} \right) \ (r \geq 1).$$

This completes the proof of the theorem. □

6. Conclusions

The result established here is more general than some earlier existing results in the sense that, for $p_n = 1$ and $q_n = 1$ our proposed mean reduces to $(\bar{N}, q_n)(E, q)$ mean and $(\bar{N}, p_n)(E, q)$ respectively. Moreover, for $p_n = 1$ and $q_n = 1$ our proposed mean reduces to $(C, 1)(E, q)$ mean. In particular, if we substitute $p_n = 2$, $q_n = 1$ and $q = 1$, then our proposed $(\bar{N}, p_n, q_n)(E, q)$ mean reduces to $(C, 2)(E, 1)$ mean of Nigam (see [11]). Thus, our result is a non-trivial extension the result of Nigam [11].

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